THEORY AND CALCULATION

OF

ALTERNATING CURRENT PHENOMENA

BY

CHARLES PROTEUS STEINMETZ

FOURTH EDITION
THOROUGHLY REVISED AND GREATLY ENLARGED

SECOND IMPRESSION

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DEDICATED

TO THE

MEMORY OF MY FATHER

CARL HEINRICH STEINMETZ

PREFACE TO THE FOURTH EDITION.

Eight years have elapsed since the appearance of the third edition, during which time the book has been reprinted without change, and a revision therefore became greatly desired.

It was very gratifying, however, to find that none of the contents of the former edition had to be dropped as superseded or antiquated. However, very much new material had to be added. During these eight years the electrical industry has progressed at least as rapidly as in any previous period, and apparatus and phenomena which at the time of the third edition were of theoretical interest only, or of no interest at all, have assumed great industrial importance, as, for instance, the single-phase commutator motor or high frequency phenomena, and therefore require an extensive recognition.

Besides rewriting and enlarging numerous paragraphs throughout the text, nine entirely new chapters have been added, on single-phase commutator motors and alternating-current motors in general, on induction machines, armature reaction of alternators, surging of synchronous machines, phase control, constant-current transformation, etc., and six other chapters greatly enlarged. There has also been added to the theory of apparatus and phenomena a short outline of the method of calculating the constants entering into the theory, and of their determination by experimental tests.

The greatest advance of late years has been in the study of transient phenomena. While the phenomena of the flow of electric energy in stationary or permanent condition were already well understood eight years ago, with the exception of the classic book of Drs. Bedell and Crehore practically no engineering work had been done at that time on the flow of electric energy under transient conditions, as the starting of circuits, short-circuit currents of alternators, inductive discharges, free oscillations of transmission lines, high frequency currents, surges, oscillations, traveling waves, etc., and very little on circuits containing distributed capacity and inductance, as transmission lines, cables, telephone lines, etc., while at

present we are rapidly approaching the time when these phenomena will be as well understood as those of the steady flow of energy. To give even a short abstract of the theory of these phenomena (which are probably the most important the engineer has to meet at present) would have been impossible without enormously increasing the already large volume, and they will therefore be published as a separate work on "Theory and Calculation of Transient Electric Phenomena and Oscillations," which is in press and may be considered as the second volume of the present work.

When reading the book, or using it as text-book, it is recommended:

First, to take up Part I of "Theoretical Elements of Electrical Engineering," which deals with general theory and is intended as an introduction to this treatise.

Then to proceed with the reading of the present volume, but when reading the chapters dealing with the theory of apparatus, to parallel their study with the reading of the corresponding chapters of Part II of "Theoretical Elements of Electrical Engineering," which deal with the same subjects in a different manner, and thus should greatly assist in imparting a clear insight into the nature of apparatus.

After finishing the main parts of these two books, the reading of "Theory and Calculation of Transient Electric Phenomena and Oscillations" may be taken up.

Where time is limited, a large part of the mathematical discussion may be skipped and in that way a general review of the material gained.

The reader is advised, after completing Chapter V, carefully to read Appendix I, on the algebra of the complex quantity, before continuing with the main text.

Great thanks are due to the technical staff of the McGraw Publishing Company, which has spared no effort to produce the fourth edition in as perfect and systematic a manner as possible, and to the numerous engineers who have greatly assisted me by pointing out typographical and other errors in the previous edition.

CHARLES PROTEUS STEINMETZ.

CAMP MOHAWK, VIELE'S CREEK, SCHENECTADY, N. Y.
August, 1908.

PREFACE TO FIRST EDITION.

THE following volume is intended as an exposition of the methods which I have found useful in the theoretical investigation and calculation of the manifold phenomena taking place in alternating-current circuits, and of their application to alternating-current apparatus.

While the book is not intended as first instruction for a beginner, but presupposes some knowledge of electrical engineering, I have endeavored to make it as elementary as possible, and have therefore used only common algebra and trigonometry, practically excluding calculus, except in §§ **144** to **151** and Appendix II; and even §§ **144** to **151** have been paralleled by the elementary approximation of the same phenomenon in §§ **140** to **143**.

All the methods used in the book have been introduced and explicitly discussed, with examples of their application, the first part of the book being devoted to this. In the investigation of alternating-current phenomena and apparatus, one method only has usually been employed, though the other available methods are sufficiently explained to show their application.

A considerable part of the book is necessarily devoted to the application of complex imaginary quantities, as the method which I found most useful in dealing with alternating-current phenomena; and in this regard the book may be considered as an expansion and extension of my paper on the application of complex imaginary quantities to electrical engineering, read before the International Electrical Congress at Chicago, 1893. The complex imaginary quantity is gradually introduced, with full explanations, the algebraic operations with complex quantities being discussed in Appendix I, so as not to require from the reader any previous knowledge of the algebra of the complex imaginary plane.

While those phenomena which are characteristic of polyphase

systems, as the resultant action of the phases, the effects of unbalancing, the transformation of polyphase systems, etc., have been discussed separately in the last chapters, many of the investigations in the previous parts of the book apply to polyphase systems as well as single-phase circuits, as the chapters on induction motors, generators, synchronous motors, etc.

A part of the book is original investigation, either published here for the first time, or collected from previous publications and more fully explained. Other parts have been published before by other investigators, either in the same, or more frequently in a different form.

I have, however, omitted altogether literary references, for the reason that incomplete references would be worse than none, while complete references would entail the expenditure of much more time than is at my disposal, without offering sufficient compensation; since I believe that the reader who wants information on some phenomenon or apparatus is more interested in the information than in knowing who first investigated the phenomenon.

Special attention has been given to supply a complete and extensive index for easy reference, and to render the book as free from errors as possible. Nevertheless, it probably contains some errors, typographical and otherwise; and I will be obliged to any reader who on discovering an error or an apparent error will notify me.

I take pleasure here in expressing my thanks to Messrs. W. D. Weaver, A. E. Kennelly, and Townsend Wolcott, for the interest they have taken in the book while in the course of publication, as well as for the valuable assistance given by them in correcting and standardizing the notation to conform to the international system, and numerous valuable suggestions regarding desirable improvements.

Thanks are due also to the publishers, who have spared no effort or expense to make the book as creditable as possible mechanically.

CHARLES PROTEUS STEINMETZ.

January, 1897.

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THEORY AND CALCULATION
OF
ALTERNATING-CURRENT PHENOMENA

CHAPTER I.

INTRODUCTION.

1. In the practical applications of electrical energy, we meet with two different classes of phenomena, due respectively to the continuous current and to the alternating current.

The continuous-current phenomena have been brought within the realm of exact analytical calculation by a few fundamental laws:

(1) Ohm's law: $i = \frac{e}{r}$, where r, the resistance, is a constant of the circuit.

(2) Joule's law: $P = i^2r$, where P is the power, or the rate at which energy is expended by the current, i, in the resistance, r.

(3) The power equation: $P_0 = ei$, where P_0 is the power expended in the circuit of e.m.f., e, and current, i.

(4) Kirchhoff's laws:

(*a*) The sum of all the e.m.fs. in a closed circuit $= 0$, if the e.m.f. consumed by the resistance, ir, is also considered as a counter e.m.f., and all the e.m.fs. are taken in their proper direction.

(*b*) The sum of all the currents directed towards a distributing point $= 0$.

In alternating-current circuits, that is, in circuits in which the currents rapidly and periodically change their direction, these laws cease to hold. Energy is expended, not only in the conductor through its ohmic resistance, but also outside of it; energy is stored up and returned, so that large currents may

exist simultaneously with high e.m.fs., without representing any considerable amount of expended energy, but merely a surging to and fro of energy; the ohmic resistance ceases to be the determining factor of current value; currents may divide into components, each of which is larger than the undivided current, etc.

2. In place of the above-mentioned fundamental laws of continuous currents, we find in alternating-current circuits the following:

Ohm's law assumes the form, $i = \frac{e}{z}$, where z, the apparent resistance, or *impedance*, is no longer a constant of the circuit, but depends upon the frequency of the currents; and in circuits containing iron, etc., also upon the e.m.f.

Impedance, z, is, in the system of absolute units, of the same dimension as resistance (that is, of the dimension $\cancel{LT^{-1}}$ = velocity), and is expressed in ohms. $|R| = ML^2T^{-1}Q^{-2}$

It consists of two components, the resistance, r, and the reactance, x, or —

$$z = \sqrt{r^2 + x^2}.$$

The resistance, r, in circuits where energy is expended only in heating the conductor, is the same as the ohmic resistance of continuous-current circuits. In circuits, however, where energy is also expended outside of the conductor by magnetic hysteresis, mutual inductance, dielectric hysteresis, etc., r is larger than the true ohmic resistance of the conductor, since it refers to the total expenditure of energy. It may be called then the *effective resistance.* It is no longer a constant of the circuit.

The reactance, x, does not represent the expenditure of energy as does the effective resistance, r, but merely the surging to and fro of energy. It is not a constant of the circuit, but depends upon the frequency, and frequently, as in circuits containing iron, or in electrolytic conductors, upon the e.m.f. also. Hence, while the effective resistance, r, refers to the power component of e.m.f., or the e.m.f. in phase with the current; the reactance, x, refers to the wattless component of e.m.f., or the e.m.f. in quadrature with the current.

3. The principal sources of reactance are electromagnetism and capacity.

Electromagnetism.

An electric current, i, in a circuit, produces a magnetic flux surrounding the conductor in lines of magnetic force (or more correctly, lines of magnetic induction), of closed, circular, or other form, which alternate with the alternations of the current, and thereby generate an e.m.f. in the conductor. Since the magnetic flux is in phase with the current, and the generated e.m.f. 90°, or a quarter period, behind the flux, this *e.m.f. of self-induction* lags 90°, or a quarter period, behind the current; that is, is in quadrature therewith, and therefore wattless.

If now ϕ = the magnetic flux produced by, and interlinked with, the current i (where those lines of magnetic force, which are interlinked n-fold, or pass around n turns of the conductor, are counted n times), the ratio, $\frac{\phi}{i}$, is denoted by L, and called the *inductance* of the circuit. It is numerically equal, in absolute units, to the interlinkages of the circuit with the magnetic flux produced by unit current, and is, in the system of absolute units, of the dimension of length. Instead of the inductance, L, sometimes its ratio with the ohmic resistance, r, is used, and is called the *time-constant* of the circuit,

$$T = \frac{L}{r}.$$

If a conductor surrounds with n turns a magnetic circuit of reluctance, $\mathcal{R}$, the current, i, in the conductor represents the m.m.f. of ni ampere-turns, and hence produces a magnetic flux of $\frac{ni}{\mathcal{R}}$ lines of magnetic force, surrounding each n turns of the conductor, and thereby giving $\phi = \frac{n^2 i}{\mathcal{R}}$ interlinkages between the magnetic and electric circuits. Hence the inductance is

$$L = \frac{\phi}{i} = \frac{n^2}{\mathcal{R}}.$$

The fundamental law of electromagnetic induction is, that the e.m.f. generated in a conductor by a magnetic field is proportional to the rate of cutting of the conductor through the magnetic field.

Hence, if i is the current, and L is the inductance of a circuit, the magnetic flux interlinked with a circuit of current, i, is Li, and $4\,fLi$ is consequently the average rate of cutting; that is, the number of lines of force cut by the conductor per second, where f = frequency, or number of complete periods (double reversals) of the current per second, i = maximum value of current.

Since the maximum rate of cutting bears to the average rate the same ratio as the quadrant to the radius of a circle (a sinusoidal variation supposed), that is the ratio $\frac{\pi}{2} \div 1$, the maximum rate of cutting is $2\,\pi f$, and, consequently, the maximum value of e.m.f. generated in a circuit of maximum current value, i, and inductance, L, is

$$e = 2\,\pi fLi.$$

Since the maximum values of sine waves are proportional (by factor $\sqrt{2}$) to the effective values (square root of mean squares), if i = effective value of alternating current, $e = 2\,\pi fLi$ is the effective value of e.m.f. of self-induction, and the ratio, $\frac{e}{i} = 2\,\pi fL$, is the *inductive reactance,*

$$x_m = 2\,\pi fL.$$

Thus, if r = resistance, x_m = reactance, z = impedance,
the e.m.f. consumed by resistance is: $e_1 = ir$;
the e.m.f. consumed by reactance is: $e_2 = ix_m$;

and, since both e.m.fs. are in quadrature to each other, the total e.m.f. is

$$e = \sqrt{e_1^2 + e_2^2} = i\sqrt{r^2 + x_m^2} = iz;$$

that is, the impedance, z, takes in alternating-current circuits the place of the resistance, r, in continuous-current circuits.

Capacity.

4. If upon a condenser of capacity, C, an e.m.f., e, is impressed, the condenser receives the electrostatic charge, Ce.

If the e.m.f., e, alternates with the frequency, f, the average rate of charge and discharge is $4\,f$, and $2\,\pi f$ the maximum rate of charge and discharge, sinusoidal waves supposed; hence,

$i = 2\,\pi f Ce$, the current to the condenser, which is in quadrature to the e.m.f., and leading.

It is then . $$x_c = \frac{e}{i} = \frac{1}{2\,\pi f C},$$

the "*condensive reactance.*"

Polarization in electrolytic conductors acts to a certain extent like capacity.

The condensive reactance is inversely proportional to the frequency and represents the leading out-of-phase wave; the inductive reactance is directly proportional to the frequency, and represents the lagging out-of-phase wave. Hence both are of opposite sign with regard to each other, and the total reactance of the circuit is their difference, $x = x_m - x_c$.

The total resistance of a circuit is equal to the sum of all the resistances connected in series; the total reactance of a circuit is equal to the algebraic sum of all the reactances connected in series; the total impedance of a circuit, however, is not equal to the sum of all the individual impedances, but in general less, and is the resultant of the total resistance and the total reactance. Hence it is not permissible directly to add impedances, as it is with resistances or reactances.

A further discussion of these quantities will be found in the later chapters.

5. In Joule's law, $P = i^2 r$, r is not the true ohmic resistance, but the "effective resistance"; that is, the ratio of the power component of e.m.f. to the current. Since in alternating-current circuits, in addition to the energy expended in the ohmic resistance of the conductor, energy is expended, partly outside, partly inside, of the conductor, by magnetic hysteresis, mutual induction, dielectric hysteresis, etc., the effective resistance, r, is in general larger than the true resistance of the conductor, sometimes many times larger, as in transformers at open secondary circuit, and is no longer a constant of the circuit. It is more fully discussed in Chapter VIII.

In alternating-current circuits the power equation contains a third term, which, in sine waves, is the cosine of the angle of the difference of phase between e.m.f. and current:

$$P_0 = ei \cos \theta.$$

Consequently, even if e and i are both large, P_0 may be very small, if $\cos\theta$ is small, that is, θ near 90°.

Kirchhoff's laws become meaningless in their original form, since these laws consider the e.m.fs. and currents as directional quantities, counted positive in the one, negative in the opposite direction, while the alternating current has no definite direction of its own.

6. The alternating waves may have widely different shapes; some of the more frequent ones are shown in a later chapter.

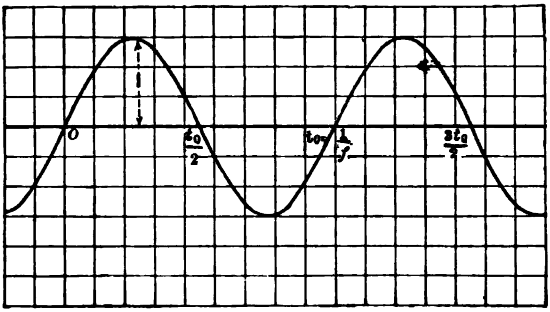


FIG. 1. — Sine Wave.

The simplest form, however, is the sine wave, shown in Fig. 1, or, at least, a wave very near sine shape, which may be represented analytically by

$$i = I \sin \frac{2\pi}{t_0}(t - t_1) = I \sin 2\pi f (t - t_1),$$

where I is the maximum value of the wave, or its *amplitude*; t_0 is the time of one complete cyclic repetition, or the *period* of the wave, or $f = \frac{1}{t_0}$ is the *frequency* or number of complete periods per second; and t_1 is the time, where the wave is zero, or the *epoch* of the wave, generally called the *phase*.*

* "Epoch" is the time where a periodic function reaches a certain value, for instance, zero; and "phase" is the angular position, with respect to a datum position, of a periodic function at a given time. Both are in alternate-current phenomena only different ways of expressing the same thing.

Obviously, "phase" or "epoch" attains a practical meaning only when several waves of different phases are considered, as "difference of phase." When dealing with one wave only, we may count the time from the moment when the wave is zero, or from the moment of its maximum, representing it respectively by

$$i = I \sin 2\pi ft,$$

and

$$i = I \cos 2\pi ft.$$

Since it is a univalent function of time, that is, can at a given instant have one value only, by Fourier's theorem, any alternating wave, no matter what its shape may be, can be represented by a series of sine functions of different frequencies and different phases, in the form:

$$\begin{aligned} i &= I_1 \sin 2\pi f (t - t_1) + I_2 \sin 4\pi f (t - t_2) \\ &\quad + I_3 \sin 6\pi f (t - t_3) + \dots \end{aligned}$$

where $I_1, I_2, I_3, \dots$ are the maximum values of the different components of the wave, $t_1, t_2, t_3 \dots$ the times, where the respective components pass the zero value.

The first term, $I_1 \sin 2\pi f (t - t_1)$, is called the *fundamental wave*, or the *first harmonic*; the further terms are called the *higher harmonics*, or "overtones," in analogy to the overtones of sound waves. $I_n \sin 2n\pi f (t - t_n)$ is the n^{th} harmonic.

By resolving the sine functions of the time differences, $t - t_1$, $t - t_2 \dots$, we reduce the general expression of the wave to the form:

$$\begin{aligned} i &= A_1 \sin 2\pi ft + A_2 \sin 4\pi ft + A_3 \sin 6\pi ft + \dots \\ &\quad + B_1 \cos 2\pi ft + B_2 \cos 4\pi ft + B_3 \cos 6\pi ft + \dots \end{aligned}$$

The two half-waves of each period, the *positive wave* and the *negative wave* (counting in a definite direction in the circuit), are almost always identical, because, for reasons inherent in their construction, practically all alternating-current machines generate e.m.fs. in which the negative half-wave is identical with the positive. Hence the even higher harmonics, which cause a difference in the shape of the two half-waves, disappear, and only the odd harmonics exist, except in very special cases.

Hence the general alternating-current wave is expressed by:

$$i = I_1 \sin 2\pi f(t - t_1) + I_3 \sin 6\pi f(t - t_3) + I_5 \sin 10\pi f(t - t_5) + \ldots,$$

or,

$$i = A_1 \sin 2\pi ft + A_3 \sin 6\pi ft + A_5 \sin 10\pi ft + \ldots + B_1 \cos 2\pi ft + B_3 \cos 6\pi ft + B_5 \cos 10\pi ft + \ldots$$

Such a wave is shown in Fig. 2, while Fig. 3 shows a wave whose half-waves are different. Figs. 2 and 3 represent the

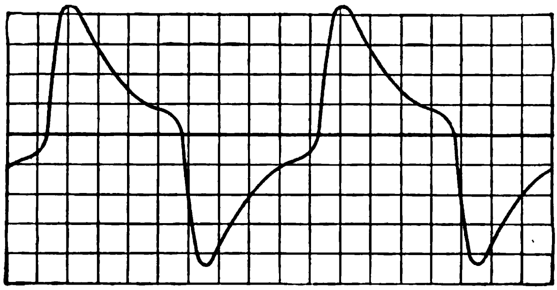


FIG. 2. — Wave without Even Harmonics.

secondary currents of a Ruhmkorff coil, whose secondary coil is closed by a high external resistance; Fig. 3 is the coil operated

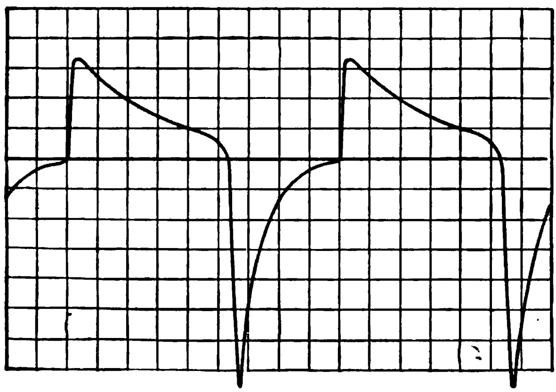


FIG. 3. — Wave with Even Harmonics.

in the usual way, by make and break of the primary battery current; Fig. 2 is the coil fed with reversed currents by a commutator from a battery.

7. Inductive reactance, or electromagnetic momentum, which is always present in alternating-current circuits, — to a large extent in generators, transformers, etc., — tends to suppress the higher harmonics of a complex harmonic wave more than the fundamental harmonic, since the inductive reactance is proportional to the frequency, and is thus greater with the higher harmonics, and thereby causes a general tendency towards simple sine shape, which has the effect, that, in general, the alternating currents in our light and power circuits are sufficiently near sine waves to make the assumption of sine shape permissible.

Hence, in the calculation of alternating-current phenomena, we can safely assume the alternating wave as a sine wave, without making any serious error; and it will be sufficient to keep the distortion from sine shape in mind as a possible disturbing factor, which, however, is in practice generally negligible — except in the case of low-resistance circuits containing large inductive reactance and large condensive reactance in series with each other, so as to produce resonance effects of these higher harmonics, and also under certain conditions of long-distance power transmission and high-potential distribution.

8. Experimentally, the impedance, effective resistance, inductance, capacity, etc., of a circuit or a part of a circuit are conveniently determined by impressing a sine wave of alternating e.m.f. upon the circuit and measuring with alternating-current ammeter, voltmeter and wattmeter, the current, i, in the circuit, the potential difference, e, across the circuit, and the power, p, consumed in the circuit.

Then,

$$\text{The impedance, } z = \frac{e}{i};$$

$$\text{The phase angle, } \cos\theta = \frac{p}{ei};$$

$$\text{The effective resistance, } r = \frac{p}{i^2}.$$

From these equations,

$$\text{The reactance, } x = \sqrt{z^2 - r^2};$$

If the reactance is inductive, the inductance is

$$L = \frac{x}{2\pi f}.$$

If the reactance is condensive, the capacity or its equivalent is

$$C = \frac{1}{2\pi f x},$$

wherein f = the frequency of the impressed e.m.f. If the reactance is the resultant of inductive and condensive reactances connected in series, *i.e.*,

$$x = 2\pi f L - \frac{1}{2\pi f C},$$

L and C can be found by measuring the reactance at two different frequencies, f_1 and f_2, and writing

$$x_1 = 2\pi f_1 L - \frac{1}{2\pi f_1 C},$$

$$x_2 = 2\pi f_2 L - \frac{1}{2\pi f_2 C},$$

then,

$$L = \frac{x_1 f_1 - x_2 f_2}{2\pi (f_1^2 - f_2^2)},$$

and

$$C = \frac{f_1^2 - f_2^2}{2\pi f_1 f_2 (x_1 f_2 - x_2 f_1)}.$$

A moderate deviation of the wave of alternating impressed e.m.f. from sine-shape does not cause any serious error as long as the circuit contains no capacity.

In the presence of capacity, however, even a very slight distortion of wave-shape may cause an error of some hundred per cent.

To measure capacity and condensive reactance by ordinary alternating currents, it is therefore advisable to insert in series with the condensive reactance a non-inductive resistance or induc-

tive reactance, which is larger than the condensive reactance, or to use a source of alternating current, in which the higher harmonics are suppressed, as the T connection of Constant Potential — Constant-Current Transformation, paragraph 69.

In ironclad inductive reactances, or reactances containing iron in the magnetic circuit, the reactance varies with the magnetic induction in the iron, and thereby with the current and the impressed e.m.f. Therefore the impressed e.m.f. or the magnetic induction must be given, to which the ohmic reactance refers, or preferably a curve is plotted from test (or calculation), giving the ohmic reactance, or, as usually done, the impressed e.m.f. as function of the current. Such a curve is called an *excitation curve* or *impedance curve,* and has the general character of the magnetic characteristic. The same also applies to electrolytic reactances, etc.

The calculation of an inductive reactance is accomplished by calculating the magnetic circuit, that is, determining the ampere-turns m.m.f. required to send the magnetic flux through the magnetic reluctance. In the air part of the magnetic circuit, unit permeability (or, referred to ampere-turns as m.m.f., reluctivity $\frac{10}{4\pi}$) is used, for the iron part, the ampere-turns are taken from the curve of the magnetic characteristic, as discussed in the following.

CHAPTER II.

INSTANTANEOUS VALUES AND INTEGRAL VALUES.

9. In a periodically varying function, as an alternating current, we have to distinguish between the *instantaneous value*, which varies constantly as function of the time, and the *integral value*, which characterizes the wave as a whole.

As such integral value, almost exclusively the *effective value* is used, that is, the square root of the mean square; and wher-

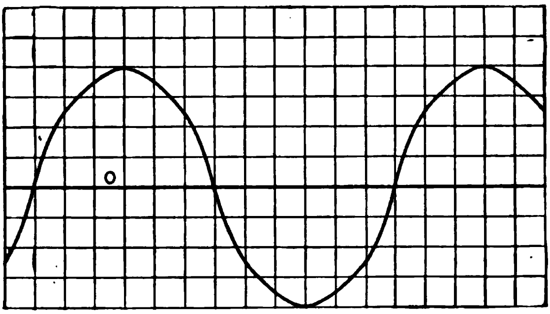


FIG. 4.—Alternating Wave.

ever the intensity of an electric wave is mentioned without further reference, the effective value is understood.

The *maximum value* of the wave is of practical interest only in few cases, and may, besides, be different for the two half-waves, as in Fig. 3.

As *arithmetic mean*, or *average value*, of a wave as in Figs. 4 and 5, the arithmetical average of all the instantaneous values during one complete period is understood.

This arithmetic mean is either = 0, as in Fig. 4, or it differs from 0, as in Fig. 5. In the first case, the wave is called an *alternating wave*, in the latter a *pulsating wave.*

Thus, an alternating wave is a wave whose positive values give the same sum total as the negative values; that is, whose

two half-waves have in rectangular coordinates the same area, as shown in Fig. 4.

A pulsating wave is a wave in which one of the half-waves preponderates, as in Fig. 5.

By electromagnetic induction, pulsating waves are produced only by commutating and unipolar machines (or by the superposition of alternating upon direct currents, etc.).

All inductive apparatus without commutation give exclusively alternating waves, because, no matter what conditions

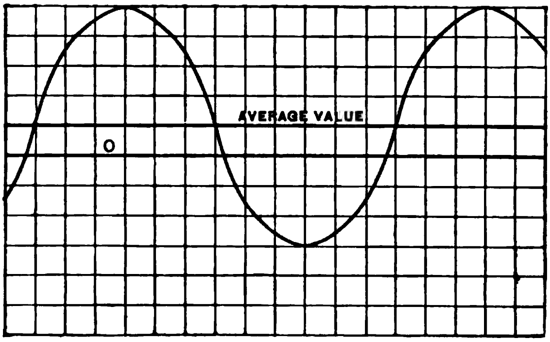


Fig. 5. — Pulsating Wave.

may exist in the circuit, any line of magnetic force, which during a complete period is cut by the circuit, and thereby generates an e.m.f., must during the same period be cut again in the opposite direction, and thereby generate the same total amount of e.m.f. (Obviously, this does not apply to circuits consisting of different parts movable with regard to each other, as in unipolar machines.)

Pulsating currents, and therefore pulsating potential differences across parts of a circuit can, however, be produced from an alternating induced e.m.f. by the use of asymmetrical circuits, as arcs, some electrochemical cells, as the aluminum-carbon cell, etc. Most of the alternating-current rectifiers are based on the use of such asymmetrical circuits.

In the following we shall almost exclusively consider the alternating wave, that is, the wave whose true arithmetic mean value $= 0$.

Frequently, by mean value of an alternating wave, the average

of one half-wave only is denoted, or rather the average of all instantaneous values without regard to their sign. This *mean value* of one half-wave is of importance mainly in the rectification of alternating e.m.fs., since it determines the unidirectional value derived therefrom.

10. In a sine wave, the relation of the mean to the maximum value is found in the following way:

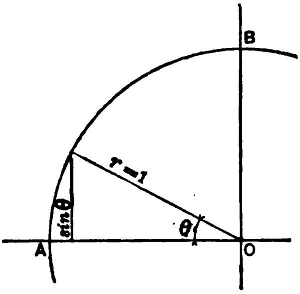


FIG. 6.

Let, in Fig. 6, AOB represent a quadrant of a circle with radius 1.

Then, while the angle θ traverses the arc $\frac{\pi}{2}$ from A to B, the sine varies from 0 to $OB = 1$. Hence the average variation of the sine bears to that of the corresponding arc the ratio $1 \div \frac{\pi}{2}$, or $\frac{2}{\pi} \div 1$. The maximum variation of the sine takes place about its zero value, where the sine is equal to the arc. Hence the maximum variation of the sine is equal to the variation of the corresponding arc, and consequently the maximum variation of the sine bears to its average variation the same ratio as the average variation of the arc to that of the sine; that is, $1 \div \frac{2}{\pi}$, and since the variations of a sine function are sinusoidal also, we have

$$\text{Mean value of sine wave} \div \text{maximum value} = \frac{2}{\pi} \div 1$$
$$= .63663.$$

The quantities, "current," "e.m.f.," "magnetism," etc., are in reality mathematical fictions only, as the components of the entities, "energy," "power," etc.; that is, they have no independent existence, but appear only as squares or products.

Consequently, the only integral value of an alternating wave which is of practical importance, as directly connected with the mechanical system of units, is that value which represents the same *power* or *effect* as the periodical wave. This is called

the *effective value*. Its square is equal to the mean square of the periodic function, that is,

The effective value of an alternating wave, or the value representing the same effect as the periodically varying wave, is the square root of the mean square.

In a sine wave, its relation to the maximum value is found in the following way:

Let, in Fig. 7, AOB represent a quadrant of a circle with radius 1.

Then, since the sines of any angle, θ, and its complementary angle, $90° - \theta$, fulfill the condition, —

$$\sin^2\theta + \sin^2(90 - \theta) = 1,$$

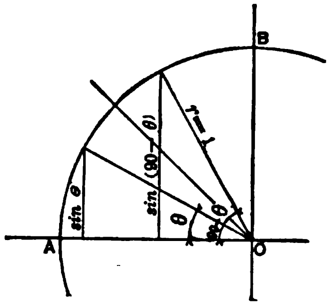


FIG. 7.

the sines in the quadrant, AOB, can be grouped into pairs, so that the sum of the squares of any pair $= 1$; or, in other words, the mean square of the sine $= \frac{1}{2}$, and the square root of the mean square, or the effective value of the sine, $= \frac{1}{\sqrt{2}}$. That is,

The effective value of a sine function bears to its maximum value the ratio,

$$\frac{1}{\sqrt{2}} \div 1 = 0.70711.$$

Hence, we have for the sine wave the following relations:

MAX.	EFF.	ARITH. MEAN. Half Period.	ARITH. MEAN. Whole Period.
1	$\frac{1}{\sqrt{2}}$	$\frac{2}{\pi}$	0
1.0	0.7071	0.63663	0
1.4142	1.0	0.90034	0
1.5708	1.1107	1.0	0

11. Coming now to the general alternating wave,

$$\begin{aligned} i &= A_1 \sin 2\pi ft + A_2 \sin 4\pi ft + A_3 \sin 6\pi ft + \dots \\ &\quad + B_1 \cos 2\pi ft + B_2 \cos 4\pi ft + B_3 \cos 6\pi ft + \dots, \end{aligned}$$

we find, by squaring this expression and cancelling all the products which give 0 as mean square, the *effective value*

$$I = \sqrt{\tfrac{1}{2}\left(A_1^2 + A_2^2 + A_3^2 + \ldots + B_1^2 + B_2^2 + B_3^2 \ldots\right)}.$$

The *mean value* does not give a simple expression, and is of no general interest.

12. All alternating-current instruments, as ammeter, voltmeter, etc., measure and indicate the *effective value.* The maximum value and the mean value can be derived from the curve of instantaneous values, as determined by wave-meter or oscillograph.

Measurement of the alternating wave after rectification by a unidirectional conductor, as an arc, gives the *mean* value with direct-current instruments, that is, instruments employing a permanent magnetic field, and the *effective value* with alternating-current instruments.

Voltage determination by spark-gap, that is, by the striking distance between needle points, gives a value depending essentially on the *maximum,* and less on the mean value of the wave.

CHAPTER III.

LAW OF ELECTROMAGNETIC INDUCTION.

13. If an electric conductor moves relatively to a magnetic field, an e.m.f. is generated in the conductor which is proportional to the intensity of the magnetic field, to the length of the conductor, and to the speed of its motion perpendicular to the magnetic field and the direction of the conductor; or, in other words, proportional to the number of lines of magnetic force cut per second by the conductor.

As a practical unit of e.m.f., the *volt* is defined as the e.m.f. generated in a conductor, which cuts $10^8 = 100{,}000{,}000$ lines of magnetic flux per second.

If the conductor is closed upon itself, the e.m.f. produces a current.

A closed conductor may be called a turn or a convolution. In such a turn, the number of lines of magnetic force cut per second is the increase or decrease of the number of lines inclosed by the turn, or n times as large with n turns.

Hence the e.m.f. in volts generated in n turns, or convolutions, is n times the increase or decrease, per second, of the flux inclosed by the turns, times 10^{-8}.

If the change of the flux inclosed by the turn, or by n turns, does not take place uniformly, the product of the number of turns, times change of flux per second, gives the average e.m.f.

If the magnetic flux, Φ, alternates relatively to a number of turns, n, — that is, when the turns either revolve through the flux, or the flux passes in and out of the turns, — the total flux is cut four times during each complete period or cycle, twice passing into, and twice out of, the turns.

Hence, if f = number of complete cycles per second, or the frequency of the flux, Φ, the average e.m.f. generated in n turns is

$$E_{avg.} = 4\, n\Phi f\, 10^{-8} \text{ volts.}$$

This is the fundamental equation of electrical engineering, and applies to continuous-current, as well as to alternating-current, apparatus.

14. In continuous-current machines and in many alternators, the turns revolve through a constant magnetic field; in other alternators and in induction motors, the magnetic field revolves; in transformers, the field alternates with respect to the stationary turns; in other apparatus, alternation and rotation occur simultaneously, as in alternating-current commutator motors.

Thus, in the continuous-current machine, if n = number of turns in series from brush to brush, Φ = flux inclosed per turn, and f = frequency, the e.m.f. generated in the machine is $E = 4\, n\Phi f\, 10^{-8}$ volts, independent of the number of poles, of series or multiple connection of the armature, whether of the ring, drum, or other type.

In an alternator or transformer, if n is the number of turns in series, Φ the maximum flux inclosed per turn, and f the frequency, this formula gives

$$E_{\text{avg.}} = 4\, n\Phi f\, 10^{-8} \text{ volts.}$$

Since the maximum e.m.f. is given by

$$E_{\text{max.}} = \frac{\pi}{2} E_{\text{avg.}},$$

we have

$$E_{\text{max.}} = 2\, \pi n\Phi f\, 10^{-8} \text{ volts.}$$

And since the effective e.m.f. is given by

$$E_{\text{eff.}} = \frac{E_{\text{max.}}}{\sqrt{2}}$$

we have

$$\begin{aligned} E_{\text{eff.}} &= \sqrt{2}\, \pi n\Phi f\, 10^{-8} \\ &= 4.44\, nf\Phi\, 10^{-8} \text{ volts,} \end{aligned}$$

which is the fundamental formula of alternating-current induction by sine waves.

15. If, in a circuit of n turns, the magnetic flux, Φ, inclosed by the circuit is produced by the current in the circuit, the ratio,

$$\frac{\text{flux} \times \text{number of turns} \times 10^{-8}}{\text{current}},$$

is called the inductance, L, of the circuit, in henrys.

(The product of the number of turns, n, into the maximum flux, Φ, produced by a current of I amperes effective, or $I\sqrt{2}$ amperes maximum, is therefore

$$n\Phi = LI\sqrt{2}\,10^8;$$

and consequently the effective e.m.f. of self-induction is

$$\begin{aligned} E &= \sqrt{2}\,\pi n\Phi f\,10^{-8} \\ &= 2\,\pi fLI \text{ volts.} \end{aligned}$$

The product, $x = 2\,\pi fL$, is of the dimension of resistance, and is called the *inductive reactance* of the circuit; and the e.m.f. of self-induction of the circuit, or the reactance voltage, is

$$E = Ix,$$

and lags 90° behind the current, since the current is in phase with the magnetic flux produced by the current, and the e.m.f. lags 90° behind the magnetic flux. The e.m.f. lags 90° behind the magnetic flux, as it is proportional to the rate of change in flux; thus it is zero when the magnetism is at its maximum value, and a maximum when the flux passes through zero, where it changes quickest.

CHAPTER IV.

GRAPHIC REPRESENTATION.

16. While alternating waves can be, and frequently are, represented graphically in rectangular coordinates, with the time as abscissas, and the instantaneous values of the wave as ordinates, the best insight with regard to the mutual relation of different alternating waves is given by their representation in polar coordinates, with the time as the angle or the amplitude, — one complete period being represented by one revolution, and the instantaneous values as radius vectors.

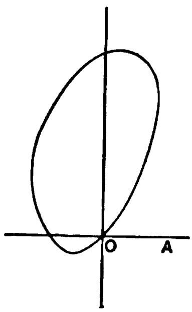


FIG. 8.

Thus the two waves of Figs. 2 and 3 are represented in polar coordinates in Figs. 8 and 9 as closed characteristic curves, which, by their intersection with the radius vector, give the instantaneous value of the wave, corresponding to the time represented by the amplitude or angle of the radius vector.

These instantaneous values are positive if in the direction of the radius vector, and negative if in opposition. Hence the two half-waves in Fig. 2 are represented by the same polar characteristic curve, which is traversed by the point of intersection of the radius vector twice per period, — once in the direction of the vector, giving the positive half-wave, and once in opposition to the vector, giving the negative half-wave. In Figs. 3 and 9, where the two half-waves are different, they give different polar characteristics.

17. The sine wave, Fig. 1, is represented in polar coordinates by one circle, as shown in Fig. 10. The diameter of the characteristic curve of the sine wave, $I = \overline{OC}$, represents the *intensity* of the wave; and the amplitude of the diameter $\overline{OC}$, $\measuredangle\, \theta_0 =$

AOC, is the *phase* of the wave, which, therefore, is represented analytically by the function:

$$i = I \cos (\theta - \theta_0),$$

where $\theta = 2\pi \frac{t}{t_0}$ is the instantaneous value of the amplitude corresponding to the instantaneous value, i, of the wave.

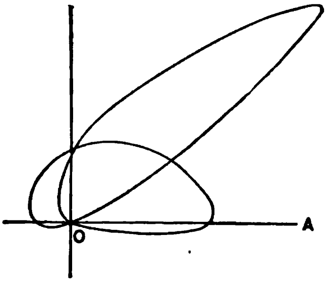


FIG. 9.

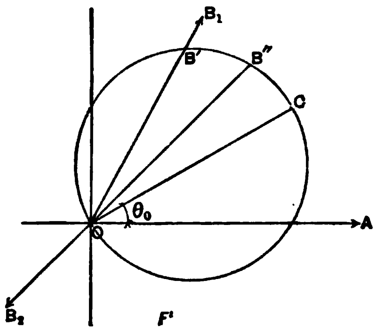


FIG. 10.

The instantaneous values are cut out on the movable radius vector by its intersection with the characteristic circle. Thus, for instance, at the amplitude, $AOB_1 = \theta_1 = 2\pi \frac{t_1}{t_0}$ (Fig. 10), the instantaneous value is $\overline{OB'}$; at the amplitude, $AOB_2 = \theta_2 = 2\pi \frac{t_2}{t_0}$, the instantaneous value is $\overline{OB''}$, and negative, since in opposition to the radius vector, OB_2.

The angle, θ_0, so represents the time, and increasing time is represented by an increase of angle θ in counter-clockwise rotation. That is, the positive direction, or increase of time, is chosen as counter-clockwise rotation.

The characteristic circle of the alternating sine wave is determined by the length of its diameter — the intensity of the wave; and by the amplitude of the diameter — the phase of the wave.

Hence wherever the integral value of the wave is considered alone, and not the instantaneous values, the characteristic circle may be omitted altogether, and the wave represented in intensity and in phase by the diameter of the characteristic circle.

Thus, in polar coordinates, the alternate wave is represented in intensity and phase by the length and direction of a vector, $\overline{OC}$, Fig. 10, and its analytical expression would then be $c = \overline{OC} \cos (\theta - \theta_0)$.

Instead of the maximum value of the wave, the *effective* value, or square root of mean square, may be used as the vector, which is more convenient; and the maximum value is then $\sqrt{2}$ times the vector $\overline{OC}$, so that the instantaneous values, when taken from the diagram, have to be increased by the factor $\sqrt{2}$.

Thus, the wave,

$$b = B \cos 2 \pi f (t - t_1)$$
$$= B \cos (\theta - \theta_1),$$

is, in Fig. 11, represented by

$$\text{vector } \overline{OB} = \frac{B}{\sqrt{2}},$$

of phase $AOB = \theta_1$;

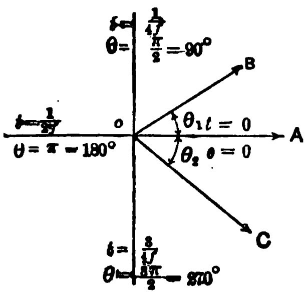


FIG. 11.

and the wave,

$$c = C \cos 2 \pi f (t + t_2)$$
$$= C \cos (\theta + \theta_2),$$

is, in Fig. 11, represented by

$$\text{vector } \overline{OC} = \frac{C}{\sqrt{2}}, \text{ of phase } AOC = - \theta_2.$$

The former is said to *lag* by angle θ_1, the latter to *lead* by angle θ_2, with regard to the zero position.

The wave b lags by angle $(\theta_1 + \theta_2)$ behind wave c, or c leads b by angle $(\theta_1 + \theta_2)$.

18. To combine different sine waves, their graphical representations, or vectors, are combined by the parallelogram law.

If, for instance, two sine waves, $\overline{OB}$ and $\overline{OC}$ (Fig. 12), are superposed,— as, for instance, two e.m.fs. acting in the same circuit,— their resultant wave is represented by $\overline{OD}$, the diagonal of a parallelogram with $\overline{OB}$ and $\overline{OC}$ as sides.

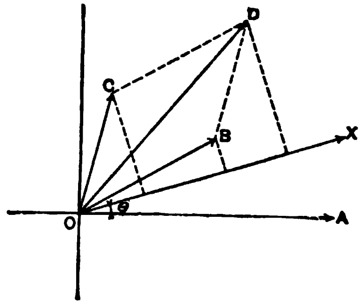


FIG. 12.

For at any time, t, represented by angle $\theta = AOX$, the instantaneous values of the three waves, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$, are their projections upon $\overline{OX}$, and the sum of the projections of $\overline{OB}$ and $\overline{OC}$ is equal to the projection of $\overline{OD}$; that is, the instantaneous values of the wave $\overline{OD}$ are equal to the sum of the instantaneous values of waves $\overline{OB}$ and $\overline{OC}$.

From the foregoing considerations we have the conclusions:

The sine wave is represented graphically in polar coordinates by a vector, which by its length, $\overline{OC}$, denotes the intensity, and by its amplitude, AOC, the phase, of the sine wave.

Sine waves are combined or resolved graphically, in polar coordinates, by the law of the parallelogram or the polygon of sine waves.

Kirchhoff's laws now assume, for alternating sine waves, the form:

(*a*) The resultant of all the e.m.fs. in a closed circuit, as found by the parallelogram of sine waves, is zero if the counter e.m.fs. of resistance and of reactance are included.

(*b*) The resultant of all the currents toward a distributing point, as found by the parallelogram of sine waves, is zero.

The power equation expressed graphically is as follows:

The power of an alternating-current circuit is represented in polar coordinates by the product of the current, I, into the projection of the e.m.f., E, upon the current, or by the e.m.f., E, into the projection of the current, I, upon the e.m.f., or by $IE \cos \theta$, where θ = angle of time-phase displacement.

19. Suppose, as an example, that in a line having the resistance, r, and the reactance, $x = 2\pi fL$, — where f = frequency and L = inductance, — there exists a current of I amperes, the line being connected to a non-inductive circuit operating at an e.m.f. of E volts. What will be the e.m.f. required at the generator end of the line?

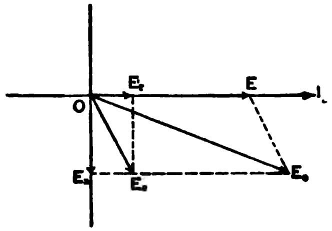


FIG. 13.

In the polar diagram, Fig. 13, let the phase of the current be assumed as the initial or zero line, $\overline{OI}$. Since the receiving circuit is non-inductive, the current is in time-phase with its e.m.f. Hence the e.m.f., E, at the end of the line, impressed upon the receiving circuit, is represented by a vector, $\overline{OE}$. To overcome the resistance, r, of the line, an e.m.f., Ir, is required in time-phase with the current, represented by $\overline{OE_r}$ in the diagram. The inductive reactance of the line generates an e.m.f. which is proportional to the current, I, and reactance, x, and lags a quarter of a period, or 90°, behind the current. To overcome this counter e.m.f. of inductive reactance, an e.m.f. of the value Ix is required, in time-phase 90° ahead of the current, hence represented by vector $\overline{OE_x}$. Thus resistance consumes e.m.f. in phase, and reactance an e.m.f. 90 time-degrees ahead of the current. The e.m.f. of the generator, E_0, has to give the three e.m.fs., E, E_r, and E_x, hence it is determined as their resultant. Combining by the parallelogram law, $\overline{OE_r}$ and $\overline{OE_x}$, give $\overline{OE_z}$, the e.m.f. required to overcome the impedance of the line, and similarly $\overline{OE_z}$ and $\overline{OE}$ give $\overline{OE_0}$, the e.m.f. required at the generator side of the line, to yield the e.m.f., E, at the receiving end of the line. Algebraically, we get from Fig. 13

$$E_0 = \sqrt{(E + Ir)^2 + (Ix)^2};$$

or

$$E = \sqrt{E_0^2 - (Ix)^2} - Ir.$$

In this example we have considered the e.m.f. consumed by the resistance (in time-phase with the current) and the e.m.f. consumed by the reactance (90 time-degrees ahead of the current)

as parts, or components, of the impressed e.m.f., E_0, and have derived E_0 by combining E_r, E_x, and E.

20. We may, however, introduce the effect of the inductive reactance directly as an e.m.f., E_x', the counter e.m.f. of inductive reactance $= Ix$, and lagging 90 time-degrees behind the current; and the e.m.f. consumed by the resistance as a counter e.m.f.,

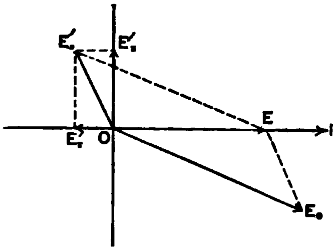


Fig. 14.

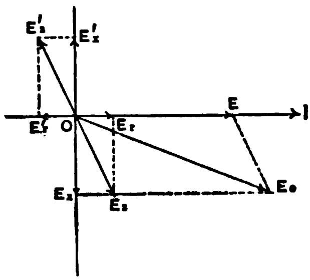


Fig. 15.

$E_r' = Ir$, but in opposition to the current, as is done in Fig. 14; and combine the three e.m.fs. E_0, E_r', E_x', to form a resultant e.m.f., E, which is left at the end of the line. E_r' and E_x' combine to form E_z', the counter e.m.f. of impedance; and since E_z' and E_0 must combine to form E, E_0 is found as the side of a parallelogram, OE_0EE_z', whose other side, $\overline{OE_z'}$, and diagonal, $\overline{OE}$, are given.

Or we may say (Fig. 15), that to overcome the counter e.m.f. of impedance, $\overline{OE_z'}$, of the line, the component, $\overline{OE_z}$, of the impressed e.m.f. is required which, with the other component $\overline{OE}$, must give the impressed e.m.f., $\overline{OE_0}$.

As shown, we can represent the e.m.fs. produced in a circuit in two ways — either as counter e.m.fs., which combine with the impressed e.m.f., or as parts, or components, of the impressed e.m.f., in the latter case being of opposite phase. According to the nature of the problem, either the one or the other way may be preferable.

As an example, the e.m.f. consumed by the resistance is Ir, and in time-phase with the current; the counter e.m.f. of resistance is in opposition to the current. The e.m.f. consumed by

the reactance is Ix, and 90 time-degrees ahead of the current, while the counter e.m.f. of reactance is 90 time-degrees behind the current; so that, if, in Fig. 16, $\overline{OI}$ is the current,

$\overline{OE_r}$ = e.m.f. consumed by resistance,
$\overline{OE_r'}$ = counter e.m.f. of resistance,
$\overline{OE_x}$ = e.m.f. consumed by inductive reactance,
$\overline{OE_x'}$ = counter e.m.f. of inductive reactance,
$\overline{OE_z}$ = e.m.f. consumed by impedance,
$\overline{OE_z'}$ = counter e.m.f. of impedance.

Obviously, these counter e.m.fs. are different from, for instance, the counter e.m.f. of a synchronous motor, in so far as they have no independent existence, but exist only through, and as long as the current exists. In this respect they are analogous to the opposing force of friction in mechanics.

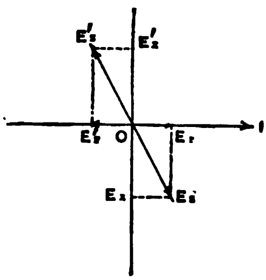


FIG. 16.

21. Coming back to the equation found for the e.m.f. at the generator end of the line,

$$E_0 = \sqrt{(E + Ir)^2 + (Ix)^2},$$

we find, as the drop of potential in the line,

$$e = E_0 - E = \sqrt{(E + Ir)^2 + (Ix)^2} - E.$$

This is different from, and less than, the e.m.f. of impedance,

$$E_z = Iz = I\sqrt{r^2 + x^2}.$$

Hence it is wrong to calculate the drop of potential in a circuit by multiplying the current by the impedance; and the drop of potential in the line depends, with a given current fed over the line into a non-inductive circuit, not only upon the constants of the line, r and x, but also upon the e.m.f., E, at end of line, as can readily be seen from the diagrams.

22. If the receiver circuit is inductive, that is, if the current, I, lags behind the e.m.f., E, by a time-angle, θ, and we choose again as the zero line, the current $\overline{OI}$ (Fig. 17), the e.m.f., $\overline{OE}$, is ahead of the current by the time-angle, θ. The e.m.f. consumed by the resistance, Ir, is in time-phase with the current, and represented by $\overline{OE_r}$; the e.m.f. consumed by the reactance, Ix, is 90 time-degrees ahead of the current, and represented by $\overline{OE_x}$. Combining $\overline{OE}$, $\overline{OE_r}$, and $\overline{OE_x}$, we get $\overline{OE_0}$, the e.m.f. required at the generator end of the line. Comparing Fig. 17 with Fig. 14, we see that in the former $\overline{OE_0}$ is larger; or conversely, if E_0 is the same, E will be less with an inductive load. In other words, the drop of potential in an inductive line is greater if the receiving circuit is inductive than if it is non-inductive. From Fig. 17,

$$E_0 = \sqrt{(E \cos\theta + Ir)^2 + (E \sin\theta + Ix)^2}.$$

If, however, the current in the receiving circuit is leading, as is the case when feeding condensers or synchronous motors whose counter e.m.f. is larger than the impressed e.m.f., then

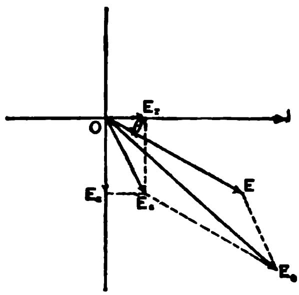


FIG. 17.

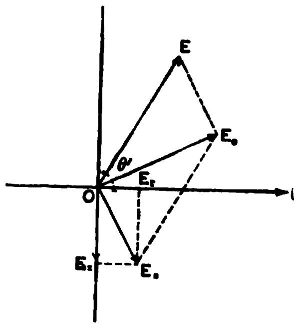


FIG. 18.

the e.m.f. will be represented, in Fig. 18, by a vector, $\overline{OE}$, lagging behind the current, $\overline{OI}$, by the time-angle of lead, θ'; and in this case we get, by combining $\overline{OE}$ with $\overline{OE_r}$, in time-phase with the current, and $\overline{OE_x}$, 90 time-degrees ahead of the current, the generator e.m.f., $\overline{OE_0}$, which in this case is not only less than in

Fig. 17 and in Fig. 14, but may be even less than E; that is, the potential rises in the line. In other words, in a circuit with leading current, the inductive reactance of the line raises the potential, so that the drop of potential is less than with a non-inductive load, or may even be negative, and the voltage at the generator lower than at the other end of the line.

These diagrams, Figs. 14 to 18, can be considered polar diagrams of an alternating-current generator of an e.m.f., E_0, a resistance e.m.f., $E_r = Ir$, a reactance e.m.f., $E_x = Ix$, and a difference of potential, E, at the alternator terminals; and we see, in this case, that with an inductive load the potential difference at the alternator terminals will be lower than with a non-inductive load, and that with a non-inductive load it will be lower than when feeding into a circuit with leading current, as, for instance, a synchronous motor circuit under the circumstances stated above.

23. As a further example, we may consider the diagram of an alternating-current transformer, feeding through its secondary circuit an inductive load.

For simplicity, we may neglect here the magnetic hysteresis, the effect of which will be fully treated in a separate chapter on this subject.

Let the time be counted from the moment when the magnetic flux is zero. The time-phase of the flux, that is, the amplitude of its maximum value, is 90° in this case, and, consequently, the time-phase of the generated e.m.f. is 180°, since the generated e.m.f. lags 90 time-degrees behind the generating flux. Thus the e.m.f., E_1, generated in the secondary, will be represented by a vector, $\overline{OE}_1$, in Fig. 19, displaced by 180 time-degrees from the zero of magnetic flux. The secondary current, I_1, lags behind the e.m.f., E_1, by an angle, θ_1, which is determined by the resistance and inductive reactance of the secondary circuit; that is, by the load in the secondary circuit, and is represented in the diagram by the vector, $\overline{OF}_1$, of phase $180 + \theta_1$.

Instead of the secondary current, I_1, we plot, however, the secondary m.m.f., $F_1 = n_1 I_1$, where n_1 is the number of secondary turns, and F_1 is given in ampere-turns. This makes us independent of the ratio of transformation.

From the secondary e.m.f., E_1, we get the flux, Φ, required to induce this e.m.f., from the equation

$$E_1 = \sqrt{2}\,\pi n_1 f \Phi\, 10^{-8};$$

where

E_1 = secondary e.m.f., in effective volts,
f = frequency, in cycles per second,
n_1 = number of secondary turns,
Φ = maximum value of magnetic flux, in lines of magnetic force.

The derivation of this equation has been given in a preceding chapter.

This magnetic flux, Φ, is represented by a vector, $\overline{O\Phi}$, displaced 90° in time-phase, and to produce it an m.m.f., F, is required, which is determined by the magnetic characteristic of the iron, and the section and length of the magnetic circuit of the transformer; this m.m.f. is in time-phase with the flux, Φ, and is represented by the vector, $\overline{OF}$, in effective ampere-turns.

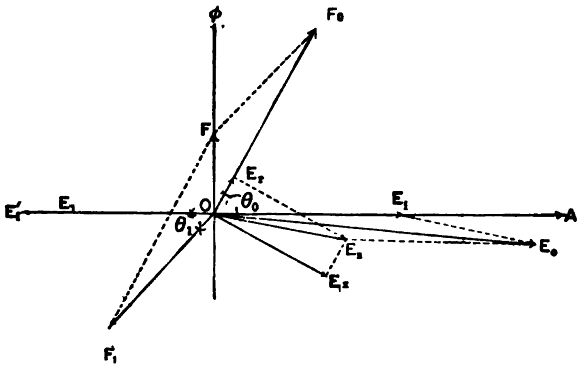


Fig. 19.

The effect of hysteresis, neglected at present, is to shift $\overline{OF}$ ahead of $\overline{O\Phi}$, by an angle, α, the angle of hysteretic lead. (See Chapter on Hysteresis.)

This m.m.f., F, is the resultant of the secondary m.m.f., F_1, and the primary m.m.f., F_0; or graphically, $\overline{OF}$ is the diagonal

of a parallelogram with $\overline{OF}_1$ and $\overline{OF}_0$ as sides. $\overline{OF}_1$ and $\overline{OF}$ being known, we find $\overline{OF}_0$, the primary ampere-turns, and therefrom, and the number of primary turns, n_0, the primary current, $I_0 = \frac{F_0}{n_0}$, which corresponds to the secondary current, I_1.

To overcome the resistance, r_0, of the primary coil, an e.m.f., $E_r = I_0 r_0$, is required, in time-phase with the current, I_0, and represented by the vector, $\overline{OE}_r$.

To overcome the reactance, $x_0 = 2\,\pi f L_0$, of the primary coil, an e.m.f., $E_x = I_0 x_0$ is required, 90° ahead of the current, I_0, and represented by vector, $\overline{OE}_x$.

The resultant magnetic flux, Φ, which generates in the secondary coil the e.m.f., E_1, generates in the primary coil an e.m.f. proportional to E_1 by the ratio of turns $\frac{n_0}{n_1}$, and in phase with E_1, or,

$$E_i' = \frac{n_0}{n_1} E_1,$$

which is represented by the vector $\overline{OE_i'}$. To overcome this counter e.m.f., E_i', a primary e.m.f., E_i, is required, equal but in time-phase opposition to E_i', and represented by the vector, $\overline{OE}_i$.

The primary impressed e.m.f., E_0, must thus consist of the three components, $\overline{OE}_i$, $\overline{OE}_r$, and $\overline{OE}_x$, and is, therefore, their resultant $\overline{OE}_0$, while the difference of phase in the primary circuit is found to be

$$\theta_0 = E_0 O F_0.$$

24. Thus, in Figs. 19 to 21, the diagram of a transformer is drawn for the same secondary e.m.f., E_1, secondary current, I_1, and therefore secondary m.m.f., F_1, but with different conditions of secondary phase displacement:

In Fig. 19, the secondary current, I_1, lags 60 time-degrees behind the secondary e.m.f., E_1.

In Fig. 20, the secondary current, I_1, is in time-phase with the secondary e.m.f., E_1.

In Fig. 21, the secondary current, I_1, leads by 60 time-degrees the secondary e.m.f., E_1.

These diagrams show that time-lag of the current in the secondary circuit increases and lead decreases the primary current and primary e.m.f. required to produce in the secondary

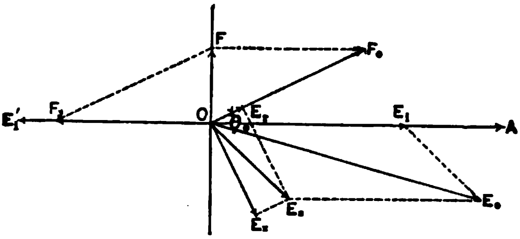


Fig. 20.

circuit the same e.m.f. and current; or conversely, at a given primary impressed e.m.f., E_0, the secondary e.m.f., E_1, will be smaller with an inductive, and larger with a condensive (leading current) load, than with a non-inductive load.

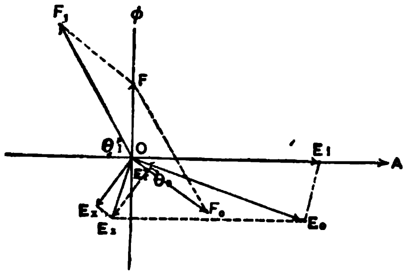


Fig. 21.

At the same time we see that a difference of time-phase existing in the secondary circuit of a transformer reappears in the primary circuit, somewhat decreased if the current is leading, and slightly increased if lagging, in time-phase. Later we shall see that hysteresis reduces the displacement in the primary circuit, so that, with an excessive lag in the secondary circuit, the lag in the primary circuit may be less than in the secondary.

A conclusion from the foregoing is that the transformer is not suitable for producing currents of displaced phase, since primary and secondary current are, except at very light loads, very nearly in phase, or rather, in opposition, to each other.

CHAPTER V.

SYMBOLIC METHOD.

25. The graphical method of representing alternating-current phenomena by polar coordinates of time affords the best means for deriving a clear insight into the mutual relation of the different alternating sine waves entering into the problem. For numerical calculation, however, the graphical method is generally not well suited, owing to the widely different magnitudes of the alternating sine waves represented in the same diagram, which make an exact diagrammatic determination impossible. For instance, in the transformer diagrams (*cf.* Figs. 19–21), the different magnitudes have numerical values in practice, somewhat like $E_1 = 100$ volts, and $I_1 = 75$ amperes. For a non-inductive secondary load, as of incandescent lamps, the only reactance of the secondary circuit thus is that of the secondary coil, or, $x_1 = 0.08$ ohms, giving a lag of $\theta_1 = 3.6°$. We have also,

$$
\begin{aligned}
n_1 &= 30 \text{ turns.}\\
n_0 &= 300 \text{ turns.}\\
F_1 &= 2250 \text{ ampere-turns.}\\
F &= 100 \text{ ampere-turns.}\\
E_r &= 10 \text{ volts.}\\
E_x &= 60 \text{ volts.}\\
E_i &= 1000 \text{ volts.}
\end{aligned}
$$

The corresponding diagram is shown in Fig. 22. Obviously, no exact numerical values can be taken from a parallelogram as flat as OF_1FF_0, and from the combination of vectors of the relative magnitudes 1 : 6 : 100.

Hence the importance of the graphical method consists not so much in its usefulness for practical calculation, as to aid in the simple understanding of the phenomena involved.

26. Sometimes we can calculate the numerical values trigonometrically by means of the diagram. Usually, however, this becomes too complicated, as will be seen by trying to calculate,

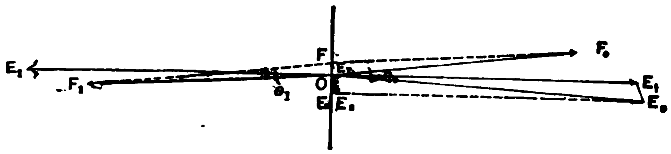


FIG. 22.

from the above transformer diagram, the ratio of transformation. The primary m.m.f. is given by the equation,

$$F_0 = \sqrt{F^2 + F_1^2 + 2\,FF_1 \sin \theta_1},$$

an expression not well suited as a starting-point for further calculation.

A method is therefore desirable which combines the exactness of analytical calculation with the clearness of the graphical representation.

27. We have seen that the alternating sine wave is represented in intensity, as well as phase, by a vector, $\overline{OI}$, which is determined analytically by two numerical quantities — the length, $\overline{OI}$, or intensity; and the amplitude, AOI, or phase θ, of the wave, I.

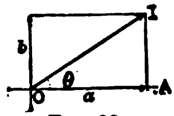


FIG. 23.

Instead of denoting the vector which represents the sine wave in the polar diagram by the polar coordinates, I and θ, we can represent it by its rectangular coordinates, a and b (Fig. 23), where

$a = I \cos \theta$ is the horizontal component,

$b = I \sin \theta$ is the vertical component of the sine wave.

This representation of the sine wave by its rectangular components is very convenient, in so far as it avoids the use of trigonometric functions in the combination or resolution of sine waves.

Since the rectangular components, a and b, are the horizontal and the vertical projections of the vector representing the sine wave, and the projection of the diagonal of a parallelogram is equal to the sum of the projections of its sides, the combination of sine waves by the parallelogram law is reduced to the addition, or subtraction, of their rectangular components. That is,

Sine waves are combined, or resolved, by adding, or subtracting, their rectangular components.

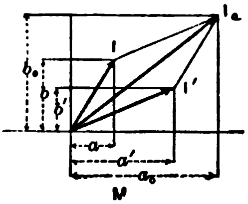


FIG. 24.

For instance, if a and b are the rectangular components of a sine wave, I, and a' and b' the components of another sine wave, I' (Fig. 24), their resultant sine wave, I_0, has the rectangular components $a_0 = (a + a')$, and $b_0 = (b + b')$.

To get from the rectangular components, a and b, of a sine wave, its intensity, i, and phase, θ, we may combine a and b by the parallelogram, and derive

$$i = \sqrt{a^2 + b^2};$$

$$\tan \theta = \frac{b}{a}.$$

Hence we can analytically operate with sine waves, as with forces in mechanics, by resolving them into their rectangular components.

28. To distinguish, however, the horizontal and the vertical components of sine waves, so as not to be confused in lengthier calculation, we may mark, for instance, the vertical components, by a distinguishing index, or the addition of an otherwise meaningless symbol, as the letter j, and thus represent the sine wave by the expression

$$\underset{\cdot}{I} = a + jb,$$

which now has the meaning, that a is the horizontal and b the vertical component of the sine wave I, and that both components are to be combined in the resultant wave of intensity

$$i = \sqrt{a^2 + b^2},$$

and of phase, $\tan \theta = \dfrac{b}{a}$.

Similarly, $a - jb$ means a sine wave with a as horizontal, and $-b$ as vertical, components, etc.

Obviously, the plus sign in the symbol, $a + jb$, does not imply simple addition, since it connects heterogeneous quantities — horizontal and vertical components — but implies combination by the parallelogram law.

For the present, j is nothing but a distinguishing index, and otherwise free for definition except that it is not an ordinary number.

29. A wave of equal intensity, and differing in phase from the wave, $a + jb$, by 180 time-degrees, or one-half period, is represented in polar coordinates by a vector of opposite direction, and denoted by the symbolic expression, $-a - jb$. Or,

Multiplying the symbolic expression, $a + jb$, of a sine wave by -1 means reversing the wave, or rotating it through 180 time-degrees, or one-half period.

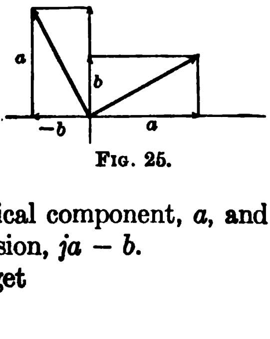


FIG. 25.

A wave of equal intensity, but lagging 90 time-degrees, or one-quarter period, behind $a + jb$, has (Fig. 25) the horizontal component, $-b$, and the vertical component, a, and is represented symbolically by the expression, $ja - b$.

Multiplying, however, $a + jb$ by j, we get

$$ja + j^2b;$$

therefore, if we define the heretofore meaningless symbol, j, by the condition,

$$j^2 = -1,$$

we have

$$j(a + jb) = ja - b;$$

hence,

Multiplying the symbolic expression, $a + jb$, of a sine wave by j means rotating the wave through 90 time-degrees, or one-quarter period; that is, retarding the wave through one-quarter period.

Similarly, —

Multiplying by $-j$ means advancing the wave through one-quarter period.

Since $$j^2 = -1, \qquad j = \sqrt{-1};$$
that is,

j is the imaginary unit, and the sine wave is represented by a complex imaginary quantity, $a + jb$.

As the imaginary unit, j, has no numerical meaning in the system of ordinary numbers, this definition of $j = \sqrt{-1}$ does not contradict its original introduction as a distinguishing index. For the Algebra of Complex Quantities see Appendix I. For a more complete discussion thereof see "Electrical Engineering Mathematics."

30. In the polar diagram of time, the sine wave is represented in intensity as well as phase by one complex quantity,

$$a + jb,$$

where a is the horizontal and b the vertical component of the wave; the intensity is given by

$$i = \sqrt{a^2 + b^2},$$

the phase by

$$\tan \theta = \frac{b}{a}.$$

and

$$a = i \cos \theta,$$
$$b = i \sin \theta;$$

hence the wave, $a + jb$, can also be expressed by

$$i (\cos \theta + j \sin \theta),$$

or, by substituting for $\cos \theta$ and $\sin \theta$ their exponential expressions, we obtain

$$i \epsilon^{j\theta}.*$$

Since we have seen that sine waves may be combined or resolved by adding or subtracting their rectangular components, consequently,

* In this representation of the sine wave by the exponential expression of the complex quantity, the angle θ necessarily must be expressed in *radians*, and not in degrees, that is, with one complete revolution or cycle as 2π, or with $\frac{180}{\pi} = 57.3$ degrees as unit.

Sine waves may be combined or resolved by adding or subtracting their complex algebraic expressions.

For instance, the sine waves,

$$a + jb$$

and

$$a' + jb',$$

combined give the sine wave,

$$\dot{I} = (a + a') + j(b + b').$$

It will thus be seen that the combination of sine waves is reduced to the elementary algebra of complex quantities.

31. If $\dot{I} = i + ji'$ is a sine wave of alternating current, and r is the resistance, the e.m.f. consumed by the resistance is in phase with the current, and equal to the product of the current and resistance. Or

$$r\dot{I} = ri + jri'.$$

If L is the inductance, and $x = 2\pi fL$ the inductive reactance, the e.m.f. produced by the reactance, or the counter e.m.f. of self-induction, is the product of the current and reactance, and lags in time-phase 90° behind the current; it is, therefore, represented by the expression

$$jx\dot{I} = jxi - xi'.$$

The e.m.f. required to overcome the reactance is consequently 90 time-degrees ahead of the current (or, as usually expressed, the current lags 90 time-degrees behind the e.m.f.), and represented by the expression

$$-jx\dot{I} = -jxi + xi'.$$

Hence, the e.m.f. required to overcome the resistance, r, and the reactance, x, is

$$(r - jx)\dot{I};$$

that is,

$Z = r - jx$ is the expression of the impedance of the circuit in complex quantities.

Hence, if $\dot{I} = i + ji'$ is the current, the e.m.f. required to overcome the impedance, $Z = r - jx$, is

$$\dot{E} = Z\dot{I} = (r - jx)(i + ji')$$
$$= (ri - j^2 i') + j(ri' - xi);$$

hence, since $\quad j^2 = -1$

$$\dot{E} = (ri + xi') + j(ri' - xi);$$

or, if $\dot{E} = e + je'$ is the impressed e.m.f. and $Z = r - jx$ the impedance, the current through the circuit is

$$\dot{I} = \frac{\dot{E}}{Z} = \frac{e + je'}{r - jx};$$

or, multiplying numerator and denominator by $(r + jx)$ to eliminate the imaginary from the denominator, we have

$$\dot{I} = \frac{(e + je')(r + jx)}{r^2 + x^2} = \frac{er - e'x}{r^2 + x^2} + j\frac{e'r + ex}{r^2 + x^2};$$

or, if $\dot{E} = e + je'$ is the impressed e.m.f. and $\dot{I} = i + ji'$ the current in the circuit, its impedance is

$$Z = \frac{\dot{E}}{\dot{I}} = \frac{e + je'}{i + ji'} = \frac{(e + je')(i - ji')}{i^2 + i'^2} = \frac{ei + e'i'}{i^2 + i'^2} + j\frac{e'i - ei'}{i^2 + i'^2}.$$

32. If C is the capacity of a condenser in series in a circuit in which exists a current $\dot{I} = i + ji'$, the e.m.f. impressed upon the terminals of the condenser is $\dot{E} = \frac{\dot{I}}{2\pi fC}$, 90 time-degrees behind the current; and may be represented by $\frac{j\dot{I}}{2\pi fC}$ or $jx_1\dot{I}$, where $x_1 = \frac{1}{2\pi fC}$ is the *condensive reactance* or *condensance* of the condenser.

Condensive reactance is of opposite sign to inductive reactance; both may be combined in the name reactance.

We therefore have the conclusion that

If r = resistance and L = inductance,

thus $\quad x = 2\pi fL$ = inductive reactance.

If C = capacity, $x_1 = \frac{1}{2\pi f C}$ = condensive reactance,

$Z = r - j(x - x_1)$ is the impedance of the circuit.

Ohm's law is then re-established as follows:

$$\dot{E} = Z\dot{I}, \quad \dot{I} = \frac{\dot{E}}{Z}, \quad Z = \frac{\dot{E}}{\dot{I}}.$$

The more general form gives not only the intensity of the wave but also its phase, as expressed in complex quantities.

33. Since the combination of sine waves takes place by the addition of their symbolic expressions, Kirchhoff's laws are now re-established in their original form:

(*a*) The sum of all the e.m.fs. acting in a closed circuit equals zero, if they are expressed by complex quantities, and if the resistance and reactance e.m.fs. are also considered as counter e.m.fs.

(*b*) The sum of all the currents directed towards a distributing point is zero, if the currents are expressed as complex quantities.

If a complex quantity equals zero, the real part as well as the imaginary part must be zero individually; thus, if

$$a + jb = 0, \qquad a = 0, b = 0.$$

Resolving the e.m.fs. and currents in the expression of Kirchhoff's law, we find:

(*a*) The sum of the components, in any direction, of all the e.m.fs. in a closed circuit, equals zero, if the resistance and reactance are represented as counter e.m.fs.

(*b*) The sum of the components, in any direction, of all the currents at a distributing point, equals zero.

Joule's law and the power equation do not give a simple expression in complex quantities, since the effect or power is a quantity of double the frequency of the current or e.m.f. wave, and therefore requires for its representation as a vector a transition from single to double frequency, as will be shown in Chapter XV.

In what follows, complex vector quantities will always be denoted by dotted capitals when not written out in full; absolute quantities and real quantities by undotted letters.

34. Referring to the example given in the fourth chapter, of a circuit supplied with an e.m.f., $\dot{E}$, and a current, $\dot{I}$, over an inductive line, we can now represent the impedance of the line by $Z = r - jx$, where r = resistance, x = reactance of the line, and have thus as the e.m.f. at the beginning of the line, or at the generator, the expression

$$\dot{E}_0 = \dot{E} + Z\dot{I}.$$

Assuming now again the current as the zero line, that is, $\dot{I} = i$, we have in general

$$\dot{E}_0 = \dot{E} + ir - jix;$$

hence, with non-inductive load, or $\dot{E} = e$,

$$\dot{E}_0 = (e + ir) - jix,$$

or
$$e_0 = \sqrt{(e + ir)^2 + (ix)^2}, \quad \tan\theta_0 = \frac{ix}{e + ir}.$$

In a circuit with lagging current, that is, with leading e.m.f., $\dot{E} = e - je'$, and

$$\begin{aligned}\dot{E}_0 &= e - je' + (r - jx)\,i \\ &= (e + ir) - j\,(e' + ix),\end{aligned}$$

or
$$e_0 = \sqrt{(e + ir)^2 + (e' + ix)^2}, \quad \tan\theta_0 = \frac{e' + ix}{e + ir}.$$

In a circuit with leading current, that is, with lagging e.m.f., $\dot{E} = e + je'$, and

$$\begin{aligned}\dot{E}_0 &= (e + je') + (r - jx)\,i \\ &= (e + ir) + j\,(e' - ix),\end{aligned}$$

or
$$e_0 = \sqrt{(e + ir)^2 + (e' - ix)^2}, \quad \tan\theta_0 = -\frac{e' - ix}{e + ir},$$

values which easily permit calculation.

35. When transferring from complex quantities to absolute values, it must be kept in mind that:

The absolute value of a product or a ratio of complex quantities is the product or ratio of their absolute values.

The phase angle of a product or a ratio of complex quantities is the sum or difference of their phase angles.

That is, if

$$\begin{aligned} \dot{A} &= a' + ja'' = a(\cos\alpha + j\sin\alpha) \\ \dot{B} &= b' + jb'' = b(\cos\beta + j\sin\beta) \\ \dot{C} &= c' + jc'' = c(\cos\gamma + j\sin\gamma) \end{aligned}$$

the absolute value of $\frac{\dot{A}\dot{B}}{\dot{C}}$ is given by $\frac{ab}{c}$, and its phase angle by $\alpha + \beta - \gamma$, that is, it is

$$\frac{\dot{A}\dot{B}}{\dot{C}} = \frac{ab}{c}[\cos(\alpha + \beta - \gamma) + j\sin(\alpha + \beta - \gamma)],$$

where

$$\begin{aligned} a &= \sqrt{a'^2 + a''^2} \\ b &= \sqrt{b'^2 + b''^2} \\ c &= \sqrt{c'^2 + c''^2} \end{aligned}$$

are the absolute values of $\dot{A}$, $\dot{B}$ and $\dot{C}$.

This rule frequently simplifies greatly the derivation of the absolute value and phase angle, from a complicated complex expression.

CHAPTER VI.

POLAR DIAGRAM AND CRANK DIAGRAM.

36. In the graphic representation of alternating-current and e.m.f. waves, described in Chapter IV, and used exclusively in the following, alternating-current quantities are represented in polar coordinates, with the instantaneous values as radius vectors, and the time as angles. The periodic function of current or e.m.f. is represented by a closed curve, as in Figs. 8 and 9, with an angle of 360 degrees representing a complete period or cycle. The sine wave is given by a circle, and this circle of instantaneous values of the sine wave is represented, in size or position, by its diameter. That is, the position of this diameter denotes the time, t, or angle, $\theta = 2\pi ft$, at which the sine wave reaches its maximum value, and the length of this diameter denotes the intensity of the maximum value.

If then, in polar coordinate representation Fig. 26, $\overline{OE}$ denotes an e.m.f., $\overline{OI}$ a current, this means, that the maximum value of e.m.f. equals $\overline{OE}$, and is reached at the time, t_1, represented by angle $AOE = \theta_1 = 2\pi ft_1$. The current in this diagram then has a maximum value equal to $\overline{OI}$, and this maximum value is reached at the time, t_2, represented by angle $AOI = \theta_2 = 2\pi ft_2$.

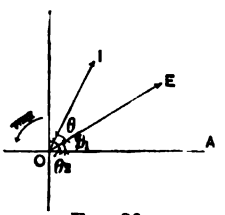


FIG. 26.

If then angle $\theta_2 > \theta_1$, this means that the current reaches its maximum value later than the e.m.f., that is, the current in Fig. 26 lags behind the e.m.f., by angle $EOI = \theta_2 - \theta_1 = 2\pi f\,(t_2 - t_1)$, or by time $t_2 - t_1$.

Occasionally, sine waves of alternating currents and e.m.fs. are represented by the so-called crank diagram, as projections of a revolving vector upon the horizontal. That is, a vector, equal in length to the alternating wave, revolves at uniform speed so as to make a complete revolution per period, and the

projections of this revolving vector upon the horizontal then denote the instantaneous values of the wave.

Obviously, by the crank diagram only sine waves can be represented.

Let, for instance, $\overline{OI}$ represent in length the maximum value of current, $i = I \cos(\theta - \theta_2)$. Assuming, then, a vector, $\overline{OI}$, to revolve, left handed or in positive direction, so that it makes a complete revolution during each cycle or period. If then at a certain moment of time, this vector stands in position $\overline{OI_1}$, (Fig. 27), the projection, $\overline{OA_1}$, of $\overline{OI}$ on $\overline{OA}$ represents the instantaneous value of the current at this moment. At a later moment, $\overline{OI}$ has moved farther, to $\overline{OI_2}$, and the projection, $\overline{OA_2}$, of $\overline{OI_2}$ on $\overline{OA}$ is the instantaneous value. The diagram so shows an instantaneous condition of the sine waves. Each sine wave reaches the maximum at the moment where its revolving vector, $\overline{OI}$, passes the horizontal.

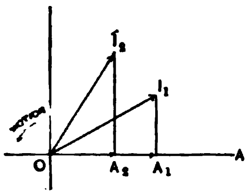


FIG. 27.

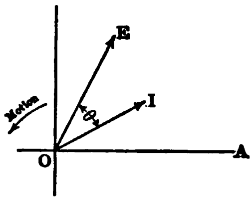


FIG. 28.

If Fig. 28 represents the crank diagram of an e.m.f., $\overline{OE}$, and a current, $\overline{OI}$, and if angle $AOE > AOI$, this means, the e.m.f., $\overline{OE}$, is ahead of the current, $\overline{OI}$, passes during the revolution the zero line or line of maximum intensity, $\overline{OA}$, earlier than the current, or leads; that is, the current lags behind the e.m.f. The same Fig. 28, considered as polar diagram, would mean that the current leads the e.m.f.; that is, the maximum value of current, $\overline{OI}$, occurs at a smaller angle, AOI, that is, at an earlier time, than the maximum value of the e.m.f., $\overline{OE}$.

In the crank diagram, the first quantity therefore can be put in any position. For instance, the current, $\overline{OI}$, in Fig. 28, could be drawn in position $\overline{OI}$, Fig. 29. The e.m.f. then, being ahead

of the current by angle $EOI = \theta_1$, would come into the position, $\overline{OE}$, Fig. 29.

A polar diagram, Fig. 26, with the current, $\overline{OI}$, lagging behind the e.m.f., $\overline{OE}$, by the angle, θ, thus considered as crank diagram would represent the current leading the e.m.f. by the angle, θ, and a crank diagram, Fig. 28 or 29, with the current lagging behind the e.m.f. by the angle, θ, would as polar diagram represent a current leading the e.m.f. by the angle, θ.

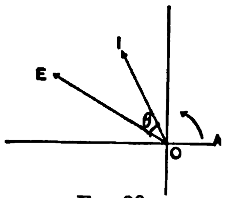


FIG. 29.

37. The main difference in appearance between the crank diagram and the polar diagram therefore is that, with the same direction of rotation, lag in the one diagram is represented in the same manner as lead in the other diagram, and inversely. Or, a representation by the crank diagram looks like a representation by the polar diagram, with reversed direction of rotation, and vice versa. Or, the one diagram is the image of the other and can be transformed into it by reversing right and left, or top and bottom. So the crank diagram, Fig. 29, is the image of the polar diagram, Fig. 26.

In symbolic representation, based upon the crank diagram, the impedance is denoted by

$$Z = r + jx,$$

where x = inductive reactance.

That is, the symbolic representation of the crank diagram differs from that of the polar diagram by a reversal of the sign of j, and by substituting $-j$ for $+j$, the symbolism of the crank diagram can be transformed into that of the polar diagram, and inversely.

38. The crank diagram offers the disadvantage, that it can be applied to sine waves only, while the polar diagram permits the construction of the curve of waves of any shape, as those in Figs. 8 and 9.

This objection does not appear serious, since in diagrammatic and symbolic representation, alternating quantities are usually

assumed as sine waves, that is, the general wave represented by the equivalent sine wave, that is, the sine wave of the same effective value as the general wave.

The transformation of the general wave into the equivalent sine wave, however, has to be carried out algebraically in the crank diagram, while the polar diagram permits a graphical transformation of the general wave into the equivalent sine wave.

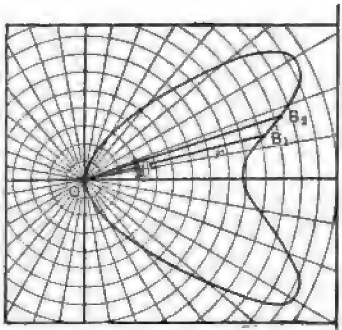


FIG. 30.

Let Fig. 30 represent a general alternating wave. An element B_1OB_2 of this wave then has the area

$$dA = \frac{r^2\, d\theta}{2},$$

and the total area of the polar curve is

$$A = \int_0^{2\pi} \frac{r^2}{2}\, d\theta.$$

The effective value of the wave is

$$R = \sqrt{\text{mean square}}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} r^2\, d\theta};$$

hence,

$$R^2\pi = \tfrac{1}{2}\int_0^{2\pi} r^2\, d\theta = A.$$

The area of the polar curve of the general periodic wave, as measured by planimeter, therefore equals the area of a circle with the effective value of the wave as radius.

The effective value of the equivalent sine wave therefore is

the radius of a circle having the same area as the general wave, in polar coordinates:

$$R = \sqrt{\frac{A}{\pi}}.$$

The diameter of the general polar circle therefore is

$$R\sqrt{2} = \sqrt{\frac{2A}{\pi}}.$$

And the phase of the equivalent sine wave, or the direction of the diameter of its polar circle, is the vector bisecting the area of the general wave, in polar coordinates.

The transformation of the general alternating wave in polar coordinates, into the equivalent sine wave, therefore, is carried out by measuring the area of the general wave in polar coordinates, and drawing the sine wave circle of half this area.

Since the polar coordinate system is universally used in all other sciences, to represent periodic functions, as in astronomy, it will be used exclusively in the following.

CHAPTER VII.

TOPOGRAPHIC METHOD

39. In the representation of alternating sine waves by vectors in a polar diagram, a certain ambiguity exists, in so far as one and the same quantity — an e.m.f., for instance — can be represented by two vectors of opposite direction, according as to whether the e.m.f. is considered as a part of the impressed e.m.f., or as a counter e.m.f. This is analogous to the distinction between action and reaction in mechanics.

Further, it is obvious that if in the circuit of a generator, G (Fig. 31), the current in the direction from terminal A over

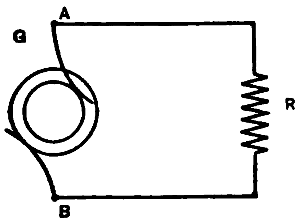


FIG. 31.

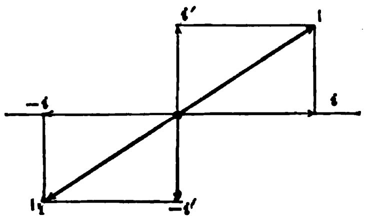


FIG. 32.

resistance R to terminal B, is represented by a vector, $\overline{OI}$ (Fig. 32), or by $\underset{\cdot}{I} = i + ji'$, the same current can be considered as being in the opposite direction, from terminal B to terminal A in opposite phase, and therefore represented by a vector $\overline{OI_1}$ (Fig. 32), or by $\underset{\cdot}{I}_1 = - i - ji'$.

Or, if the difference of potential from terminal B to terminal A is denoted by the $\underset{\cdot}{E} = e + je'$, the difference of potential from A to B is $E_1 = - e - je'$.

Hence, in dealing with alternating-current sine waves, it is necessary to consider them in their proper direction with regard to the circuit. Especially in more complicated circuits, as interlinked polyphase systems, careful attention has to be paid to this point.

40. Let, for instance, in Fig. 33, an interlinked three-phase system be represented diagrammatically, as consisting of three e.m.fs., of equal intensity, differing in phase by one-third of a period. Let the e.m.fs. in the direction from the common connection, O, of the three-branch circuits to the terminals, A_1, A_2, A_3, be represented by $\underset{.}{E}_1$, $\underset{.}{E}_2$, $\underset{.}{E}_3$. Then the difference of potential from A_2 to A_1 is $\underset{.}{E}_2 - \underset{.}{E}_1$, since the two e.m.fs., $\underset{.}{E}_1$ and $\underset{.}{E}_2$, are connected in circuit between the terminals, A_1 and A_2, in the direction, $A_1 — O — A_2$; that is, the one, E_2, in the direction, OA_2, from the common connection to terminal, the other, E_1, in the opposite direction, A_1O, from the terminal to common connection, and represented by $- E_1$. Conversely, the difference of potential from A_1 to A_2 is $\underset{.}{E}_1 - \underset{.}{E}_2$.

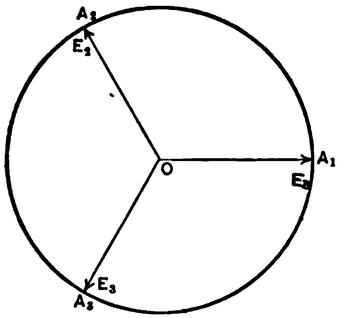


FIG. 33.

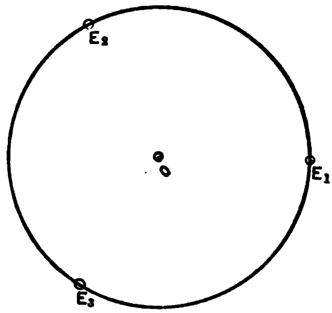


FIG. 34.

It is then convenient to go still a step farther, and drop the vector line altogether, in the diagrammatic representation; that is, denote the sine wave by a point only, the end of the corresponding vector.

Looking at this from a different point of view, it means that we choose one point of the system — for instance, the common connection, O — as a zero point, or point of zero potential, and represent the potentials of all the other points of the circuit by points in the diagram, such that their distances from the zero point gives the intensity; their amplitude the phase of the difference of potential of the respective point with regard to the zero point; and their distance and amplitude with regard to other points of the diagram, their difference of potential from these points in intensity and phase.

Thus, for example, in an interlinked three-phase system with three e.m.fs. of equal intensity, and differing in phase by one-third of a period, we may choose the common connection of the star-connected generator as the zero point, and represent, in Fig. 34, one of the e.m.fs., or the potential at one of the three-phase terminals, by point E_1. The potentials at the two other terminals will then be given by the points, E_2 and E_3, which have the same distance from O as E_1, and are equidistant from E_1 and from each other.

The difference of potential between any pair of terminals, for instance, E_1 and E_2, is then the distance $\overline{E_2E_1}$, or $\overline{E_1E_2}$, according to the direction considered.

41. If now the three branches, $\overline{OE_1}$, $\overline{OE_2}$ and $\overline{OE_3}$, of the three-phase system are loaded equally by three currents equal

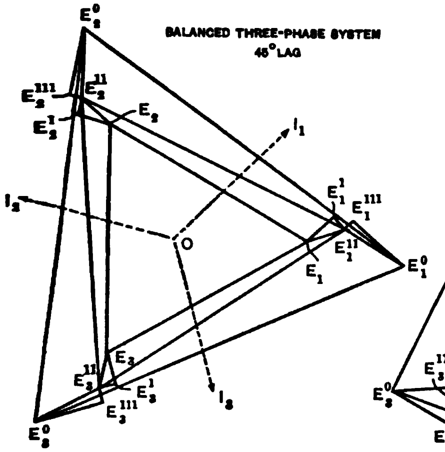


FIG. 35.

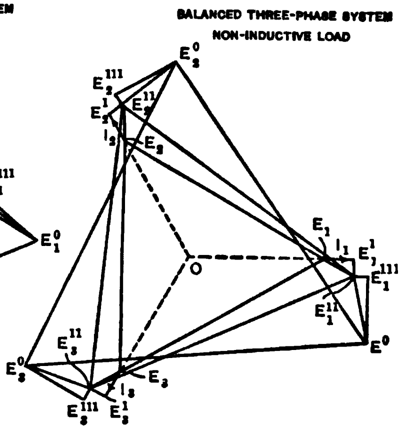


FIG. 36.

in intensity and in difference of phase against their e.m.fs., these currents are represented in Fig. 35 by the vectors $\overline{OI_1} = \overline{OI_2} = \overline{OI_3} = I$, lagging behind the e.m.fs. by angles $E_1OI_1 = E_2OI_2 = E_3OI_3 = \theta$.

Let the three-phase circuit be supplied over a line of impedance, $Z_1 = r_1 - jx_1$, from a generator of internal impedance, $Z_0 = x_0 - jx_0$.

In phase $\overline{OE}_1$ the e.m.f. consumed by resistance r_1 is represented by the distance, $\overline{E_1E_1{}^1} = Ir_1$, in phase, that is, parallel with current $\overline{OI}_1$. The e.m.f. consumed by reactance x_1 is represented by $\overline{E_1{}^1E_1{}^{11}} = Ix_1$, 90 time-degrees ahead of current $\overline{OI}_1$. The same applies to the other two phases, and it thus follows that to produce the e.m.f. triangle, $E_1E_2E_3$, at the terminals of the consumer's circuit, the e.m.f. triangle, $E_1{}^{11}E_2{}^{11}E_3{}^{11}$ is required at the generator terminals.

Repeating the same operation for the internal impedance of the generator, we get $\overline{E^{11}E^{111}} = Ir_0$, and parallel to $\overline{OI}_1$, $\overline{E^{111}E^0} = Ix_0$, and 90 time-degrees ahead of $\overline{OI}_1$, and thus as triangle of (nominal) generated e.m.fs. of the generator, $E_1{}^0E_2{}^0E_3{}^0$.

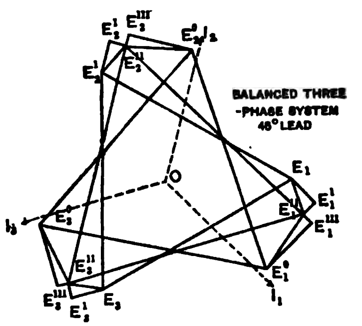


FIG. 37.

In Fig. 35 the diagram is shown for 45 time-degrees lag, in Fig. 36 for non-inductive load, and in Fig. 37 for 45 time-degrees lead of the currents with regard to their e.m.fs.

As seen, the generated e.m.f. and thus the generator excitation with lagging current must be higher, and with leading current lower, than at non-inductive load, or conversely with the same generator excitation, that is, the same internal generator e.m.f. triangle, $E_1{}^0E_2{}^0E_3{}^0$, the e.m.fs. at the receiver's circuit, E_1, E_2, E_3, fall off more with lagging, and less with leading current, than with non-inductive load.

42. As a further example may be considered the case of a single-phase alternating-current circuit supplied over a cable containing resistance and distributed capacity.

Let, in Fig. 38, the potential midway between the two terminals be assumed as zero point 0. The two terminal voltages at the receiver circuit are then represented by the points, E and E^1, equidistant from 0 and opposite each other, and the two currents at the terminals are represented by the points, I and I^1, equidistant from 0 and opposite each other, and under angle θ with E and E^1 respectively.

Considering first an element of the line or cable next to the receiver circuit. In this an e.m.f., $\overline{EE_1}$, is consumed by the resistance of the line element, in phase with the current, $\overline{OI}$, and proportional thereto, and a current, $\overline{II_1}$, consumed by the capacity, as charging current of the line element, 90° ahead in time-phase of the e.m.f., $\overline{OE}$, and proportional thereto, so that at the generator end of this cable element current and e.m.f. are $\overline{OI_1}$ and $\overline{OE_1}$ respectively.

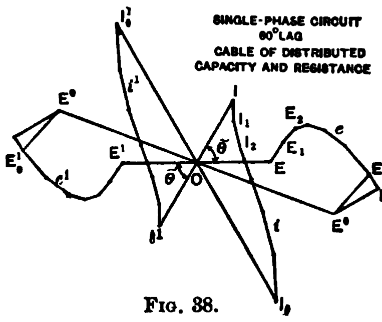


FIG. 38.

Passing now to the next cable element we have again an e.m.f., $\overline{E_1E_2}$, proportional to and in phase with the current, $\overline{OI_1}$, and a current, $\overline{I_1I_2}$, proportional to and 90 time-degrees ahead of the e.m.f., $\overline{OE_1}$, and thus passing from element to element along the cable to the generator, we get curves of e.m.fs., e and e^1, and curves of currents, i and i^1, which can be called the topographical circuit characteristics, and which correspond to each other, point for point, until the generator terminal voltages, $\overline{OE_0}$ and $\overline{OE_0}^1$, and the generator currents, $\overline{OI_0}$ and $\overline{OI_0}^1$, are reached.

Again, adding $\overline{E_0E^{11}} = I_0r_0$ and parallel to $\overline{OI_0}$ and $\overline{E^{11}E^0} = I_0x_0$ and 90° ahead of $\overline{OI_0}$, gives the (nominal) generated e.m.f. of the generator $\overline{OE^0}$, where $Z_0 = r_0 - jx_0$ = internal impedance of the generator.

In Fig. 38 is shown the circuit characteristics for 60° lag, of a cable containing only resistance and capacity.

Obviously by graphical construction the circuit characteristics appear more or less as broken lines, due to the necessity of using finite line elements, while in reality they are smooth curves when calculated by the differential method, as explained in Section III of "Theory and Calculation of Transient Electric Phenomena and Oscillations."

43. As further example may be considered a three-phase circuit supplied over a long-distance transmission line of distributed capacity, self-induction, resistance, and leakage.

Let, in Fig. 39, $\overline{OE_1}$, $\overline{OE_2}$, $\overline{OE_3}$ = three-phase e.m.fs. at receiver circuit, equidistant from each other and $= E$.

Let $\overline{OI_1}$, $\overline{OI_2}$, $\overline{OI_3}$ = three-phase currents in the receiver circuit equidistant from each other and $= I$, and making with E the time-phase angle, θ.

Considering again as in § 42 the transmission line, element by element, we have in every element an e.m.f., $\overline{E_1E_1^1}$, consumed by the resistance in phase with the current, $\overline{OI_1}$, and proportional

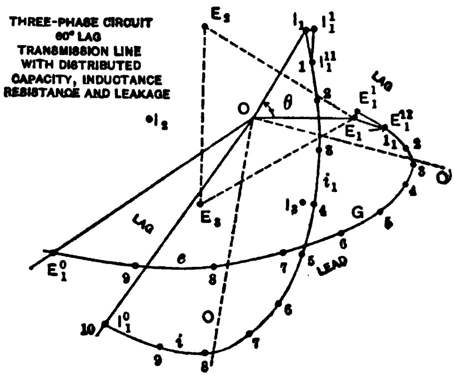


Fig. 39.

thereto, and an e.m.f., $\overline{E_1^1, E_1^{11}}$, consumed by the reactance of the line element, 90 time-degrees ahead of the current, $\overline{OI_1}$, and proportional thereto.

In the same line element we have a current, $\overline{I_1I_1^1}$, in time-phase with the e.m.f., $\overline{OE_1}$, and proportional thereto, representing the loss of current by leakage, dielectric hysteresis, etc., and a current, $\overline{I_1^1\,I_1^{11}}$, 90 time-degrees ahead of the e.m.f., $\overline{OE_1}$, and proportional thereto, the charging current of the line element as condenser, and in this manner passing along the line, element by element, we ultimately reach the generator terminal voltages, E_1^0, E_2^0, E_3^0, and generator currents, I_1^0, I_2^0, I_3^0, over the topographical characteristics of e.m.f., e_1, e_2, e_3, and of current, i_1, i_2, i_3, as shown in Fig. 39.

The circuit characteristics of current, i, and of e.m.f., e, correspond to each other, point for point, the one giving the current and the other the e.m.f in the line element.

Only the circuit characteristics of the first phase are shown, as e_1 and i_1. As seen, passing from the receiving end towards the generator end of the line, potential and current alternately rise and fall, while their time-phase angle changes periodically between lag and lead.

44. *a.* More markedly this is shown in Fig. 40, the topographic circuit characteristic of one of the lines with 90 time-degrees lag in the receiver circuit. Corresponding points of the two characteristics, *e* and *i*, are marked by corresponding figures 0 to 16, representing equidistant points of the line. The values of e.m.f. current and their difference of phase are plotted in Fig. 41 in rectangular coordinates with the distance as abscissas, counting from the receiving circuit towards the generator. As seen from Fig. 41, e.m.f. and current periodically but alternately rise and fall, a maximum of one approximately coinciding with a minimum of the other, and with a point of zero phase displacement. The time-phase angle between current and e.m.f. changes from 90° lag to 72° lead, 44° lag, 34° lead, etc., gradually decreasing in the amplitude of its variation.

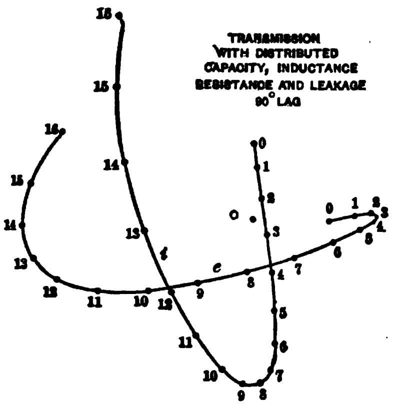


FIG. 40.

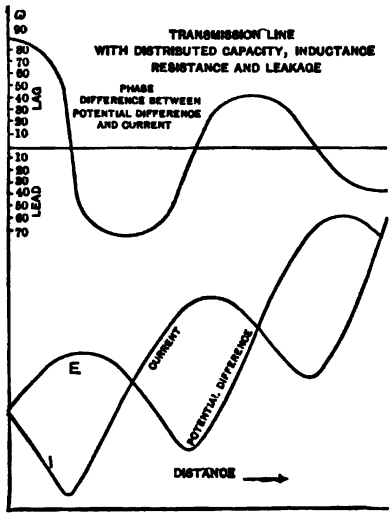


FIG. 41.

CHAPTER VIII.

ADMITTANCE, CONDUCTANCE, SUSCEPTANCE.

45. If in a continuous-current circuit, a number of resistances, $r_1, r_2, r_3, \ldots$, are connected in series, their joint resistance, R, is the sum of the individual resistances, $R = r_1 + r_2 + r_3 + \ldots$.

If, however, a number of resistances are connected in multiple or in parallel, their joint resistance, R, cannot be expressed in a simple form, but is represented by the expression

$$R = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \ldots} \cdot$$

Hence, in the latter case it is preferable to introduce, instead of the term *resistance*, its reciprocal, or inverse value, the term *conductance*, $g = \frac{1}{r} \cdot$ If, then, a number of conductances, $g_1, g_2, g_3, \ldots$ are connected in parallel, their joint conductance is the sum of the individual conductances, or $G = g_1 + g_2 + g_3 + \ldots$ When using the term conductance, the joint conductance of a number of series-connected conductances becomes similarly a complicated expression

$$G = \frac{1}{\frac{1}{g_1} + \frac{1}{g_2} + \frac{1}{g_3} + \ldots} \cdot$$

Hence the term *resistance* is preferable in case of series connection, and the use of the reciprocal term *conductance* in parallel connections; therefore,

The joint resistance of a number of series-connected resistances is equal to the sum of the individual resistances; the joint conductance of a number of parallel-connected conductances is equal to the sum of the individual conductances.

46. In alternating-current circuits, instead of the term *resistance* we have the term *impedance*, $Z = r - jx$, with its two components, the *resistance*, r, and the *reactance*, x, in the formula of Ohm's law, $\dot{E} = \dot{I}Z$. The resistance, r, gives the component of e.m.f. in phase with the current, or the power component of the e.m.f., Ir; the reactance, x, gives the component of the e.m.f. in quadrature with the current, or the wattless component of e.m.f., $\dot{I}x$; both combined give the total e.m.f.,

$$Iz = I\sqrt{r^2 + x^2}.$$

Since e.m.fs. are combined by adding their complex expressions, we have:

The joint impedance of a number of series-connected impedances is the sum of the individual impedances, when expressed in complex quantities.

In graphical representation impedances have not to be added, but are combined in their proper phase by the law of parallelogram in the same manner as the e.m.fs. corresponding to them.

The term impedance becomes inconvenient, however, when dealing with parallel-connected circuits; or, in other words, when several currents are produced by the same e.m.f., such as in cases where Ohm's law is expressed in the form,

$$\dot{I} = \frac{\dot{E}}{Z}.$$

It is preferable, then, to introduce the reciprocal of impedance, which may be called the *admittance* of the circuit, or

$$Y = \frac{1}{Z}.$$

As the reciprocal of the complex quantity, $Z = r - jx$, the admittance is a complex quantity also, or $Y = g + jb$; it consists of the component, g, which represents the coefficient of

current in time-phase with the e.m.f., or the power component, $g\dot{E}$, of the current, in the equation of Ohm's law,

$$\dot{I} = Y\dot{E} = (g + jb)\dot{E},$$

and the component, b, which represents the coefficient of current in quadrature with the e.m.f., or wattless component, $b\dot{E}$, of the current.

g is called the *conductance*, and b the *susceptance*, of the circuit. Hence the conductance, g, is the power component, and the susceptance, b, the wattless component, of the admittance, $Y = g + jb$, while the numerical value of admittance is

$$y = \sqrt{g^2 + b^2};$$

the resistance, r, is the power component, and the reactance, x, the wattless component, of the impedance, $Z = r - jx$, the numerical value of impedance being

$$z = \sqrt{r^2 + x^2}.$$

47. As shown, the term *admittance* implies resolving the current into two components, in time-phase and in time-quadrature with the e.m.f., or the power component and the wattless or reactive component; while the term *impedance* implies resolving the e.m.f. into two components, in time-phase and in time-quadrature with the current, or the power component and the wattless or reactive component.

It must be understood, however, that the conductance is not the reciprocal of the resistance, but depends upon the reactance as well as upon the resistance. Only when the reactance $x = 0$, or in continuous-current circuits, is the conductance the reciprocal of resistance.

Again, only in circuits with zero resistance ($r = 0$) is the susceptance the reciprocal of reactance; otherwise, the susceptance depends upon reactance and upon resistance.

The conductance is zero for two values of the resistance:

(1) If $r = \infty$, or $x = \infty$, since in this case there is no current, and either component of the current $= 0$.

(2) If $r = 0$, since in this case the current in the circuit is in time-quadrature with the e.m.f., and thus has no power component.

Similarly, the susceptance, b, is zero for two values of the reactance:

(1) If $x = \infty$, or $r = \infty$.

(2) If $x = 0$.

From the definition of admittance, $Y = g + jb$, as the reciprocal of the impedance, $Z = r - jx$, we have

$$Y = \frac{1}{Z}, \text{ or } g + jb = \frac{1}{r - jx};$$

or, multiplying numerator and denominator on the right side by $(r + jx)$,

$$g + jb = \frac{r + jx}{(r - jx)(r + jx)};$$

hence, since

$$(r - jx)(r + jx) = r^2 + x^2 = z^2,$$

$$g + jb = \frac{r}{r^2 + x^2} + j\frac{x}{r^2 + x^2} = \frac{r}{z^2} + j\frac{x}{z^2};$$

or

$$g = \frac{r}{r^2 + x^2} = \frac{r}{z^2},$$

$$b = \frac{x}{r^2 + x^2} = \frac{x}{z^2};$$

and conversely

$$r = \frac{g}{g^2 + b^2} = \frac{g}{y^2},$$

$$x = \frac{b}{g^2 + b^2} = \frac{b}{y^2}.$$

By these equations, the conductance and susceptance can be calculated from resistance and reactance, and conversely.

Multiplying the equations for g and r, we get

$$gr = \frac{rg}{z^2 y^2};$$

hence, $z^2 y^2 = (r^2 + x^2)(g^2 + b^2) = 1;$

and $z = \frac{1}{y} = \frac{1}{\sqrt{g^2 + b^2}},$ } the absolute value of impedance;

$y = \frac{1}{z} = \frac{1}{\sqrt{r^2 + x^2}},$ } the absolute value of admittance.

48. If, in a circuit, the reactance, x, is constant, and the resistance, r, is varied from $r = 0$ to $r = \infty$, the susceptance, b, decreases from $b = \frac{1}{x}$ at $r = 0$, to $b = 0$ at $r = \infty$; while the conductance, $g = 0$ at $r = 0$, increases, reaches a maximum for $r = x$, where $g = \frac{1}{2r}$ is equal to the susceptance, or $g = b$, and then decreases again, reaching $g = 0$ at $r = \infty$.

In Fig. 42, for constant reactance $x = 0.5$ ohm, the variation of the conductance, g, and of the susceptance, b, are shown as functions of the varying resistance, r. As shown, the absolute value of admittance, susceptance, and conductance are plotted in full lines, and in dotted line the absolute value of impedance,

$$z = \sqrt{r^2 + x^2} = \frac{1}{y}.$$

Obviously, if the resistance, r, is constant, and the reactance, x, is varied, the values of conductance and susceptance are merely exchanged, the conductance decreasing steadily from $g = \frac{1}{r}$ to 0, and the susceptance passing from 0 at $x = 0$ to the maximum, $b = \frac{1}{2r} = g = \frac{1}{2x}$ at $x = r$, and to $b = 0$ at $x = \infty$.

The resistance, r, and the reactance, x, vary as functions of the conductance, g, and the susceptance, b, in the same manner as g and b vary as functions of r and x.

The sign in the complex expression of admittance is always opposite to that of impedance; this is obvious, since if the current lags behind the e.m.f., the e.m.f. leads the current, and conversely.

We can thus express Ohm's law in the two forms,

$$\dot{E} = \dot{I}Z,$$
$$\dot{I} = \dot{E}Y,$$

and therefore,

The joint impedance of a number of series-connected impedances is equal to the sum of the individual impedances; the joint admittance of a number of parallel-connected admittances, if expressed

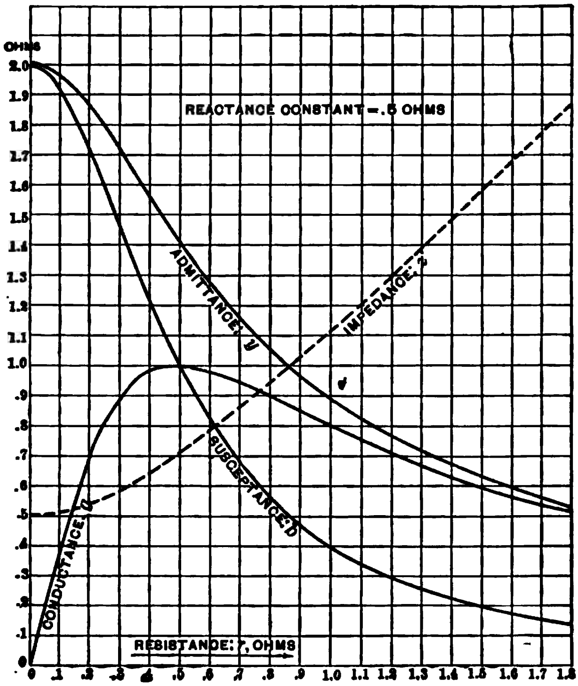


FIG. 42.

in complex quantities, is equal to the sum of the individual admittances. In diagrammatic representation, combination by the parallelogram law takes the place of addition of the complex quantities.

49. Experimentally, impedances and admittances are most conveniently determined by establishing an alternating current in the circuit, and measuring by voltmeter, ammeter and wattmeter, the volts, e, the amperes, i, and the watts, p.

It is then,

$$\text{Impedance: } z = \frac{e}{i}.$$

$$\text{Resistance (effective): } r = \frac{p}{i^2}.$$

$$\text{Reactance: } x = \sqrt{z^2 - r^2}.$$

$$\text{Admittance: } y = \frac{i}{e}.$$

$$\text{Conductance: } g = \frac{p}{e^2}.$$

$$\text{Susceptance: } b = \sqrt{y^2 - g^2}.$$

Regarding their calculation, see "Theoretical Elements of Electrical Engineering."

CHAPTER IX.

CIRCUITS CONTAINING RESISTANCE, INDUCTIVE REACTANCE, AND CONDENSIVE REACTANCE.

50. Having, in the foregoing, re-established Ohm's law and Kirchhoff's laws as being also the fundamental laws of alternating-current circuits, when expressed in their complex form,

$$\dot{E} = Z\dot{I}, \qquad \text{or, } \dot{I} = Y\dot{E},$$

and

$$\Sigma \dot{E} = 0 \text{ in a closed circuit,}$$

$$\Sigma \dot{I} = 0 \text{ at a distributing point,}$$

where $\dot{E}$, $\dot{I}$, Z, Y, are the expressions of e.m.f., current, impedance, and admittance in complex quantities, — these values representing not only the intensity, but also the phase, of the alternating wave, — we can now — by application of these laws, and in the same manner as with continuous-current circuits, keeping in mind, however, that $\dot{E}$, $\dot{I}$, Z, Y, are complex quantities — calculate alternating-current circuits and networks of circuits containing resistance, inductive reactance, and condensive reactance in any combination, without meeting with greater difficulties than when dealing with continuous-current circuits.

It is obviously not possible to discuss with any completeness all the infinite varieties of combinations of resistance, inductive reactance, and condensive reactance which can be imagined, and which may exist, in a system of network of circuits; therefore only some of the more common or more interesting combinations will here be considered.

(1) *Resistance in series with a circuit.*

51. In a constant-potential system with impressed e.m.f.,

$$\dot{E}_0 = e_0 + je_0', \qquad E_0 = \sqrt{e_0^2 + e_0'^2},$$

let the receiving circuit of impedance,

$$Z = r - jx, \qquad z = \sqrt{r^2 + x^2},$$

be connected in series with a resistance, r_0.

The total impedance of the circuit is then

$$Z + r_0 = r + r_0 - jx;$$

hence the current is

$$\underset{\cdot}{I} = \frac{\dot{E}_0}{Z + r_0} = \frac{\dot{E}_0}{r + r_0 - jx} = \frac{\dot{E}_0 (r + r_0 + jx)}{(r + r_0)^2 + x^2};$$

and the e.m.f. of the receiving circuit becomes

$$\underset{\cdot}{E} = \underset{\cdot}{I} Z = \frac{\dot{E}_0 (r - jx)}{r + r_0 - jx} = \frac{\dot{E}_0 \{r(r + r_0) + x^2 - jr_0 x\}}{(r + r_0)^2 + x^2}$$

$$= \frac{\dot{E}_0 \{z^2 + rr_0 - jr_0 x\}}{z^2 + 2rr_0 + r_0^2};$$

or, in absolute values we have the following:

Impressed e.m.f.,

$$E_0 = \sqrt{e_0^2 + e_0'^2};$$

current,

$$I = \frac{E_0}{\sqrt{(r + r_0)^2 + x^2}} = \frac{E_0}{\sqrt{z^2 + 2rr_0 + r_0^2}};$$

e.m.f. at terminals of receiver circuit,

$$E = E_0 \sqrt{\frac{r^2 + x^2}{(r + r_0)^2 + x^2}} = \frac{E_0 z}{\sqrt{z^2 + 2rr_0 + r_0^2}};$$

difference of phase in receiver circuit, $\tan\theta = \frac{x}{r}$;

difference of phase in supply circuit, $\tan\theta_0 = \frac{x}{r + r_0}$

since in general,

$$\tan(\text{phase}) = \frac{\text{imaginary component}}{\text{real component}}.$$

(*a*) If x is negligible with respect to r, as in a non-inductive receiving circuit,

$$I = \frac{E_0}{r + r_0}, \qquad E = E_0 \frac{r}{r + r_0},$$

and the current and e.m.f. at receiver terminals decrease steadily with increasing r_0.

(*b*) If r is negligible compared with x, as in a wattless receiver circuit,

$$I = \frac{E_0}{\sqrt{r_0^2 + x^2}}, \quad E = E_0 \frac{x}{\sqrt{r_0^2 + x^2}};$$

or, for small values of r_0,

$$I = \frac{E_0}{x}, \qquad E = E_0;$$

that is, the current and e.m.f. at receiver terminals remain approximately constant for small values of r_0, and then decrease with increasing rapidity.

In the general equations, x appears in the expressions for I and E only as x^2, so that I and E assume the same value when x is negative, as when x is positive; or, in other words, series-resistance acts upon a circuit with leading current, or in a condenser circuit, in the same way as upon a circuit with lagging current, or an inductive circuit.

For a given impedance, z, of the receiver circuit, the current, I, and e.m.f., E, are smaller, the larger the value of r; that is, the less the difference of phase in the receiver circuit.

As an instance, in Fig. 43 is shown the e.m.f., E, at the receiver circuit, for E_0 = const. = 100 volts, $z = 1$ ohm; hence $I = E$, and

(*a*) $r_0 = 0.2$ ohm (Curve I.)
(*b*) $r_0 = 0.8$ ohm (Curve II.)

with values of reactance, $x = \sqrt{z^2 - r^2}$, for abscissæ, from $x = +1.0$ to $x = -1.0$ ohm.

As shown, I and E are smallest for $x = 0$, $r = 1.0$, or for the non-inductive receiver circuit, and largest for $x = \pm 1.0$,

$r = 0$, or for the wattless circuit, in which latter a series resistance causes but a very small drop of potential.

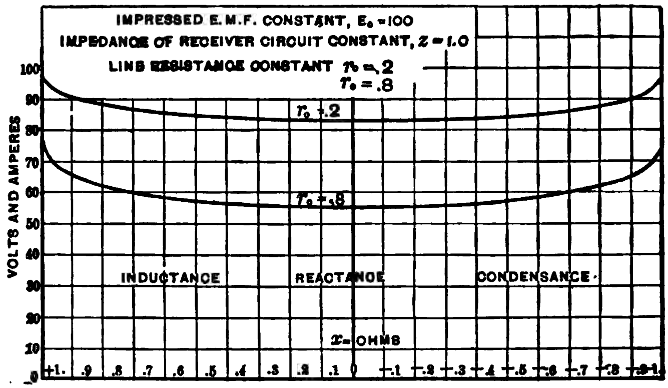


Fig. 43.—Variation of Voltage at Constant Series Resistance with Phase Relation of Receiver Circuit.

Hence the control of a circuit by series resistance depends upon the difference of phase in the circuit.

For $r_0 = 0.8$, and $x = 0$, $x = +0.8$, $x = -0.8$, the polar diagrams are shown in Figs. 44 to 46.

(2) *Reactance in series with a circuit.*

52. In a constant potential system of impressed e.m.f.,

$$E_0 = e_0 + je_0', \qquad E_0 = \sqrt{e_0^2 + e_0'^2},$$

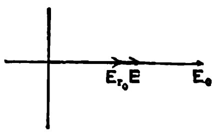


Fig. 44.

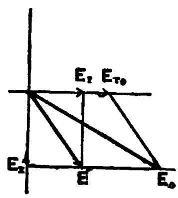


Fig. 45.

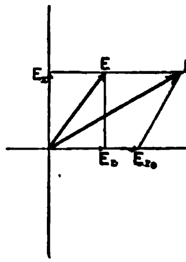


Fig. 46.

let a reactance, x_0, be connected in series in a receiver circuit of impedance,

$$Z = r - jx, \qquad z = \sqrt{r^2 + x^2}.$$

Then, the total impedance of the circuit is

$$Z - jx_0 = r - j(x + x_0),$$

and the current is

$$\dot{I} = \frac{\dot{E}_0}{Z - jx_0} = \frac{\dot{E}_0}{r - j(x + x_0)},$$

while the difference of potential at the receiver terminals is

$$\dot{E} = \dot{I}Z = \dot{E}_0 \frac{r - jx}{r - j(x + x_0)}.$$

Or, in absolute quantities,

current,

$$I = \frac{E_0}{\sqrt{r^2 + (x + x_0)^2}} = \frac{E_0}{\sqrt{z^2 + 2\,xx_0 + x_0^2}};$$

e.m.f. at receiver terminals,

$$E = E_0 \sqrt{\frac{r^2 + x^2}{r^2 + (x + x_0)^2}} = \frac{E_0 z}{\sqrt{z^2 + 2\,xx_0 + x_0^2}};$$

difference of phase in receiver circuit,

$$\tan\theta = \frac{x}{r};$$

difference of phase in supply circuit,

$$\tan\theta_0 = \frac{x + x_0}{r}.$$

(*a*) If x is small compared with r, that is, if the receiver circuit is non-inductive, I and E change very little for small values of x_0; but if x is large, that is, if the receiver circuit is of large reactance, I and E change considerably with a change of x_0.

(*b*) If x is negative, that is, if the receiver circuit contains condensers, synchronous motors, or other apparatus which produce leading currents — above a certain value of x the

denominator in the expression of E becomes $< z$, or $E > E_0$; that is, the reactance, x_0, raises the potential.

(*c*) $E = E_0$, or the insertion of a series reactance, x_0, does not affect the potential difference at the receiver terminals, if

$$\sqrt{z^2 + 2\,xx_0 + x_0^2} = z;$$

or,

$$x_0 = -2\,x.$$

That is, if the reactance which is connected in series in the circuit is of opposite sign, but twice as large as the reactance of the receiver circuit, the voltage is not affected, but $E = E_0$, $I = \frac{E_0}{z}$. If $x_0 < -2\,x$, it raises, if $x_0 > -2\,x$, it lowers, the voltage.

We see, then, that a reactance inserted in series in an alternating-current circuit will always lower the voltage at the receiver terminals, when of the same sign as the reactance of the receiver circuit; when of opposite sign, it will lower the voltage if larger, raise the voltage if less, than twice the numerical value of the reactance of the receiver circuit.

(*d*) If $x = 0$, that is, if the receiver circuit is non-inductive, the e.m.f. at receiver terminals is

$$E = \frac{E_0 r}{\sqrt{r^2 + x_0^2}} = \frac{E_0}{\sqrt{1 + \left(\frac{x_0}{r}\right)^2}}$$

$$= E_0 \left\{1 - \frac{1}{2}\left(\frac{x_0}{r}\right)^2 + \frac{3}{8}\left(\frac{x_0}{r}\right)^4 - + \ldots\right\}$$

$\left(\frac{1}{\sqrt{1 + x}} = (1 + x)^{-\frac{1}{2}}\right.$ expanded by the binomial theorem

$$\left.(1 + x)^n = 1 + nx + \frac{n\,(n-1)}{1.2}\,x^2 + \ldots\right).$$

Therefore, if x_0 is small compared with r,

$$E = E_0\left(1 - \frac{1}{2}\left(\frac{x_0}{r}\right)^2\right),$$

$$\frac{E_0 - E}{E_0} = \frac{1}{2}\left(\frac{x_0}{r}\right)^2.$$

That is, the percentage drop of potential by the insertion of reactance in series in a non-inductive circuit is, for small values of reactance, independent of the sign, but proportional to the square of the reactance, or the same whether it be inductive reactance or condensive reactance.

53. As an example, in Fig. 47 the changes of current, I, and of e.m.f. at receiver terminals, E, at constant impressed e.m.f., E_0, are shown for various conditions of a receiver circuit and amounts of reactance inserted in series.

Fig. 47 gives for various values of reactance, x_0 (if positive, inductive — if negative, condensive), the e.m.fs., E, at receiver

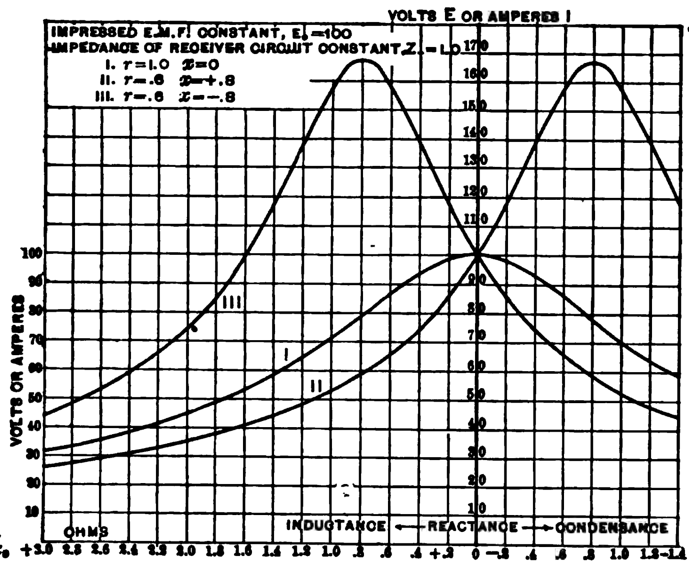


FIG. 47.

terminals, for constant impressed e.m.f., $E_0 = 100$ volts, and the following conditions of receiver circuit:

$$z = 1.0,\ r = 1.0,\ x = 0 \text{ (Curve I.)}$$
$$z = 1.0,\ r = 0.6,\ x = 0.8 \text{ (Curve II.)}$$
$$z = 1.0,\ r = 0.6,\ x = -0.8 \text{ (Curve III.)}$$

As seen, curve I is symmetrical, and with increasing x_0 the voltage E remains first almost constant, and then drops off with increasing rapidity.

In the inductive circuit series inductive reactance, or, in a condenser circuit series condensive reactance, causes the voltage to drop off very much faster than in a non-inductive circuit.

Series inductive reactance in a condenser circuit, and series condensive reactance in an inductive circuit, cause a rise of potential. This rise is a maximum for $x_0 = \pm 0.8$, or, $x_0 = -x$ (the condition of resonance), and the e.m.f. reaches the value, $E = 167$ volts, or, $E = E_0 \frac{z}{r}$. This rise of potential by series reactance continues up to $x_0 = \pm 1.6$, or, $x_0 = -2x$, where $E = 100$ volts again; and for $x_0 > 1.6$ the voltage drops again.

At $x_0 = \pm 0.8$, $x = \mp 0.8$, the total impedance of the circuit is $r - j(x + x_0) = r = 0.6$, $x + x_0 = 0$, and $\tan \theta_0 = 0$; that is, the current and e.m.f. in the supply circuit are in phase with each other, or the circuit is in *electrical resonance.*

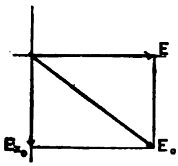


FIG. 48.

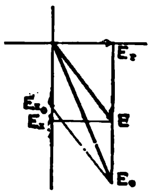


FIG 49.

Since a synchronous motor in the condition of efficient working acts as a condensive reactance, we get the remarkable result that, in synchronous motor circuits, choking coils, or reactive coils, can be used for raising the voltage.

In Figs. 48 to 50, the polar diagrams are shown for the conditions

$$
\begin{array}{llll}
E_0 = 100,\ x_0 = 0.6, & x = \ \ \ \ 0 & \text{(Fig. 48)}\ E = & 85.7 \\
 & x = +0.8 & \text{(Fig. 49)}\ E = & 65.7 \\
 & x = -0.8 & \text{(Fig. 50)}\ E = & 158.1
\end{array}
$$

54. In Fig. 51 the dependence of the potential, E, upon the difference of phase, θ, in the receiver circuit is shown for the constant impressed e.m.f., $E_0 = 100$; for the constant receiver impedance, $z = 1.0$ (but of various phase differences θ), and for various series reactances, as follows:

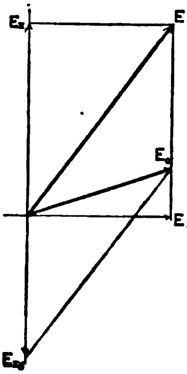


FIG. 50.

$x_0 = 0.2$ (Curve I.)
$x_0 = 0.6$ (Curve II.)
$x_0 = 0.8$ (Curve III.)
$x_0 = 1.0$ (Curve IV.)
$x_0 = 1.6$ (Curve V.)
$x_0 = 3.2$ (Curve VI.)

Since $z = 1.0$, the current, I, in all these diagrams has the same value as E.

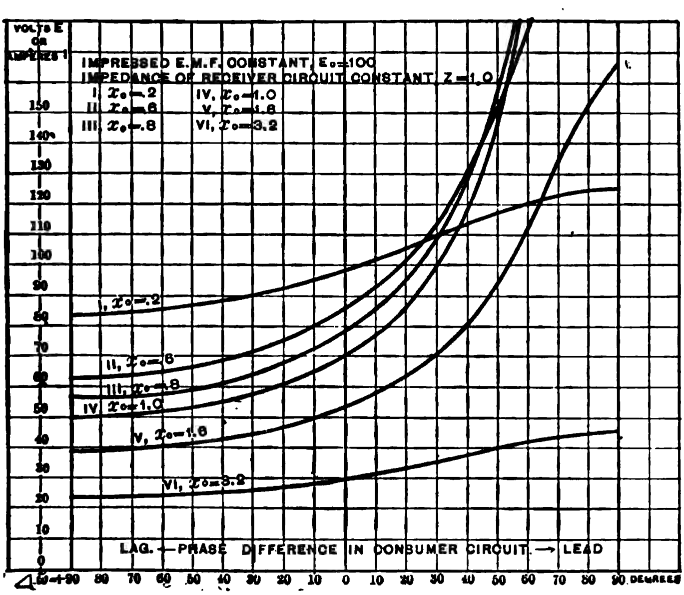


FIG. 51. — Variation of Voltage at Constant Series Reactance with Phase Angle of Receiver Circuit.

In Figs. 52 and 53, the same curves are plotted as in Fig. 51, but in Fig. 52 with the reactance, x, of the receiver circuit as

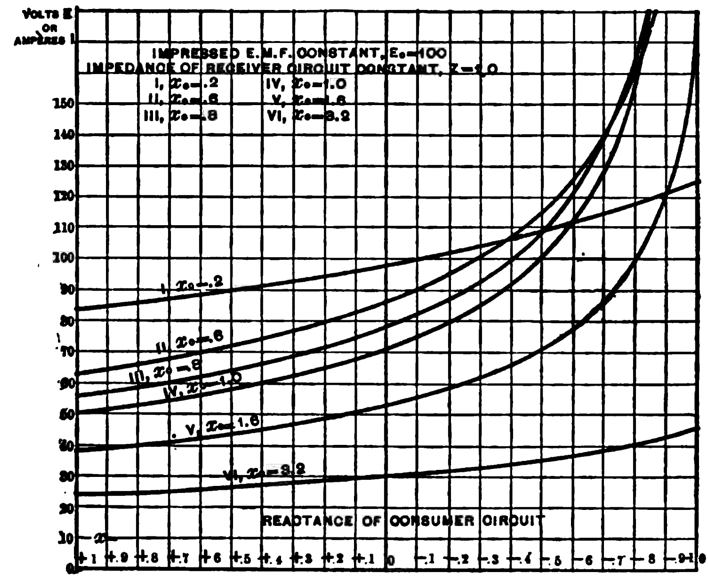


Fig. 52. — Variation of Voltage at Constant Series Reactance with Reactance of Receiver Circuit.

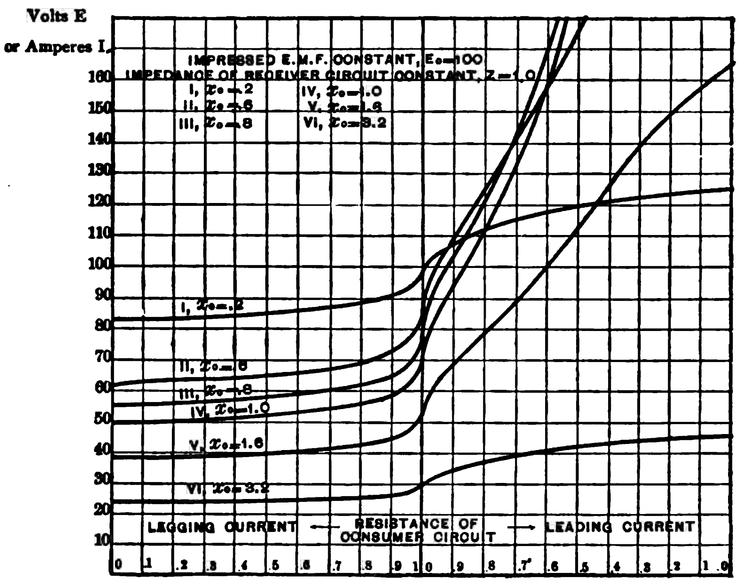


Fig. 53. — Variation of Voltage at Constant Series Reactance with Resistance of Receiver Circuit.

abscissas; and in Fig. 53 with the resistance, r, of the receiver circuit as abscissas.

As shown, the receiver voltage, E, is always lowest when x_0 and x are of the same sign, and highest when they are of opposite sign.

The rise of voltage due to the balance of x_0 and x is a maximum for $x_0 = +1.0$, $x = -1.0$, and $r = 0$, where $E = \infty$; that is, absolute resonance takes place. Obviously, this condition cannot be completely reached in practice.

It is interesting to note, from Fig. 53, that the largest part of the drop of potential due to inductive reactance, and rise to condensive reactance — or conversely — takes place between $r = 1.0$ and $r = 0.9$; or, in other words, a circuit having a power-factor $\cos\theta = 0.9$, gives a drop several times larger than a non-inductive circuit, and hence must be considered as an inductive circuit.

(3) *Impedance in series with a circuit.*

55. By the use of reactance for controlling electric circuits, a certain amount of resistance is also introduced, due to the ohmic resistance of the conductor and the hysteretic loss, which, as will be seen hereafter, can be represented as an effective resistance.

Hence the impedance of a reactive coil (choking coil) may be written thus:

$$Z_0 = r_0 - jx_0, \qquad z_0 = \sqrt{r_0^2 + x_0^2},$$

where r_0 is in general small compared with x_0.

From this, if the impressed e.m.f. is

$$\dot{E}_0 = e_0 + je_0', \qquad E_0 = \sqrt{e_0^2 + e_0'^2},$$

and the impedance of the consumer circuit is

$$Z = r - jx, \qquad z = \sqrt{r^2 + x^2},$$

we get the current

$$\dot{I} = \frac{\dot{E}_0}{Z + Z_0} = \frac{\dot{E}_0}{(r + r_0) - j(x + x_0)}$$

and the e.m.f. at receiver terminals,

$$\dot{E} = \dot{E}_0 \frac{Z}{Z + Z_0} = \dot{E}_0 \frac{r - jx}{(r + r_0) - j(x + x_0)}.$$

Or, in absolute quantities,
the current is,

$$I = \frac{E_0}{\sqrt{(r + r_0)^2 + (x + x_0)^2}} = \frac{E_0}{\sqrt{z^2 + z_0^2 + 2\,(rr_0 + xx_0)}}\,,$$

the e.m.f. at receiver terminals is

$$E = \frac{E_0 z}{\sqrt{(r + r_0)^2 + (x + x_0)^2}} = \frac{E_0 z}{\sqrt{z^2 + z_0^2 + 2\,(rr_0 + xx_0)}}\,;$$

the difference of phase in receiver circuit is

$$\tan \theta = \frac{x}{r}\,;$$

and the difference of phase in the supply circuit is

$$\tan \theta = \frac{x + x_0}{r + r_0}\,.$$

56. In this case, the maximum drop of potential will not take place for either $x = 0$, as for resistance in series, or for $r = 0$, as for reactance in series, but at an intermediate point. The drop of voltage is a maximum; that is, E is a minimum if the denominator of E is a maximum; or, since z, z_0, r_0, x_0 are constant, if $rr_0 + xx_0$ is a maximum, that is, since $x = \sqrt{z^2 - r^2}$, if $rr_0 + x_0\sqrt{z^2 - r^2}$ is a maximum. A function, $f = rr_0 + x_0 \sqrt{z^2 - r^2}$ is a maximum when its differential coefficient equals zero. For, plotting f as curve with values of r as abscissas, at the point where f is a maximum or a minimum, this curve is for a short distance horizontal, hence the tangens-function of its tangent equals zero. The tangens-function of the tangent of a curve, however, is the ratio of the change of ordinates to the change of abscissas, or is the differential coefficient of the function represented by the curve.

Thus we have:

$$f = rr_0 + x_0 \sqrt{z^2 - r^2}$$

is a maximum or minimum, if

$$\frac{df}{dr} = 0.$$

Differentiating, we get

$$r_0 + \frac{1}{2}\frac{x_0}{\sqrt{z^2 - r^2}}(-2r) = 0;$$

or, expanded,

$$r_0\sqrt{z^2 - r^2} - x_0 r = r_0 x - x_0 r = 0,$$

or, $$r \div x = r_0 \div x_0.$$

That is, the drop of potential is a maximum, if the reactance factor, $\frac{x}{r}$, of the receiver circuit equals the reactance factor, $\frac{x_0}{r_0}$, of the series impedance.

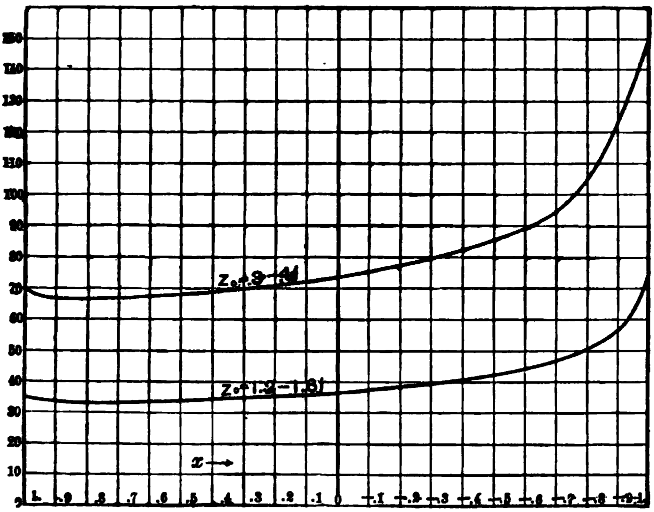


FIG. 54.

57. As an example, Fig. 54 shows the e.m.f., E, at the receiver terminals, at a constant impressed e.m.f., $E_0 = 100$, a constant impedance of the receiver circuit, $z = 1.0$, and constant series impedances,

$$Z_0 = 0.3 - j\,0.4 \qquad \text{(Curve I.)}$$
$$Z_0 = 1.2 - j\,1.6 \qquad \text{(Curve II.)}$$

as functions of the reactance, x, of the receiver circuit.

Figs. 55 to 57 give the polar diagram for $E_0 = 100, x = 0.95$, $x = 0, x = -0.95$, and $Z_0 = 0.3 - 0.4j$.

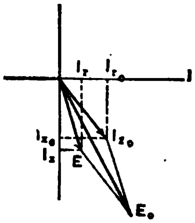


Fig. 55.

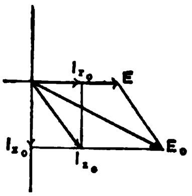


Fig. 56.

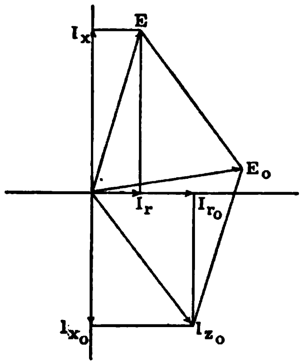


Fig. 57.

(4) *Compensation for Lagging Currents by Shunted Condensive Reactance.*

58. We have seen in the preceding paragraphs, that in a constant potential alternating-current system, the voltage at the terminals of a receiver circuit can be varied by the use of a variable reactance in series with the circuit, without loss of energy except the unavoidable loss due to the resistance and hysteresis of the reactance; and that, if the series reactance is very large compared with the resistance of the receiver circuit, the current in the receiver circuit becomes more or less independent of the resistance, — that is, of the power consumed in the receiver circuit, which in this case approaches the conditions of a constant alternating-current circuit, whose current is

$$I = \frac{E_0}{\sqrt{r^2 + x_0^2}}, \text{ or, approximately, } I = \frac{E_0}{x_0}.$$

This potential control, however, causes the current taken from the mains to lag greatly behind the e.m.f., and thereby

requires a much larger current than corresponds to the power consumed in the receiver circuit.

Since a condenser draws from the mains a current in leading time-phase, a condenser shunted across such a circuit carrying current in lagging time-phase compensates for the lag, the leading and the lagging current combining to form a resultant current more or less in time-phase with the e.m.f., and therefore proportional to the power expended.

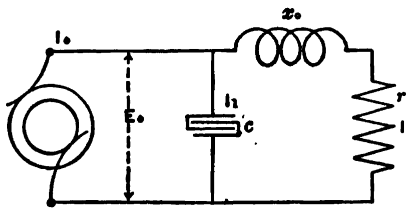


Fig. 58.

In a circuit shown diagrammatically in Fig. 58, let the non-inductive receiver circuit of resistance, r, be connected in series with the inductive reactance, x_0, and the whole shunted by a condenser of condensive reactance, x_c, entailing but a negligible loss of power.

Then, if $\dot{E}_0$ = impressed e.m.f., the current in receiver circuit is

$$\dot{I} = \frac{\dot{E}_0}{r - jx_0}, \quad I = \frac{E_0}{\sqrt{r^2 + x_0^2}};$$

the current in condenser circuit is

$$\dot{I}_1 = \frac{\dot{E}_0}{jx_c}, \quad I_1 = \frac{E_0}{x_c};$$

and the total current is

$$\dot{I}_0 = \dot{I} + \dot{I}_1 = \dot{E}_0 \left\{ \frac{1}{r - jx_0} + \frac{1}{jx_c} \right\}$$

$$= \dot{E}_0 \left\{ \frac{r}{r^2 + x_0^2} + j\left(\frac{x_0}{r^2 + x_0^2} - \frac{1}{x_c} \right) \right\},$$

or, in absolute terms, $I_0 \doteq E_0 \sqrt{\left(\frac{r}{r^2 + x_0^2} \right)^2 + \left(\frac{x_0}{r^2 + x_0^2} - \frac{1}{x_c} \right)^2}$;

while the e.m.f. at receiver terminals is

$$\dot{E} = \dot{I} r = \dot{E}_0 \frac{r}{r - jx_0}, \qquad E = \frac{E_0 r}{\sqrt{r^2 + x_0^2}}.$$

59. The main current, $\dot{I}_0$, is in time-phase with the impressed e.m.f., $\dot{E}_0$, or the lagging current is completely balanced, or supplied by, the condensive reactance, if the imaginary term in the expression of $\dot{I}_0$ disappears; that is, if

$$\frac{x_0}{r^2 + x_0^2} - \frac{1}{x_c} = 0.$$

This gives, expanded, $x_c = \dfrac{r^2 + x_0^2}{x_0}$.

Hence the capacity required to compensate for the lagging current produced by the insertion of inductive reactance in series with a non-inductive circuit depends upon the resistance and the inductive reactance of the circuit. x_0 being constant, with increasing resistance, r, the condensive reactance has to be increased, or the capacity decreased, to keep the balance.

Substituting
$$x_c = \frac{r^2 + x_0^2}{x_0},$$

we get, as the equations of the inductive circuit balanced by condensive reactance,

$$\dot{I} = \frac{\dot{E}_0}{r - jx_0} = \frac{\dot{E}_0(r + jx_0)}{r^2 + x_0^2}, \qquad I = \frac{E_0}{\sqrt{r^2 + x_0^2}};$$

$$\dot{I}_1 = -\frac{j\dot{E}_0 x_0}{r^2 + x_0^2}, \qquad I_1 = \frac{E_0 x_0}{r^2 + x_0^2};$$

$$\dot{I}_0 = \frac{\dot{E}_0 r}{r^2 + x_0^2}, \qquad I_0 = \frac{E_0 r}{r^2 + x_0^2};$$

$$\dot{E} = \frac{\dot{E}_0 r}{r - jx_0}, \qquad E = \frac{E_0 r}{\sqrt{r^2 + x_0^2}};$$

and for the power expended in the receiver circuit,

$$I^2 r = \frac{E_0^2 r}{r^2 + x_0^2} = I_0 E_0;$$

that is, the main current is proportional to the expenditure of power.

For $r = 0$, we have $x_c = x_0$, as the condition of balance.

Complete balance of the lagging component of current by shunted capacity thus requires that the condensive reactance x_c

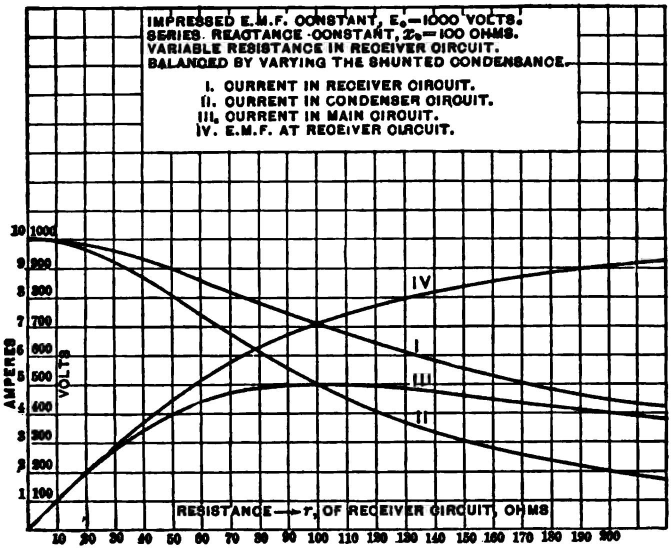


FIG. 59. — Compensation of Lagging Currents in Receiving Circuit by Variable Shunted Condensance.

be varied with the resistance, r; that is, with the varying load on the receiver circuit.

In Fig. 59 are shown, for a constant impressed e.m.f., $E_0 =$ 1000 volts, and a constant series reactance, $x_0 = 100$ ohms, values for the balanced circuit of

current in receiver circuit	(Curve I.),
current in condenser circuit	(Curve II.),
current in main circuit	(Curve III.),
e.m.f. at receiver terminals	(Curve IV.),

with the values the resistance, r, of the receiver circuit as abscissas.

60. If, however, the condensive reactance is left unchanged, $x_c = x_0$ at the no-load value, the circuit is balanced for $r = 0$,

but will be overbalanced for $r > 0$, and the main current will become leading.

We get in this case,

$$x_c = x_0;$$

$$\dot{I} = \frac{\dot{E}_0}{r - jx_0}, \qquad I = \frac{E_0}{\sqrt{r^2 + x_0^2}};$$

$$\dot{I}_1 = -\frac{j\dot{E}_0}{x_0}, \qquad I_1 = \frac{E_0}{x_0};$$

$$\dot{I}_0 = \dot{I} + \dot{I}_1 = \frac{\dot{E}_0 r}{x_0 (x_0 + jr)}, \quad I_0 = \frac{E_0 r}{x_0 \sqrt{r^2 + x_0^2}};$$

$$\dot{E} = \dot{I} r = \frac{\dot{E}_0 r}{r - jx_0}, \qquad E = \frac{E_0 r}{\sqrt{r^2 + x_0^2}}.$$

The difference of phase in the main circuit is

$$\tan \theta_0 = -\frac{r}{x_0},$$

which is $= 0$, when $r = 0$ or at no-load, and increases with increasing resistance, as the lead of the current. At the same time, the current in the receiver circuit, I, is approximately constant for small values of r, and then gradually decreases.

In Fig. 60 are shown the values of I, I_1, I_0, E, in Curves I, II, III, IV, similarly as in Fig. 60, for $E_0 = 1000$ volts, $x_c = x_0 = 100$ ohms, and r as abscissas.

(5) *Constant Potential — Constant-Current Transformation.*

61. In a constant potential circuit containing a large and constant reactance, x_0, and a varying resistance, r, the current is approximately constant, and only gradually drops off with increasing resistance, r, — that is, with increasing load, — but the current lags greatly behind the e.m.f. This lagging current in the receiver circuit can be supplied by a shunted condensance. Leaving, however, the condensance constant, $x_c = x_0$, so as to balance the lagging current at no-load, that is, at $r = 0$, it will overbalance with increasing load, that is, with increasing r, and

thus the main current will become leading, while the receiver current decreases if the impressed e.m.f., E_0, is kept constant. Hence, to keep the current in the receiver circuit entirely constant, the impressed e.m.f., E_0, has to be increased with increasing

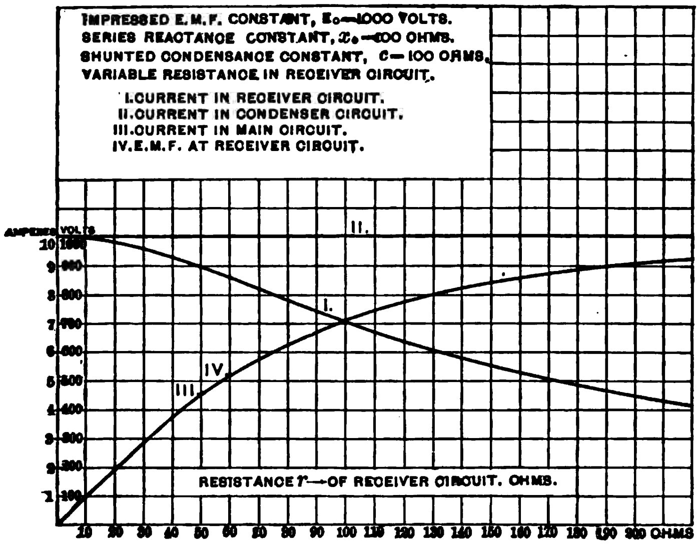


FIG. 60.

resistance, r; that is, with increasing lead of the main current. Since, as explained before, in a circuit with leading current, a series inductive reactance raises the potential, to maintain the

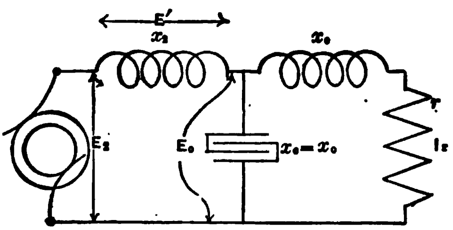


FIG. 61.

current in the receiver circuit constant under all loads, an inductive reactance, x_2, inserted in the main circuit, as shown in the diagram, Fig. 61, can be used for raising the potential, E_0, with increasing load.

62. The generation of alternating-current electric power almost always takes place at constant potential. For some purposes, however, as for operating series arc circuits, and to a limited extent also for electric furnaces, a constant, or approximately constant, alternating current is required.

Such constant alternating currents, therefore, are usually produced from constant potential circuits by means of inductive reactances, or combinations of inductive and condensive reactances; and the investigation of different methods of producing constant alternating current from constant alternating potential, or inversely, constitutes a good illustration of the application of the terms "impedance," "reactance," etc., and offers a large number of problems or examples for the application of the method of complex quantities.

As seen in the preceding paragraphs, with an inductive reactance inserted in series with an alternating-current non-inductive circuit, the current in this circuit is approximately constant, as long as the resistance of the circuit is small compared with the series inductive reactance.

Let

$E_0 = e_0$ = constant = impressed e.m.f.

r = resistance of non-inductive receiver circuit.

x_0 = inductive reactance inserted in series with this circuit.

The impedance of the total circuit then is

$$Z = r - jx_0, \text{ and } z = \sqrt{r^2 + x_0^2};$$

and thus, the current

$$\dot{I} = \frac{e_0}{Z} = \frac{e_0}{r - jx_0}; \tag{1}$$

and the absolute value is

$$i = \frac{e_0}{z} = \frac{e_0}{\sqrt{r^2 + x_0^2}}. \tag{2}$$

The tangent of the phase angle of the supply circuit is

$$\tan\theta_0 = \frac{x_0}{r}, \tag{3}$$

and the power-factor,

$$\cos\theta_0 = \frac{r}{z} \tag{4}$$

If in this case, r is small compared with x_0, it is

$$i = \frac{e_0}{x_0} \frac{1}{\sqrt{1 + \left(\frac{r}{x_0}\right)^2}}; \tag{5}$$

or, expanded by the binomial theorem,

$$\frac{1}{\sqrt{1 + \left(\frac{r}{x_0}\right)^2}} = \left\{1 + \left(\frac{r}{x_0}\right)^2\right\}^{-\frac{1}{2}} = 1 - \frac{r^2}{2\,x_0^2} + \frac{3\,r^4}{8\,x_0^4} - + \ldots$$

hence,

$$i = \frac{e_0}{x_0}\left\{1 - \frac{r^2}{2\,x_0^2} + \frac{3\,r^4}{8\,x_0^4} - + \ldots\right\}; \tag{6}$$

that is, for small values of r, the current, i, is approximately constant, and is

$$i = \frac{e_0}{x_0}.$$

For small values of r, the power-factor

$$\cos\theta = \frac{r}{z}$$

is very low, however.

Allowing a variation of current of 10 per cent from short-circuit or no-load, $r = 0$, to full load, or $r = r_1$, it is, substituted in (2):

No load current:

$$i_0 = \frac{e_0}{x_0}.$$

Full load current:

$$i = \frac{e_0}{\sqrt{r_1^2 + x_0^2}} = 0.9\,i_0.$$

Hence,

$$\frac{e_0}{\sqrt{r_1^2 + x_0^2}} = 0.9\,\frac{e_0}{x_0},$$

and therefore,

$$r_1 = 0.485\,x_0,$$

and the power-factor, from (4), is 0.437.

That is, even allowing as large a variation of current, i, as 10 per cent, the maximum power-factor only reaches 43.7 per

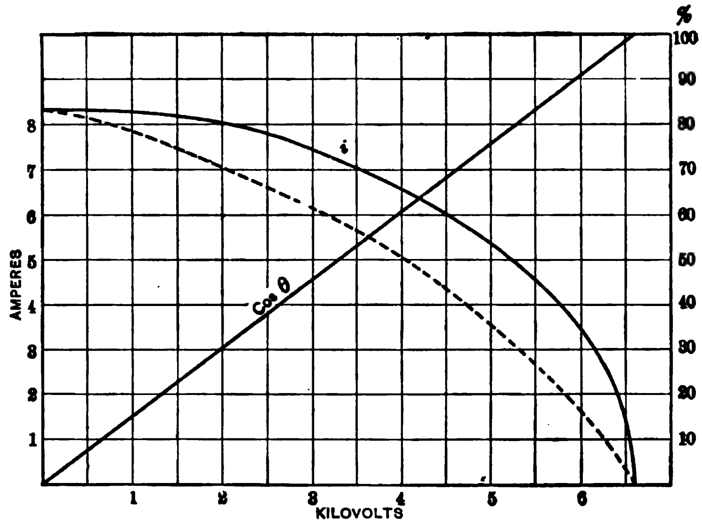


FIG. 62.

cent, when producing constant current regulation by series inductive reactance.

As illustrations are shown, in Fig. 62, for the constants:

$$e_0 = 6600 \text{ volts applied e.m.f.};$$
$$x_0 = 792 \text{ ohms series reactance};$$

the current:

$$i = \frac{6600}{\sqrt{r^2 + 792^2}}$$
$$= \frac{8.33}{\sqrt{1 + \left(\frac{r}{792}\right)^2}} \text{ amperes};$$

and the power-factor:

$$\cos\theta = \frac{r}{\sqrt{r^2 + 792^2}} = \frac{r}{792\sqrt{1 + \left(\frac{r}{792}\right)^2}};$$

with the voltage at the secondary terminals:

$$e = ri$$

as abscissas.

63. If the receiver circuit is inductive, that is, contains, in addition to the resistance, r, an inductive reactance, x, and if this reactance is proportional to the resistance,

$$x = kr,$$

as is commonly the case in arc circuits, due to the inductive reactance of the regulating mechanism of the arc lamp (the effective resistance, r, and the inductive reactance, x, in this case are both proportional to the number of lamps, hence proportional to each other), it is:
total impedance:

$$Z = r - j\,(x_0 + x) = r - j\,(x_0 + kr);$$

or the absolute value is

$$z = \sqrt{r^2 + (x_0 + x)^2} = \sqrt{r^2 + (x_0 + kr)^2};$$

thus, the current

$$I = \frac{e_0}{r - j(x_0 + kr)}, \tag{7}$$

and the absolute value is

$$i = \frac{e_0}{\sqrt{r^2 + (x_0 + kr)^2}} = \frac{e_0}{x_0} \frac{1}{\sqrt{1 + \frac{2\,kr}{x_0} + \frac{r^2\,(1 + k^2)}{x_0^2}}}; \tag{8}$$

and the power-factor:

$$\cos\theta_0 = \frac{r}{z} = \frac{r}{\sqrt{r^2 + (x_0 + kr)^2}}. \tag{9}$$

By the binomial theorem, it is

$$\frac{i}{\sqrt{1 + \frac{2\,kr}{x_0} + \frac{r^2\,(1 + k^2)}{x_0^2}}} = 1 - \frac{kr}{x_0} - \frac{r^2\,(2 - k^2)}{4\,x_0^2} + - \dots$$

Hence, the current

$$i = \frac{e_0}{x_0}\left\{1 - \frac{kr}{x_0} - \frac{r^2(2 - k^2)}{4\,x_0^2} + - \dots\right\} \tag{10}$$

that is, the expression of the current, i (10), contains the ratio, $\frac{r}{x_0}$, in the first power, with k as coefficient; and if therefore k is not very small, that is, the inductive reactance, $x = kr$, a very small fraction of the resistance, r, the current, i, is not even approximately constant, but begins to fall off immediately, even at small values of r.

Assuming, for instance,

$$k = 0.4.$$

That is, the inductive reactance x of the receiver circuit equals 40 per cent of its resistance r, and the power-factor of the receiver circuit accordingly is

$$\begin{aligned}\cos\theta &= \frac{r}{r^2 + x^2} \\ &= \frac{1}{1 + k^2} \\ &= 93 \text{ per cent;}\end{aligned}$$

it is, substituted in (8),

$$I = \frac{e_0}{x_0\sqrt{\left(\frac{r}{792}\right)^2 + \left(1 + 0.4\frac{r}{x_0}\right)^2}}.$$

As illustration are shown, in the same Fig. 62, for the constants:

$$\begin{aligned}e_0 &= 6600 \text{ volts supply e.m.f.;} \\ x_0 &= 792 \text{ ohms series reactance;}\end{aligned}$$

the current:

$$i = \frac{8.33}{\sqrt{\left(\frac{r}{792}\right)^2 + \left(1 + 0.4\,\frac{r}{792}\right)^2}} \text{ amperes.}$$

This current is shown by dotted line.

In this case, in an inductive circuit, the current, i, has decreased by 10 per cent below the no-load or short-circuit value of 8.33

amperes, that is, has fallen to 7.5 amperes, at the resistance $r = 187$ ohms, or at the voltage of the receiving circuit,

$$e = i\sqrt{r^2 + x^2} = ri\sqrt{1 + k^2} = 1.077\, ri = 1500 \text{ volts};$$

while, in the case of a non-inductive load, the current has fallen off to 7.5 amperes, or by 10 per cent, at the resistance $r = 395$ ohms, or at the voltage of the receiving circuit: $e = 2950$ volts.

64. As seen, a moderate constant-current regulation can be produced in a non-inductive circuit, by a constant series inductive reactance, at a considerable sacrifice, however, of the power-factor, while in an inductive receiver circuit, the constant-current regulation is not even approximate.

To produce constant alternating current, from a constant potential supply, by a series inductive reactance, over a wide range of load and without too great a sacrifice of power-factor, therefore requires a variation of the series inductive reactance with the load. That is, with increasing load, or increasing resistance of the receiver circuit, the series inductive reactance has to be decreased, so as to maintain the total impedance of the circuit, and thereby the current, constant.

In constant-current apparatus, as transformers from constant potential to constant current, or regulators, this variation of series inductive reactance with the load is usually accomplished automatically by the mechanical motion caused by the mechanical force exerted by the magnetic field of the current, upon the conductor in which the current exists.

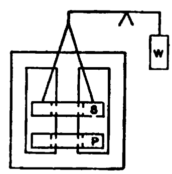


FIG. 63.

For instance, in the constant-current transformer, as shown diagrammatically in Fig. 63, the secondary coils, S, are arranged so that they can move away from the primary coils, P, or inversely. Primary and secondary currents are proportional to each other, as in any transformer, and the magnetic field between primary and secondary coils, or the magnetic stray field, in which the secondary coils float, is proportional to either current. The magnetic repulsion between primary coils and secondary coils is proportional to the current (or rather

its ampere-turns), and to the magnetic stray field, hence is proportional to the square of the current, but independent of the voltage. The secondary coils, S, are counter-balanced by a weight, W, which is adjusted so that this weight, W, plus the repulsive thrust between secondary coils, S, and primary coils, P (which, as seen above, is proportional to the square of the current), just balances the weight of the secondary coils. Any increase of secondary current, as, for instance, caused by short-circuiting a part of the secondary load, then increases the repulsion between primary and secondary coils, and the secondary coils move away from the primary; hence more of the magnetic flux produced by the primary coils passes between primary and secondary, as stray field, or self-inductive flux, less passes through the secondary coils, and therefore the secondary generated voltage decreases with the separation of the coils, and also thereby the secondary current, until it has resumed the same value, and the secondary coil is again at rest, its weight balancing counterweight plus repulsion.

Inversely, an increase of load, that is, of secondary impedance, decreases the secondary current, so causes the secondary coils to move nearer the primary, and to receive more of the primary flux; that is, generate higher voltage.

In this manner, by the mechanical repulsion caused by the current, the magnetic stray flux, or, in other words, the series inductive reactance of the constant-current transformer, varies automatically between a maximum, with the primary and secondary coils at their maximum distance apart, and a minimum with the coils touching each other. Obviously, this automatic action is independent of frequency, impressed voltage, and character of load.

Assuming then, in the constant-current transformer or other apparatus, a device to vary the series inductive reactance so as to maintain the current constant. Let

$E_0 = e_0 =$ constant $=$ impressed e.m.f.,

$Z = r - jx,$

$= r\,(1 - jk)$ the impedance of the load, and let

$x_0 =$ inductive series reactance, as the self-inductive internal reactance of the constant-current transformer.

The current in the circuit then is

$$\dot{I} = \frac{e_0}{r - j(x_0 + x)},$$

or, the absolute value:

$$i = \frac{e_0}{\sqrt{r^2 + (x_0 + x)^2}};$$

and, to maintain the current, i, constant ($i = i_0$), then requires

$$i_0 = \frac{e_0}{\sqrt{r^2 + (x_0 + x)^2}};$$

or, transposed,

$$x_0 = \sqrt{\left(\frac{e_0}{i_0}\right)^2 - r^2} - x; \tag{11}$$

or, for

$$x = kr,$$

$$x_0 = \sqrt{\left(\frac{e_0}{i_0}\right)^2 - r^2} - kr; \tag{12}$$

that is, to produce perfectly constant current by means of a variable series inductive reactance, this series reactance must be varied with the load on the circuit, according to equation (11) or (12).

For non-inductive load, or $x = 0$, it is

$$x_0 = \sqrt{\left(\frac{e_0}{i_0}\right)^2 - r^2}, \tag{13}$$

the maximum load, which can be carried, is given by

$$x_0 = 0$$

and is

$$z = \sqrt{r^2 + x^2} = r\sqrt{1 + k^2} = \frac{e_0}{i_0}. \tag{14}$$

As seen from equation (13), the decrease of inductive reactance, x_0, required to maintain constant current with non-inductive load, is small for small values of resistance, r, where the r^2 under the root is negligible. With inductive load, equation (11), the inductive reactance, x_0, has still further to be decreased by the inductive reactance of the load, x.

$$x_{00} = \frac{e_0}{i_0}$$

is the value of the series inductive reactance at no-load or short-circuit, equations (11), (12), (13) assume the form:
General inductive load:

$$x_0 = \sqrt{x_{00}^2 - r^2} - x, \tag{14}$$

Inductive load of $\frac{x}{r} = k$:

$$x_0 = \sqrt{x_{00}^2 - r^2} - kr, \tag{15}$$

Non-inductive load:

$$x_0 = \sqrt{x_{00}^2 - r^2} \tag{16}$$

65. As seen, a constant series inductive reactance gives an approximately constant regulation with non-inductive load, but if the load is inductive this regulation is spoiled. Inversely it can be shown, that condensive reactance, that is, a source of leading current in the load, improves the constant-current regulation.

With a non-inductive load, series condensive reactance exerts the same effect on the current regulation as series inductive reactance; the equations discussed in the preceding paragraphs remain the same, except that the sign of x_0 is reversed and the current always leading.

With series condensive reactance, condensive reactance in the load spoils, inductive reactance in the load improves the constant-current regulation.

That is, in general, a constant series reactance gives approximately constant-current regulation in a non-inductive circuit, and with a reactive load this regulation is impaired if the

reactance of the load is of the same sign as the series reactance, and the regulation is improved if the reactance of the load is of opposite sign as the series reactance.

Since a constant-current load is usually somewhat inductive, it follows that a constant series condensive reactance gives a better constant-current regulation, in the average case of a somewhat inductive arc-circuit, than the constant series inductive reactance.

Let $\dot{E}_0 = e_0$ = constant = impressed, or supply voltage.

$Z = r - jx$ = impedance of the load, or the receiver circuit, and

$$x = kr,$$

that is,

$$Z = r(1 - jk),$$

or, absolute,

$$z = r\sqrt{1 + k^2}.$$

Let now a constant condensive reactance be inserted in series with this circuit, of the reactance, $- x_c$, then the total impedance of the circuit is

$$Z' = r + j(x_c - kr), \tag{17}$$

The current is

$$\dot{I} = \frac{e_0}{r + j(x_c - kr)}, \tag{18}$$

or, the absolute value is

$$i = \frac{e_0}{\sqrt{r^2 + (x_c - kr)^2}}; \tag{19}$$

the phase angle is

$$\tan \theta_0 = - \frac{x_c - kr}{r}, \tag{20}$$

and the power-factor is

$$\cos \theta_0 = \frac{r}{\sqrt{r^2 + (x_c - kr)^2}}, \tag{21}$$

for

$$k = 0,$$

or non-inductive load, equations (19) and (21) assume the form:

$$i = \frac{e_0}{\sqrt{r^2 + x_c}} \text{ and } \cos \theta = \frac{r}{\sqrt{r^2 + x_c^2}},$$

that is, the same as with series inductive reactance.

From equation (19) it follows, that with increasing current, i, from no-load:

$$r = 0, \text{ hence: } i_0 = \frac{e_0}{x_c}; \tag{22}$$

the current, i_0, first increases, reaches a maximum, and then decreases again. When decreasing, it once more reaches the value i_0, for the resistance, r_1, of the load, which is given by

$$i_0 = \frac{e_0}{\sqrt{r_1^2 + (x_c - kr_1)^2}} = \frac{e_0}{x_c};$$

hence, expanded,

$$r_1 = \frac{2\, kx_c}{1 + k^2}, \tag{23}$$

and the maximum value through which i passes between $r = 0$ and $r = r_1$, is given by

$$\frac{di}{dr} = 0,$$

or

$$\frac{d}{dr}\{r^2 + (x_c - kr)^2\} = 0 = 2\,r - 2\,k\,(x_c - kr);$$

hence,

$$r_2 = \frac{kx_c}{1 + k^2} = \frac{r_1}{2}. \tag{24}$$

This maximum value is given by substituting (24) in (19), as

$$\begin{aligned} i_2 &= \frac{e_0}{x_c}\sqrt{1 + k^2}, \\ &= i_0\sqrt{1 + k^2}, \end{aligned} \tag{25}$$

for $k = 0.4$,

this value is

$$i_2 = 1.077\, i_0,$$

that is, the current rises from no load to a maximum 7.7 per cent above the no-load value, and then decreases again.

As an example, let

$$e_0 = 6{,}600 \text{ volts impressed e.m.f.}$$

and

$$x_c = 880 \text{ ohm condensive reactance,}$$

x_c being chosen so as to give

$$i_0 = \frac{e_0}{x_c} = 7.5 \text{ amperes;}$$

for

$$k = 0.4.$$

then:

$$i = \frac{6600}{\sqrt{r^2 + (880 - 0.4\,r)^2}},$$

$$\cos\theta_0 = \frac{r}{\sqrt{r^2 + (880 - 0.4\,r)^2}},$$

$$e = zi = 1.077\,ri.$$

These values of current and power-factor are plotted, with the receiver voltage as abscissas, in Fig. 64.

66. The conclusions from the preceding are that a constant series reactance, whether condensive or inductive, when inserted in a constant-potential circuit, tends towards a constant-current regulation, at least within a certain range of load. That is, at varying resistance, r, and therefore varying load, the current is approximately constant at light load, and drops off only gradually with increasing load.

This constant-current regulation, and the power-factor of the circuit, are best if the reactance of the receiver circuit is of opposite sign to the series reactance, and poorest if of the same sign. That is, series condensive reactance in an inductive circuit, and series inductive reactance in a circuit carrying leading current, give the best regulation; series inductive reactance with an inductive, and series condensive reactance with leading current in the circuit, give the poorest regulation.

Since the receiver circuit is usually inductive, to get best

regulation, either a series condensive reactance has to be used, as in Fig. 64, or, if a series inductive reactance is used, the cur-

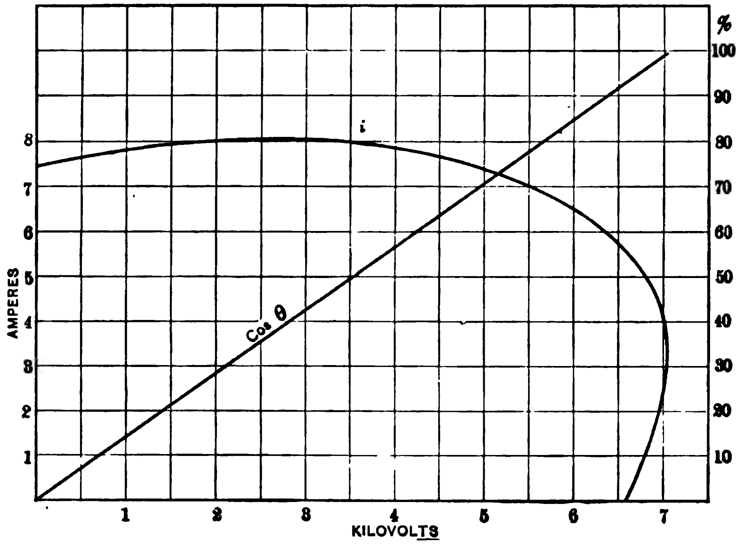


FIG. 64.

rent in the receiver circuit is made leading, as, for instance, by shunting the receiver circuit by a condensive reactance.

Assuming, then, as sketched diagrammatically in Fig. 65, in a circuit of constant impressed e.m.f., $\dot{E}_0 = e_0 =$ constant, a constant inductive reactance, x_0, inserted in series; and the receiver circuit, of impedance,

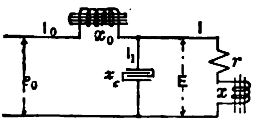


FIG. 65.

$$Z = r - jx = r(1 - jk)$$

where

$$k = \text{tangent of the angle of lag} = \frac{x}{r};$$

let the receiver circuit be shunted by a constant condensive reactance, x_c; let then:

$\dot{E}$ = potential difference of receiver circuit, and also at the condenser terminals,

$\dot{I}$ = current in the receiver circuit, or the "secondary current,"

$\dot{I}_1$ = current in the condenser,

$\dot{I}_0$ = total supply current, or "primary current."

Then
$$I_0 = I + I_1 \tag{26}$$
and the e.m.f. at receiver circuit is
$$E = ZI; \tag{27}$$
at the condenser,
$$E = jx_c I_1, \tag{28}$$
hence,
$$I_1 = \frac{Z}{jx_c} I, \tag{29}$$

and, in the main circuit, the impressed e.m.f. is

$$E_0 = e_0 = E - jx_0 I_0. \tag{30}$$

Hence, substituting (26), (27) and (29) in (30):

$$e_0 = ZI - jx_0 \left(I + \frac{Z}{jx_c} I \right)$$
$$= \left\{ Z - jx_0 - \frac{x_0}{x_c} Z \right\} I$$

or

$$e_0 = \left\{ Z \frac{k - x_0}{k} - jx_0 \right\} I \tag{31}$$

and

$$I = \frac{e_0}{Z \dfrac{k - x_0}{k} - jx_0}. \tag{32}$$

If $x_c = x_0$, that is, if the shunted condensive reactance equals the series inductive reactance, equations (32) assume the form:

$$I = - \frac{e_0}{jx_0} = j \frac{e_0}{x_0}, \tag{33}$$

and the absolute value is

$$i = \frac{e_0}{x_0}, \tag{34}$$

that is, the current, i, is constant, independent of the load and the power-factor.

That is, if in a constant-potential circuit, of impressed e.m.f., e_0, an inductive reactance, x_0, and a condensive reactance x_c, are connected in series with each other, and if $x_c = x_0$, (35), that is, the two reactances are in resonance condition with each other, any circuit shunting the capacity reactance is a constant-current circuit, and regardless of the impedance of this circuit, $Z = r - jx$, the current in the circuit is

$$i = \frac{e_0}{x_0}.$$

67. Such a combination of two equal reactances of opposite sign so can be considered as a transforming device from constant potential to constant current.

Substituting, therefore, (35) in the preceding equation gives: (33) substituted in (29):

Current in shunted capacity

$$\dot{I}_1 = \frac{Z}{x_0^2} e_0. \tag{36}$$

or, absolute,

$$i_1 = \frac{ze_0}{x_0^2}, \tag{37}$$

and, substituting (33) and (36) in (26):
primary supply current is

$$\dot{I}_0 = \frac{Z + jx_0}{x_0^2} e_0, \tag{38}$$

or the absolute value is

$$i_0 = \frac{e_0}{x_0^2}\sqrt{r^2 + (x_0 - x)^2}, \tag{39}$$

and the power-factor of the supply current is

$$\tan \theta_0 = -\frac{x_0 - x}{r},$$

$$\cos \theta_0 = \frac{r}{\sqrt{r^2 + (x_0 - x)^2}}. \tag{40}$$

In this case, the higher the inductive reactance, x, of the receiving circuit the lower is the supply current, i_0, at the same resistance, r, and the higher is the power-factor, and if $x = x_0$

$$\dot{I}_0 = \frac{e^2 r}{x_0^2} \text{ and } \cos\theta = 1, \tag{41}$$

that is, the primary, or supply circuit is non-inductive, and the primary current is in time-phase with the supply e.m.f., and the power-factor is unity, while the secondary or receiver current (33) is 90° in time-phase behind the primary impressed e.m.f., e_0.

Inserting therefore an inductive reactance, $x_1 = x_0 - x$, in series in the receiver circuit of impedance, $Z = r - jx$, raises the power-factor of the supply current, i_0, to unity, and makes this current, i_0, a minimum. Or, if the inductive reactance, x_0, is inserted in the receiver circuit, thus giving a total impedance, $Z - jx_0 = r - j(x + x_0)$ by equation (38), substituting $Z - jx_0$ instead of Z, gives the primary supply current as

$$\dot{I}_0 = \frac{Ze_0}{x_0^2} \tag{42}$$

or the absolute value as

$$i_0 = \frac{ze_0}{x_0^2} \tag{43}$$

and the tangent of the primary phase angle

$$\tan\theta_0 = \frac{x}{r} = \tan\theta,$$

that is, the primary power-factor equals that of the secondary.

Hence, as shown diagrammatically in Fig. 66, a combination of two equal inductive reactances in series with each other and with the receiver circuit, and shunted midway between the inductive reactances by a condensive reactance equal to the inductive reactance, transforms constant potential into constant current, and inversely, without any change of power-factor, that is, the primary supply current has the same power-factor as the secondary current.

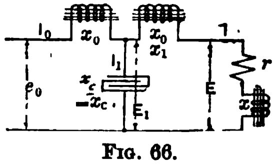


FIG. 66.

With an inductive secondary circuit, the primary power-factor can in this way be made unity, by reducing the inductive reactance of the secondary side, by the amount of secondary reactance.

68. Shunted condensive reactance, x_c, and series inductive reactance, x_0, therefore transforms from constant potential, e_0, to constant current, i, and inversely, if their reactances are equal, $x_c = x_0$, and in this case, the main current is leading, with non-inductive load, and the time lead of the main current decreases, with increasing inductive reactance, that is, increasing lag, of the receiving circuit. The constant secondary current, i, lags 90° behind the constant primary e.m.f., e_0.

Inversely, by reversing the signs of x_0 and x_c in the preceding equations, that is, exchanging inductive and condensive reactances, it follows that shunted inductive reactance, x_0, and series condensive reactance, x_c, if of equal reactance, $x_c = x_0$, transform constant potential, e_0, into constant current, i, and inversely. In this case, the main current lags the more the higher the inductive reactance of the receiving circuit, and the constant secondary current, i, is 90 time-degrees ahead of the constant primary e.m.f., e_0.

In general, it follows that, if equal inductive and condensive reactances, $x_0 = x_c$, that is, in resonance condition, are connected in series across a constant-potential circuit of impressed e.m.f., e_0, any circuit connected to the common point between the reactances is a constant-current circuit, and carries the current, $i = \frac{e_0}{x_0}$.

Instead of connecting this secondary or constant-current circuit with its other terminal to line A, so shunting the condensive reactance with it, and causing the main current to lead (I in Fig. 67), or to line, B, so shunting the inductive reactance with it, and causing the main current to lag (II in Fig. 67), it can be connected to any point intermediate between A and B, by a compensator, as in III, Fig. 67. If connected to the middle point between A and B, the main current is neither lagging nor leading, that is, is non-inductive with non-inductive load, and with inductive load, has the same power-factor as the load.

The two arrangements, I and II, can also be combined, by

connecting the constant-current circuit across, as in IV, Fig. 67, and in this case the two inductive reactances and two condensive reactances diagrammatically form a square, with the constant potential, e_0, as one, the constant current, i, as the other diagonal, as shown in Fig. 68 on page 111. This arrangement has been called the *monocyclic square.*

The insertion of an e.m.f. into the constant-current circuit, in such arrangements, obviously does not exert any effect on the constancy of the secondary current, i, but merely changes the primary current, i_0, by the amount of power supplied or consumed by the e.m.f. inserted in the secondary circuit.

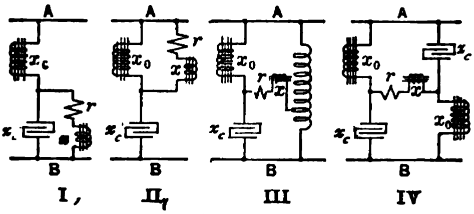


FIG. 67.

While theoretically the secondary current is absolutely constant, at constant primary e.m.f., practically, it can not be perfectly constant in the reactances due to the power consumed, but falls off slightly with increase of load, the more, the greater the loss of power in the reactances, that is, the lower the efficiency of the transforming device.

Two typical arrangements of such constant-current transforming devices are the *T-connection* or the *resonating-circuit,* diagram Fig. 66, and the *monocyclic square,* diagram Fig. 68. From these, a very large number of different combinations of inductive and condensive reactances, with addition of compensators, and of impressed e.m.fs., can be devised to transform from constant potential to constant current and inversely, and by the use of quadrature e.m.fs. taken from a second phase of the polyphase system, the secondary output, for the same amount of reactances, increased.

These combinations afford very convenient and instructive examples for accustoming oneself to the use of complex quantities in the solution of alternating-current problems.

Only two typical cases, the T-connection and the monocyclic square will be more fully discussed in the next chapter.

CHAPTER X.

CONSTANT POTENTIAL — CONSTANT-CURRENT TRANSFORMATION.

A: T-Connection or Resonating Circuit.

69. *General.* A combination, in a constant-potential circuit, of an inductive and a condensive reactance in series with each other in resonance condition, that is, with the condensive reactance equal to the inductive reactance, gives constant current in a circuit shunting the capacity. This circuit thus can be called the "secondary circuit" of the constant potential — constant-current transforming device, while the constant-potential supply circuit may be called the "primary circuit."

If the total inductive reactance in the constant-current circuit is equal to the condensive reactance, the primary supply current is in time-phase with the impressed e.m.f.

Let, as shown diagrammatically in Fig. 66,

x_0 = value of the inductive and the condensive reactances which are in series with each other.

x_1 = the additional inductive reactance inserted in the constant-current circuit.

$\dot{Z} = r - jx$, or $z = \sqrt{r^2 + x^2}$ = the absolute value of the impedance of the constant-current load.

Assuming now in the constant-current circuit the inductive reactance and the resistance as proportional to each other, as for instance is approximately the case in a series arc circuit, in which, by varying the number of lamps and therewith the load, reactance and resistance change proportionally. Let, then,

$k = \frac{x}{r}$ = ratio of inductive reactance to resistance of the load,

or tangent of the angle of lag of the constant-current circuit.

It is then

$$Z = r\,(1 - jk)$$

and
$$z = r\sqrt{1 + k^2}\,; \tag{1}$$

let, then,

$\dot{E}_0 = e_0$ = constant = primary impressed e.m.f., or supply voltage,
$\dot{E}_1$ = potential difference at condenser terminals,
$\dot{E}$ = secondary e.m.f., or voltage at constant-current circuit,
$\dot{I}_0$ = primary supply current,
$\dot{I}_1$ = condenser current,
$\dot{I}$ = secondary current,

then, in the secondary or receiver circuit,

$$\dot{E} = Z\dot{I}, \tag{2}$$

at the condenser terminals

$$\begin{aligned}\dot{E}_1 &= \dot{E} - jx_1\dot{I}\\ &= (Z - jx_1)\dot{I}\end{aligned} \tag{3}$$

and, also,

$$\dot{E}_1 = +\,jx_0\dot{I}_1; \tag{4}$$

hence,

$$\dot{I}_1 = \frac{Z - jx_1}{jx_0}\,\dot{I} \tag{5}$$

and the primary current is

$$\dot{I}_0 = \dot{I} + \dot{I}_1 = \left\{\frac{Z - jx_1}{jx_0} + 1\right\}\dot{I};$$

hence, expanded,

$$\dot{I}_0 = \frac{Z + j\,(x_0 - x_1)}{jx_0}\,\dot{I} \tag{6}$$

and the primary supply voltage is

$$e_0 = \dot{E}_1 - jx_0\dot{I}_0;$$

hence, substituting (3) and (6),

$$e_0 = [(Z - jx_1) - \{Z + j\,(x_0 - x_1)\}]\,\dot{I},$$

or, expanded,

$$e_0 = -jx_0 \dot{I}, \tag{7}$$

or, the secondary current is

$$\dot{I} = \frac{je_0}{x_0} \tag{8}$$

and, substituting (8) in (6) and (5): The primary current is

$$\dot{I} = \frac{Z + j(x_0 - x_1)}{x_0^2} e_0, \tag{9}$$

condenser current is

$$\dot{I}_1 = \frac{Z - jx_1}{x_0^2} e_0, \tag{10}$$

or, the absolute value is

$$\dot{i} = \frac{e_0}{x_0}, \tag{11}$$

$$\dot{i}_0 = \frac{\sqrt{r^2 + (x_0 - x_1 - x)^2}}{x_0^2} e_0, \tag{12}$$

$$\dot{i}_1 = \frac{\sqrt{r^2 + (x + x_1)^2}}{x_0^2} e_0, \tag{13}$$

$$\tan\theta = \frac{x}{r} = k \text{ gives the secondary phase angle,} \tag{14}$$

and

$$\tan\theta_0 = -\frac{x_0 - x_1 - x}{r} \text{ gives the primary phase angle.} \tag{15}$$

This phase angle $\theta_0 = 0$, that is, the primary supply current is non-inductive, if

$$x_0 - x_1 - x = 0,$$

that is,

$$x_1 = x_0 - x. \tag{16}$$

The primary supply can in this way be made non-inductive for any desired value of secondary load, by choosing the reactance, x_1, according to equation (16).

If $x = 0$, that is, a non-inductive secondary circuit (series incandescent lamps for instance), $x_1 = x_0$, that is, with a non-inductive secondary circuit, the primary supply current is always non-inductive, if the secondary reactance, x_1, is made equal to the primary reactance, x_0.

In this case $x_1 = x_0$, with an inductive secondary circuit $\tan \theta_0 = \frac{x}{r} = \tan \theta$: that is, the primary supply current has the same phase angle as the secondary load, if all three reactances (two inductive and one condensive reactance) are made equal.

In general, x_1 would probably be chosen so as to make $\dot{I}_0$ non-inductive at full load, or at some average load.

70. Example: A 100 lamp arc circuit of 7.5 amperes is to be operated from a 6600 volt constant-potential supply $e_0 = 6600$ volts, and $i = 7.5$ amperes.

Assuming 75 volts per lamp, including line resistance, gives a maximum secondary voltage, for 100 lamps, of $e' = 7500$ volts.

Assuming the power-factor of the arc circuit as 93 per cent lagging, gives

$$\cos \theta = 0.93, \quad \text{or} \quad \tan \theta = 0.4;$$

hence,

$$k = \frac{x}{r} = 0.4, \quad \text{and} \quad Z = r\,(1 - 0.4\,j),$$

or

$$z = 1.077\,r \quad \text{at full load},$$

of

$$e' = 7500 \text{ volts},$$

$$z' = \frac{e'}{i} = 1000 \text{ ohms},$$

hence

$$r' = 0.93\,z' = 930 \text{ ohms},$$
$$x' = 0.4\,r' = 372 \text{ ohms},$$

and

$$i = \frac{e_0}{x_0}, \quad \text{or} \quad x_0 = \frac{e_0}{i} = \frac{6600}{7.5} = 880 \text{ ohms}.$$

To make the primary current i_0 non-inductive at full load, or for $x' = 372$ ohms, this requires

$$x_1 = x_0 - x' = 508 \text{ ohms}.$$

This gives the equations

$$i = 7.5 \text{ amperes},$$
$$e = 7.5\, z = 8.08\, r \text{ volts}.$$
$$i_0 = \sqrt{r^2 + (372 - 0.4\, r)^2} \times \frac{6600}{880^2}$$
$$= 7.5 \sqrt{\left(\frac{r}{880}\right)^2 + \left(0.423 - \frac{r}{2200}\right)^2}$$
$$\tan \theta_0 = -\frac{372 - 0.4\, r}{r}$$
$$= 0.4 - \frac{372}{r},$$

hence, leading current below full load, non-inductive at full load and lagging current at overload.

71. *Apparatus Economy.* Denoting by z', r', x' the respective full-load values, the volt-ampere output at full load is

$$P_{a_0} = i^2 z' = \frac{e_0^2 z'}{x_0^2} = \frac{e_0^2 r' \sqrt{1 + k^2}}{x_0^2}, \tag{17}$$

volt-ampere input:

$$P_{a_1} = i_0 e_0 = \frac{e_0^2 r'}{x_0^2}. \tag{18}$$

That is, the volt-ampere input is less than the volt-ampere output, since the input is non-inductive, while the output is not.

The power output is

$$p = i^2 r' = \frac{e_0^2 r'}{x_0^2}, \tag{19}$$

which is equal to the volt-ampere input, since the losses of power in the reactances were neglected in the preceding equations.

The volt-amperes at the condenser are

$$P_a' = i_1^2 x_0;$$

hence, substituting (13),

$$P_a' = \frac{r'^2 + (x' + x_1)^2}{x_0^3} e_0^2 = \frac{r'^2 + (kr' + x_1)^2}{x_0^3}. \tag{20}$$

The volt-ampere consumption of the first, or primary inductive reactance, x_0, is

$$P_a'' = i_0^2 x_0;$$

hence, substituting (12),

$$P_a'' = \frac{r'^2 + (x_0 - x' - x_1)^2}{x_0^3} e_0^2 = \frac{r'^2 + (x_0 - kr' - x_1)^2}{x_0^3} e_0^2, \quad (21)$$

the volt-ampere consumption of the second, or secondary inductive reactance, x_1, is

$$P_a''' = i^2 x_1,$$

or

$$P_a''' = \frac{x_1}{x_0^2} e_0^2 . \quad (22)$$

The total volt-ampere rating of the reactances required for the transformation from constant potential to constant current then is

$$P_a = P_a' + P_a'' + P_a'''$$

$$= \frac{2 r'^2(1 + k^2) + 2 kr' (2 x_1 - x_0) + (x_0^2 - x_0 x_1 + 2 x_1^2)}{x_0^3} e_0^2, \quad (23)$$

and the apparatus economy, or the ratio of volt-amperes output to the volt-ampere rating of the apparatus is

$$f = \frac{P_{a_0}}{P_a} = \frac{r' x_0 \sqrt{1 + k^2}}{2 r'^2(1 + k^2) + 2 kr' (2 x_1 - x_0) + (x_0^2 - x_0 x_1 + 2 x_1^2)} \quad (24)$$

this apparatus economy depends upon the load, r', the power-factor or phase angle of the load, k, and the secondary additional inductive reactance, x_1.

To determine the effect of the secondary inductive reactance, x_1: The apparatus economy is a maximum for that value of secondary inductive reactance, x_1, for which: $\frac{df}{dx_1} = 0.$

Instead of directly differentiating f, it is preferable to simplify the function f first, by dropping all those factors, terms, etc.,

which inspection shows do not change the position of the maximum or the minimum value of the function. Thus the numerator can be dropped, the denominator made numerator, and its first term dropped, leaving

$$f' = 2\,kr'\,(2\,x_1 - x_0) + (x_0^2 - x_0 x_1 + 2\,x_1^2)$$

as the simplest function, which has an extreme value for the same value of x_1, as f. Then

$$\frac{df'}{dx_1} = 4\,kr' - x_0 + 4\,x_1 = 0,$$

and
$$x_1 = \frac{x_0 - 4\,kr'}{4}, \tag{25}$$

substituting (25) in (24), gives

$$f_1 = \frac{8\,r'\,x_0\sqrt{1 + k^2}}{16\,r'^2 - 8\,kr'x_0 + 7\,x_0^2}. \tag{26}$$

To determine the effect of the load r':
f_1 becomes a maximum for that load, r', which makes

$$\frac{df_1}{dr'} = 0,$$

or, simplified,
$$f_1' = \frac{16\,r'^2 - 8\,kr'x_0 + 7\,x_0^2}{r'},$$

hence
$$\frac{df_1'}{dr'} = r'(32\,r' - 8\,kx_0) - (16\,r'^2 - 8\,kr'x_0 + 7\,x_0^2) = 0,$$

hence
$$r' = \frac{x_0\sqrt{7}}{4}, \tag{27}$$

and, substituting (27) in (26),

$$f_2 = \frac{\sqrt{1 + k^2}}{\sqrt{7} - k}, \tag{28}$$

hence, for $k = 0$:

$$f_2 = \frac{1}{\sqrt{7}} = 0.378.$$

$$r' = \frac{x_0\sqrt{7}}{4} = 0.662\,x_0.$$

$$x_1 = \frac{x_0}{4} = 0.25\,x_0.$$

for $k = 0.4$:

$$f_2 = \frac{\sqrt{1.16}}{\sqrt{7}-0.4} = 0.478.$$

$$r' = \frac{x_0\sqrt{7}}{4} = 0.662\,x_0$$

$$z' = \sqrt{1.16}\,\frac{x_0\sqrt{7}}{4} = 0.712\,x_0$$

$$x_1 = \frac{x_0}{4}(1 - 0.4\sqrt{7}) = -0.016\,x_0$$

$$= \text{approximately zero.}$$

At non-inductive load

$$k = 0$$

and with non-inductive primary supply, that is,

$$x_1 = x_0,$$

by substituting these values in (24), the apparatus economy is

$$f = \frac{r'x_0}{2\,(r'^2 + x_0^2)}, \tag{29}$$

which is a maximum for $r' = x_0$, (30)

$$f_0 = \frac{1}{4} = 0.25, \tag{31}$$

which is rather low:

That is, non-inductive load and supply circuit do not give very high apparatus economy, but inductive reactance of the load,

and phase displacement in the supply circuit, gives far higher apparatus economy, that is, more output with the same volt-amperes in reactance.

By inserting in (23), with the quantities, P_a', P_a'', and P_a''', coefficients n_1, n_2, n_3, which are proportional respectively to the cost of the reactances per kilovolt-ampere, the expression

$$\frac{n_1 P_a' + n_2 P_a'' + n_3 P_a'''}{P}, \tag{32}$$

then represents the commercial economy, that is, the maximum of this expression, derived by analogous considerations as before, gives the arrangement for minimum cost at given output.

72. *Power Losses in Reactances.*

In the preceding equations, the losses of power in reactances have been neglected. However small these may be, in accurate investigations, they require consideration as to their effect on the regulation of the transforming device, and on the efficiency.

Let

a = power-factor of inductive reactance, that is, loss of power, as fraction of total volt-amperes.

b = power-factor of condensive reactance, that is, loss of power, as fraction of total volt-amperes.

Here a and b are very small quantities, in general b, the loss in the condensive reactance, being far smaller than the loss in the inductive reactance.

Approximately, the inductive reactances are $(a - j)\, x_0$ and $(a - j)\, x_1$ respectively, and the condensive reactance is $(b + j)x_0$.

Assuming the same denotations as in the preceding paragraphs, receiver circuit

$$\dot{E} = Z\, I, \tag{33}$$

at condenser terminals

$$\begin{aligned} \dot{E}_1 &= \dot{E} + (a - j)\, x_1\, \dot{I} \\ &= \{Z + (a - j)\, x_1\}\, \dot{I} \end{aligned} \tag{34}$$

and also

$$\dot{E}_1 = (b + j)\, x_0\, \dot{I}_1; \tag{35}$$

hence

$$\dot{I}_1 = \frac{Z + (a - j)\, x_1}{(b + j)\, x_0} \dot{I}, \tag{36}$$

and

$$\begin{aligned} \dot{I}_0 &= \dot{I} + \dot{I}_1 \\ &= \frac{Z + (b + j)\, x_0 + (a - j)\, x_1}{(b + j)\, x_0} \dot{I} \\ &= \frac{Z + j\,(x_0 - x_1) + (bx_0 + ax_1)}{(b + j)\, x_0} \dot{I} \end{aligned} \tag{37}$$

and the impressed e.m.f.

$$e_0 = \dot{E}_1 + (a - j)\, x_0 \dot{I}_0;$$

hence, substituting (35) and (37),

$$e_0 = \frac{x_0 + \{Z(a+b) + jx_0(a-b) - jx_1(a+b)\} + \{x_0 ab + x_1 a\,(a+b)\}}{b + j} \dot{I} \tag{38}$$

Since a and b are very small quantities, their products and squares can be neglected, then

$$e_0 = \frac{x_0 + \{\, Z\,(a + b) + jx_0\,(a - b) - jx_1\,(a + b)\}}{b + j} \dot{I} \tag{39}$$

or

$$\dot{I} = \frac{(j + b)\, e_0}{x_0 + \{Z\,(a + b) + jx_0\,(a - b) - jx_1\,(a + b)\}}; \tag{40}$$

this can be written

$$\dot{I} = \frac{je_0}{x_0} \frac{1 - jb}{1 + \left\{\frac{Z}{x_0}(a + b) + j\,(a - b) - j\frac{x_1}{x_0}(a + b)\right\}};$$

hence

$$\dot{I} = \frac{je_0}{x_0} \left\{1 - ja + j\,\frac{x_1}{x_0}\,(a + b) - \frac{Z}{x_0}\,(a + b)\right\} \tag{41}$$

that is, due to the loss of power in the reactances, the secondary current is less than it would be otherwise, and decreases with increasing load still further.

Equation (41) can also be written

$$\dot{I} = \frac{je_0}{x_0}\left\{\left[1 - \frac{r}{x_0}(a+b)\right] + j\left[\frac{x+x_1}{x_0}(a+b) - a\right]\right\} \tag{42}$$

here the imaginary component is very small in the parenthesis, that is, the secondary current remains practically in quadrature with the primary voltage.

The absolute value is, neglecting terms of secondary order,

$$i = \frac{e_0}{x_0}\left\{1 - \frac{r}{x_0}(a+b)\right\} \tag{43}$$

The primary current is, by equation (37) and (40),

$$\dot{I}_0 = \frac{Z + j(x_0 - x_1) + (bx_0 + ax_1)}{x_0 + Z(a+b) + jx_0(a-b) - jx_1(a+b)} \frac{e_0}{x_0} \tag{44}$$

$$= \frac{e_0}{x_0} \frac{\frac{Z}{x_0} + j\left(1 - \frac{x_1}{x_0}\right)\left(b + a\frac{x_1}{x_0}\right)}{1 + \frac{Z - jx_1}{x_0}(a+b) + j(a-b)}.$$

73. Example:

Considering the same example as before: a constant-potential circuit of $e_0 = 6600$ volts supplying a 100-lamp series arc-circuit, with $i' = 7.5$ amperes, and $e' = 7500$ volts at full load of 93 per cent power-factor, that is, $k = 0.4$, and $Z = (1 - 0.4\,j)r$. Assuming now, however, the loss in the inductive reactance as 3 per cent, and in the capacity as 1 per cent, that is, $a = 0.03$ $b = 0.01$, the full-load value of the secondary load impedance is: $z' = 1000$ ohms, $r' = 930$ ohms and $x' = 372$ ohms.

To give non-inductive primary supply at full load, the following equation must be fulfilled:

$$x_1 = x_0 - x' = x_0 - 372.$$

From equation (43), the secondary current, at full load, is

$$i' = \frac{e_0}{x_0}\left\{1 - \frac{r}{x_0}(a + b)\right\}$$

or

$$7.5 = \frac{6600}{x_0}\left\{1 - \frac{930 \times 0.04}{x_0}\right\};$$

hence

$$x_0 = 840 \text{ ohms, and } x_1 = 468 \text{ ohms.}$$

Substituting in (42), (43), (44),

$$\underset{\cdot}{I} = 7.86\, j\left\{\left(1 - 0.04\,\frac{r}{840}\right) - j\left(0.052 - 0.016\,\frac{r}{840}\right)\right\}$$

$$i = 7.86\left(1 - 0.04\,\frac{r}{840}\right)$$

$$e = iz = 1.077\, ri$$

$$= 8.46\, r\left(1 - 0.04\,\frac{r}{840}\right)$$

$$\underset{\cdot}{I}_0 = 7.86\,\frac{\left(\frac{r}{840} + 0.027\right) + j\left(0.443 - \frac{0.4\, r}{840}\right)}{\left(1 + \frac{0.04\, r}{840}\right) - j\left(\frac{0.016\, r}{840} + 0.002\right)}.$$

and herefrom the power-factor, efficiency, etc.

In Fig. 69, page 117, there are plotted, with the secondary, e.m.f., e, as abscissas, the values: secondary current, i; primary current, i_0; primary power-factor, $\cos\theta$, and efficiency.

74. In alternating-current circuits small variations of frequency are unavoidable, as for instance, caused by changes of load, etc., and the inductive reactance is directly proportional, the condensive reactance inversely proportional to the frequency. Wherever inductive and condensive reactances are used in series with each other and of equal or approximately equal reactance, so more or less neutralizing each other, even small changes of frequency may cause very large variations in the result, and in

such cases it is therefore necessary to investigate the effect of a change of frequency on the result: for instance, in a resonating circuit of very small power loss, a small change of frequency at constant impressed e.m.f. may change the current over an enormous range.

Since in the preceding, constant-current regulation is produced by inductive and condensive reactances in series with each other, the effect of a variation of frequency requires investigation.

Let, then, the frequency be increased by a small fraction, s.

The inductive reactance thereby changes to $x_0 (1 + s)$ and $x (1 + s)$, and $Z = r - j (1 + s) x$ respectively, and the condensive reactance to $\frac{x_0}{1 + s}$.

Leaving all the other denotations the same, and neglecting the loss of power in the reactances,

$$\begin{aligned} \dot{E} &= Z\dot{I} \\ \dot{E}_1 &= \{Z - j (1 + s) x_1\} \dot{I} \\ &= \frac{j x_0 \dot{I}_1}{1 + s}, \end{aligned}$$

hence,

$$\dot{I}_1 = \frac{(1 + s)\{Z - j (1 + s) x_1\}}{j x_0} \dot{I}$$

and

$$\dot{I}_0 = \dot{I} + \dot{I}_1 = \frac{(1 + s) Z + j \{x_0 - (1 + s)^2 x_1\}}{j x_0} \dot{I},$$

thus

$$\begin{aligned} e_0 &= \dot{E}_1 - j (1 + s) x_0 \dot{I}_0 \\ &= [Z - j (1 + s) x_1 - (1 + s)^2 Z - j (1 + s) \{x_0 - (1 + s)^2 x_1\}] \dot{I} \end{aligned}$$

hence, expanding and dropping terms of higher order,

$$e_0 = - j \dot{I} \{x_0 + s (x_0 - 2 x_1 - 4 Zj) - s^2 (3 x_1 - 2 j)\},$$

or

$$\dot{I} = \frac{j e_0}{x_0} \left\{1 - s \left(1 - 2 \frac{x_1}{x_0} - 4 \frac{4 jZ}{x_0}\right)\right\}. \tag{45}$$

Hence, the current is not greatly affected by a change of frequency. That is, the constant-current regulation of the above discussed device does not depend, or require, a constancy of frequency beyond that available in ordinary alternating-current circuits.

B. *Monocyclic Square.*

75. *General.*

A combination of four equal reactances, two condensive and two inductive, arranged in a square as shown diagrammatically in Fig. 68, transforms a constant voltage, impressed upon one diagonal, into a constant current across the other diagonal, and inversely.

Let, then,

$\dot{E}_0 = e_0 =$ constant $=$ primary impressed e.m.f., or supply voltage,

$\dot{E}$ $=$ secondary terminal voltage,

$\dot{E}_1 =$ voltage across the condensive reactance,

$\dot{E}_2 =$ voltage across the inductive reactance,

and

$\dot{I}_0 =$ primary supply current,

$\dot{I}$ $=$ secondary current,

$\dot{I}_1 =$ current in condensive reactance,

$\dot{I}_2 =$ current in inductive reactance,

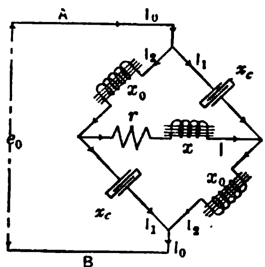


FIG. 68.

these currents and e.m.fs. being assumed in the direction as indicated by the arrows in Fig. 68.

Let

x_0 $=$ condensive and inductive reactances;

hence,

$$Z_1 \doteq + jx_0 = \text{condensive reactance,} \tag{1}$$
$$Z_2 = - jx_0 = \text{inductive reactance.} \tag{2}$$

Then, at the dividing point,

$$\dot{I}_0 = \dot{I}_1 + \dot{I}_2, \tag{3}$$

and

$$\dot{I} = \dot{I}_2 - \dot{I}_1; \tag{4}$$

hence,

$$\dot{I}_1 = \frac{\dot{I}_0 - \dot{I}}{2} \tag{5}$$

and

$$\dot{I}_2 = \frac{\dot{I}_0 + \dot{I}}{2}. \tag{6}$$

In the e.m.f. triangle,

$$e_0 = Z_1 I_1 + Z_2 I_2 \tag{7}$$

and

$$E = Z_1 I_1 - Z_2 I_2 \tag{8}$$

and

$$E = Z I; \tag{9}$$

substituting (1) and (2) in (7) and (8) gives

$$e_0 = j x_0 (I_1 - I_2) \tag{10}$$

and

$$Z I = j x_0 (I_1 + I_2), \tag{11}$$

and, substituting herein the current,

$$e_0 = - j x_0 I \tag{12}$$

and

$$Z I = + j x_0 I_0, \tag{13}$$

hence, the secondary current is

$$I = \frac{j e_0}{x_0}; \tag{14}$$

the primary current,

$$I_0 = \frac{e_0 Z}{x_0^2}; \tag{15}$$

the condenser current,

$$I_1 = \frac{Z - j x_0}{2 x_0^2}. \tag{16}$$

and the current in the inductive reactance,

$$I_2 = \frac{Z + j x_0}{2 x_0^2}. \tag{17}$$

The secondary terminal voltage is

$$E = j e_0 \frac{Z}{x_0}; \tag{18}$$

the condenser voltage,

$$E_1 = j x_0 I_1 = \frac{j (Z - j x_0)}{2 x_0} e_0; \tag{19}$$

and the inductive reactance voltage,

$$E_2 = - j x_0 I_2 = - \frac{j (Z + j x_0)}{2 x_0} e_0. \tag{20}$$

The tangent of the primary phase angle is

$$\tan \theta_0 = \frac{x}{r} = \tan \theta; \tag{21}$$

hence, the absolute value of the secondary current is

$$i = \frac{e_0}{x_0}; \tag{22}$$

of the primary current,

$$i_0 = \frac{e_0 z}{x_0^2}; \tag{23}$$

of the condenser current,

$$i_1 = \frac{\sqrt{r^2 + (x_0 + x)^2}}{2\, x_0^2} e_0, \tag{24}$$

and of the inductive reactance current

$$i_2 = \frac{\sqrt{r^2 + (x_0 - x)^2}}{2\, x_0^2} e_0. \tag{25}$$

The secondary terminal voltage is

$$e = \frac{z}{x_0} e_0; \tag{26}$$

the condenser voltage,

$$e_1 = \frac{\sqrt{r^2 + (x_0 + x)^2}}{2\, x_0} e_0, \tag{27}$$

and the inductive reactance voltage:

$$e_2 = \frac{\sqrt{r^2 + (x_0 - x)^2}}{2\, x_0} e_0. \tag{28}$$

76. From these equations follow the apparent powers, or volt-amperes of the different circuits as:

Output,

$$P_{a_0} = ei = \frac{e_0^2 z}{x_0^2}. \tag{29}$$

Input,

$$P_{a_i} = e_0 i_0 = \frac{e_0^2 z}{x_0^2}. \tag{30}$$

Hence the input is the same as the output. This is obvious, since the losses of power in the reactances are neglected, and it was found (21), that the phase angle or the power-factor of the primary circuit equals that of the secondary circuit.

Apparent power of the condensive reactance,

$$P_{a_1} = e_1 i_1 = \frac{r^2 + (x_0 + x)^2}{4\, x_0^{\,3}} e_0^{\,2}, \tag{31}$$

Inductance,

$$P_{a_2} = e_2 i_2 = \frac{r^2 + (x_0 - x)^2}{4\, x_0^{\,3}} e_0^{\,2}, \tag{32}$$

and, therefore, total volt-ampere capacity of the reactances is

$$P_a = 2\,(P_{a_1} + P_{a_2}) = \frac{r^2 + x^2 + x_0^{\,2}}{x_0^{\,3}} e_0^{\,2};$$

hence

$$P_a = \frac{z^2 + x_0^{\,2}}{x_0^{\,3}} e_0^{\,2}, \tag{33}$$

and,

apparatus economy,

$$f = \frac{P_{a_0}}{P_a} = \frac{z x_0}{z^2 + x_0^{\,2}}, \tag{34}$$

hence a maximum for $z = x_0$, (35)

and this maximum is equal to $f_0 = \frac{1}{2}$ or 50 per cent. (36)

That is, the maximum apparatus economy of the monocyclic square, as discussed here, is 50 per cent, or in other words, for every kilovolt-ampere output, two kilovolt-amperes in reactances have to be provided.

This apparatus economy is higher than that of the T-connection, in which under the same conditions, that is for $x_1 = x_0$, the apparatus economy was only 25 per cent.

The commercial, or cost economy would be given by

$$g = \frac{P_{a_0}}{2\,(n_1 P_{a_1} + n_2 P_{a_2})} = \text{maximum}, \tag{37}$$

where

n_1 = price per kilovolt-ampere of condensive reactance, n_2 = price per kilovolt-ampere of inductive reactance.

77. Example:

Considering the same problem as under A. From a constant impressed e.m.f. $e_0 = 6600$ volts, a 100-lamp arc circuit, of 93 per cent power-factor, is to be operated, requiring

$$\begin{aligned} i &= 7.5 \text{ amperes}, \\ Z &= r - jx \\ &= r\,(1 - jk) \end{aligned}$$

where

$$k = \frac{x}{r} = 0.4;$$

hence

$$Z = r\,(1 - 0.4\,j),$$

and at full load

$$e' = 7500 \text{ volts}.$$

Then, from (22),

$$x_0 = \frac{e_0}{i} = 880 \text{ ohms}, \qquad z' = \frac{e'}{i} = 1000 \text{ ohms};$$

hence $\quad r' = 930$ ohms, $\qquad x' = kr' = 372$ ohms,

and, therefore,

$$\begin{aligned} i &= 7.5 \text{ amperes}, \\ i_0 &= 7.5\,\frac{z}{880} \text{ amperes}, \\ e &= 7.5\,z, \end{aligned}$$

and at full load, or $r = 930$, when denoting full load values by prime,

$$\begin{aligned} i' &= 7.5 \text{ amperes}, \\ i_0' &= 7.93 \text{ amperes}, \\ i_1' &= 6.65 \text{ amperes}, \\ i_2' &= 4.52 \text{ amperes}, \\ e' &= 7500 \text{ volts}, \\ e_1' &= 5850 \text{ volts}, \\ e_2' &= 3980 \text{ volts}, \\ \left.\begin{aligned} P'_{a_i} &= \\ P'_{a_0} &= \end{aligned}\right\} & \ 56.25 \text{ kv-amp.} \\ P'_{a_1} &= 38.9 \text{ kv-amp.} \\ P'_{a_2} &= 18.0 \text{ kv-amp.} \\ P'_a &= 113.8 \text{ kv-amp.} \end{aligned}$$

$$f' = 0.4943$$

or 49.43 per cent, that is, practically the maximum.

78. *Power Loss in Reactances.*

In the preceding, as first approximation, the loss of power in the reactances has been neglected, and so the constancy of current, i, was perfect, and the output equal to the input. Considering, however, the loss of power in the reactances, it is found that the current, i, varies slightly, decreasing with increasing load, and the input exceeds the output.

Let, then,

$$Z_1 = (b + j)\, x_0 = \text{condensive reactance,}$$
$$Z_2 = (a - j)\, x_0 = \text{inductive reactance,}$$

otherwise retaining the same denotations as in the preceding paragraphs.

Then, substituting in (7) and (8),

$$\frac{e_0}{x_0} = (b + j)\, I_1 + (a - j)\, I_2, \tag{38}$$

$$\frac{Z\dot{I}}{x_0} = (b + j)\, I_1 - (a - j)\, I_2. \tag{39}$$

Assuming

$$\left.\begin{aligned} a &= c_1 + c_2 \\ b &= c_1 - c_2 \end{aligned}\right\}, \tag{40}$$

$$c_1 = \frac{a + b}{2}, \quad c_2 = \frac{a - b}{2}. \tag{41}$$

Substituting in (38) and (39)

$$\frac{e_0}{x_0} = j\,(I_1 - I_2) + c_1\,(I_1 + I_2) - c_2\,(I_1 - I_2)$$

$$\frac{Z\dot{I}}{x_0} = j\,(I_1 + I_2) + c_1\,(I_1 - I_2) - c_2\,(I_1 + I_2),$$

substituting herein from equations (3) and (4) gives

$$\frac{e_0}{x_0} = (c_2 - j)\, I + c_1\, I_0, \tag{42}$$

and

$$\frac{Z\dot{I}}{x_0} = -\, c_1\, I + (j - c_2)\, I_0, \tag{43}$$

and from these two equations with the two variables, $\underset{\cdot}{I}$ and $\underset{\cdot}{I}_0$, it follows from (43) that

$$\underset{\cdot}{I}_0 = \frac{Z - c_1 x_0}{(j - c_2)\, x_0} \underset{\cdot}{I}. \tag{44}$$

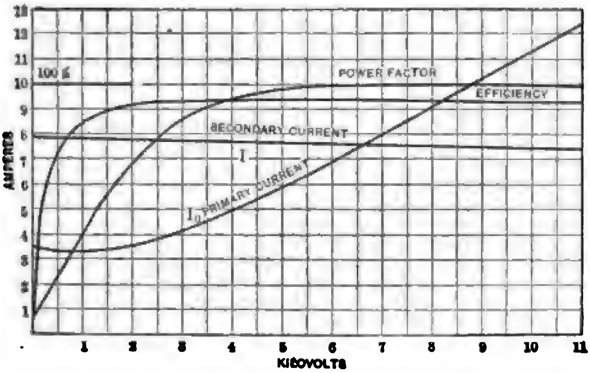


Fig. 69.

Substituting (44) in (42), transposing, and dropping terms of secondary order, that is, products and squares of c_1 and c_2, gives:

$$\underset{\cdot}{I} = \frac{j e_0}{x_0} \left\{ 1 - j c_2 - c_1 \frac{Z}{x_0} \right\}, \tag{45}$$

substituting (45) in (44), and transposing,

$$\underset{\cdot}{I}_0 = \frac{e_0}{x_0^2} \left\{ Z + c_1 \frac{x_0^2 - Z^2}{x_0} - 2 j c_2 Z \right\}, \tag{46}$$

then, substituting (45) and (46) in (5) and (6),

$$\underset{\cdot}{I}_1 = \frac{e_0}{x_0^2} \left\{ \frac{Z_1 - j x_0}{2} + \frac{c_1 - c_2}{2} x_0 - jZ \frac{2 c_2 - c_1}{2} - \frac{c_1 Z^2}{2 x_0} \right\} \tag{47}$$

$$\underset{\cdot}{I}_2 = \frac{e_0}{x_0^2} \left\{ \frac{Z_1 + j x_0}{2} + \frac{c_1 + c_2}{2} x_0 - jZ \frac{2 c_2 + c_1}{2} - \frac{c_1 Z^2}{2 x_0} \right\} \tag{48}$$

and the absolute value is

$$i = \frac{e_0}{x_0} \sqrt{\left(1 - c_1 \frac{r}{x_0}\right)^2 + \left(c_2 - c_1 \frac{x}{x_0}\right)^2}$$

or, approximately,

$$i = \frac{e_0}{x_0} \left(1 - c_1 \frac{x_0}{r}\right), \text{ etc.} \tag{49}$$

79. Example:

Considering the same example as before, of a 7.5-ampere 100-lamp arc circuit operated from a 6600-volt constant-potential supply, and assuming again as in § 73:

3 per cent power-factor of inductive reactance, or $a = 0.03$.
1 per cent power-factor of condensive reactance, or $b = 0.01$.

It is then:

$$c_1 = 0.02, \; c_2 = 0.01,$$

and at full load

$$i' = \frac{e_0}{x_0}\left(1 - c_1 \frac{r'}{x_0}\right)$$

or

$$7.5 = \frac{6600}{x_0}\left(1 - 0.02 \frac{930}{x_0}\right);$$

hence

$$x_0 = 861, \text{ and } i = 7.66 \left(1 - 0.02 \frac{r}{861}\right),$$

and we have, approximately,

$$i_0 = 7.66 \left\{ \frac{r}{861} + 0.02 - \frac{0.42\, jr}{861} \right\},$$

$$i_1 = 3.83 \left\{ \frac{r}{861} - j\left(1 + \frac{0.4\, r}{861}\right) \right\},$$

$$i_2 = 3.83 \left\{ \frac{r}{861} + j\left(1 - \frac{0.4\, r}{861}\right) \right\},$$

$$e = zi = 1.077\, ri.$$

In Fig. 70 are plotted, with the secondary terminal voltage, e, as abscissas, the values of secondary current, i; primary current, i_0; condenser current, i_1; inductive reactance current, i_2, and efficiency.

As seen, with the monocyclic square, the current regulation

is closer, and the efficiency higher than with the T-connection. This is due to the lesser amount of reactance required with the monocyclic square.

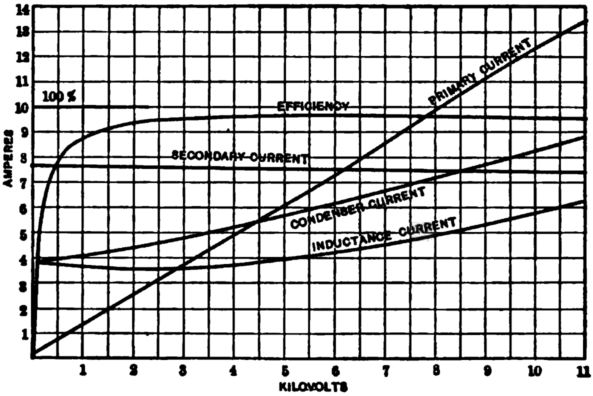


FIG. 70.

The investigation of the effect of a variation of frequency on the current regulation by the monocyclic square, now can be carried out in the analogous manner as in *A* with the T-connection.

C. General Discussion of Constant-Potential, Constant-Current Transformation.

80. In the preceding methods of transformation between constant potential and constant current by reactances, that is, by combinations of inductive and condensive reactances, the constant alternating-current is in quadrature with the constant e.m.f. Even in constant-current control by series inductive reactances, the constancy of current is most perfect for light loads, where the reactance voltage is large and thus the constant-current voltage almost in quadrature, and the constant-current control is impaired in direct proportion to the shift of phase of the constant current from quadrature relation.

The cause hereof is the storage of energy required to change the character of the flow of energy. That is, the energy supplied at constant potential in the primary circuit, is stored in the

reactances, and returned at constant current, in the secondary circuit.

The storage of the total transformed energy in the reactances allows a determination of the theoretical minimum of reactive power, that is, of inductive and condensive reactances required for constant-potential to constant-current transformation, since the energy supplied in the constant-current circuit must be stored for a quarter period after being received from the constant-potential circuit.

Let

$$p = P\,(1 + \cos 2\,\theta)$$

= Power supplied to the constant-current circuit; thus, neglecting losses,

$$p_0 = P\,(1 - \cos 2\,\theta)$$

= Power consumed from the constant-potential circuit, and

$$p_0 - p = 2\,P \cos 2\,\theta$$

= Power in the reactances.

That is, to produce the constant-current power, P, from a single-phase constant-potential circuit, the apparent power, $2\,P$, must be used in reactances; or, in other words, per kw. constant-current power produced from a single-phase constant-potential circuit, reactances rated at 2 kv-amp. as a minimum are required, arranged so as to be shifted 45 time-degrees against the constant-potential and the constant-current circuit.

The reactances used for the constant-potential, constant-current transformation may be divided between inductive and condensive reactances in any desired proportion.

The additional wattless component of constant-potential power is obviously the difference between the wattless volt-amperes of the inductive and that of the condensive reactances. That is, if the wattless volt-amperes of reactance is one-half of inductive and one-half of condensive, the resultant wattless volt-amperes of the main circuit is zero, and the constant-potential circuit is non-inductive, at non-inductive load, or consumes current proportional to the load.

If A is the condensive and B the inductive volt-amperes, the resultant wattless volt-amperes is $B-A$; that is, a lagging watt-

less volt-amperes of *B–A* (or a leading volt-ampere of *A–B*) exist in the main circuit, in addition to the wattless volt-amperes of the secondary circuit, which reappear in the primary circuit.

81. These theoretical considerations permit the criticism of the different methods of constant-potential to constant-current transformation in regard to what may be called their *apparatus economy*, that is, the kilovolt-ampere rating of the reactance used, compared with the theoretical minimum rating required.

1. *Series Inductive Reactance*, that is, a reactive coil of constant inductive reactance in series with the circuit. This arrangement obviously gives only imperfect constant-current control. Permitting a variation of five per cent in the value of the current (that is, full-load current is five per cent less than no-load current) and assuming four per cent loss in the reactive coil, a reactance rated at 3.45 kv-amp. is required per kw. constant-current load. This apparatus operates at 87.9 per cent economy and 30 per cent power-factor.

Assuming 10 per cent variation in the value of the current, reactance rated at 2.22 kv-amp. is required per kw. constant-current load. This arrangement operates at an economy of 91.8 per cent, and a power-factor of 89.5 per cent.

In the first case, the apparatus economy, that is, the ratio of the theoretical minimum kilovolt-ampere rating of the reactance to the actual rating of the reactance is 88 per cent, and in the last case 92 per cent, thus the objection to this method is not the high rating of the reactance and the economy, but the poor constant-current control, and especially the very low power-factor.

2. *Inductive and Condensive Reactances in Resonance Condition*, the condensive reactance being shunted by the constant-current circuit. In this case, condensive reactance rated at 1 kv-amp. and inductive reactance rated at 2 kv-amp. are required per kw. constant-current load, and the main circuit gives a constant wattless lagging apparent power of one kv-amp. Assuming again four per cent loss in the inductive and two per cent loss in the condensive reactances, gives a full-load efficiency of 91 per cent and a power-factor (lagging) of 74 per cent. The apparatus economy by this method is 66.7 per cent.

3. *Inductive and condensive reactances in resonance condition*, the *inductive reactance shunted* by the constant-current circuit. In this case, as a minimum, per kw. constant-current load, condensive reactance rated at 2 kv-amp. and inductive reactance rated at 1 kv-amp. is required, and the main circuit gives a constant wattless leading apparent power of 1 kv-amp. The efficiency of transformation is at full load 92.5 per cent, the power-factor (leading) 73 per cent, the apparatus economy 66.7 per cent.

4. *T-Connection*, that is, two equal inductive reactances in series to the constant-current circuit and shunted midway by an equal condensive reactance. In this case per kw. constant-current load, condensive reactance rated at 2 kv-amp. and inductive reactance rated at 2 kv-amp. are required.

The main circuit is non-inductive at all non-inductive loads, that is, the power-factor is 100 per cent.

The full-load efficiency is 89.3 per cent, apparatus economy 50 per cent.

5. *The Monocyclic Square.* In this case a condensive reactance rated at 1 kv-amp. and inductive reactance rated at 1 kv-amp. are required per kw. constant-current load. The main circuit is non-inductive at all non-inductive loads, that is, the power-factor is 100 per cent. The full-load efficiency is 94.3 per cent, the apparatus economy 100 per cent.

6. The *Monocyclic Square* in combination with a *constant-potential polyphase system* of impressed e.m.f. In this case, per kw. constant-current load, condensive reactance rated at 0.5 kv-amp. and inductive reactance rated at 0.5 kv-amp. are required. The main circuits are non-inductive at all loads, that is, the power-factor is 100 per cent. The full-load efficiency is over 97 per cent, the apparatus economy 200 per cent.

82. In the preceding, the constant-potential to constant-current transformation with a single-phase system of constant impressed e.m.f., has been discussed; and shown that as a minimum in this case, to produce 1 kw. constant-current output, reactances rated at 2 kv-amp. are required for energy storage. The constant current is in quadrature with the main or impressed e.m.f., but can be either leading or lagging. Thus the total range available is from 1 kw. leading, to zero, to 1 kw.

lagging. Hence if a constant-quadrature e.m.f. is available by the use of a polyphase system, the range of constant current

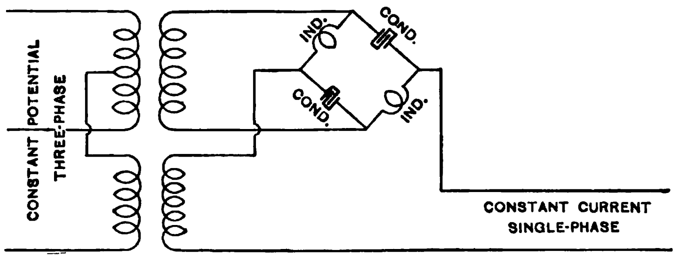


FIG. 71.

can be doubled, that is, reactance rated at 2 kv-amp. can be made to control the potential for 2 kw. constant-current output in the way shown in Fig. 71 for a three-phase, and Fig. 72 for a quarter-phase system of impressed e.m.f.

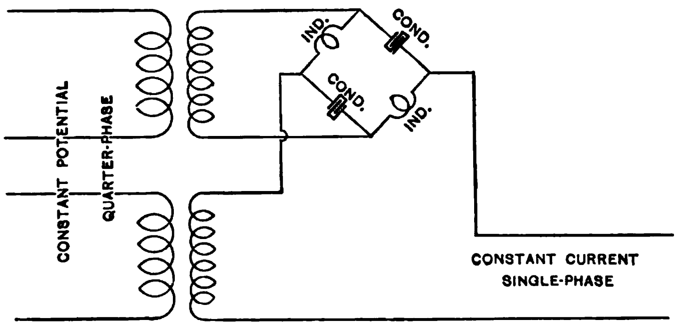


FIG. 72.

In this case, one transformer feeds a monocyclic square, the other transformer inserts an equal constant e.m.f. in quadrature with the former, which from no-load to half-load is subtractive, from half-load to full load is additive, that is, at full load, both phases are equally loaded; at half-load only one phase is loaded and at no-load one phase transforms energy into the other phase.

The monocyclic e.m.f. square in this case, when passing from full load to no-load, gradually collapses to a straight line at half-load, then overturns and opens again to a square in the opposite direction at no-load. That is, at full load the trans-

formation is from constant potential to constant current, and at no-load the transformation is from constant current to constant-potential.

Obviously with this arrangement the efficiency is greatly increased by the reduction of the losses to one-half, and the constant-current control improved.

At the same time, the sensitiveness of the arrangement for distortion of the wave-shape, as will be discussed in Chapter XXIX, is greatly reduced, due to the insertion of a constant-potential e.m.f. into the constant-current circuit.

Obviously the arrangements in Figs. 71 and 72 are not the only ones, but many arrangements of inserting a constant-quadrature e.m.f. into the monocyclic square or triangle are suitable.

Different arrangements can also be used of the constant-current control, for instance, the inductive and condensive reactances in resonance condition with their common connection connected to the center of a compensator or transformer, with the insertion of the constant-potential quadrature e.m.f. in the latter circuit as shown in Fig. 73, or the T-connection, shown applied to a quarter-phase system in Fig. 74.

Constant-potential apparatus and constant-current single-phase circuits can also be operated from the same transformer

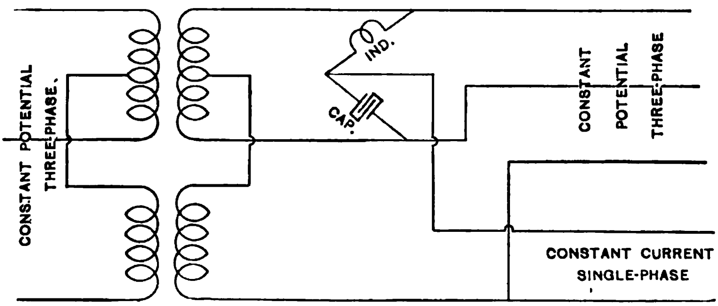


FIG. 73.

secondaries in a similar manner, as indicated in Fig. 73, for a three-phase secondary system.

In Figs. 71 to 74, the arrangement has been shown as applied to step-down transformers, but in the estimate of the efficiency the losses in these transformers have not been included, since these transformers are obviously not essential but merely for the

convenience of separating electrically the constant-current circuit from the high-potential line. It is evident, for instance, in Fig. 73, that the constant-current and constant-potential circuits

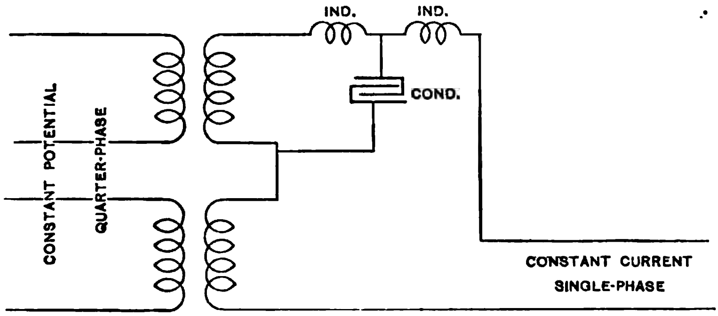


FIG. 74.

instead of being operated from the three-phase secondaries of the step-down transformers can be operated directly from the three-phase primaries by replacing the central connection of the one transformer by the central connection of the compensator.

D. Problems.

83. In the following problems referring to constant-potential to constant-current transformation by reactances, it is recommended:

a. To derive the equation of all the currents and e.m.fs., in complex quantities as well as in absolute terms, while neglecting the loss of power in the reactances.

b. To determine the volt-amperes in the different parts of the circuit, as load, reactances, etc., and therefrom derive the apparatus economy, to find its maximum value, and on which condition it depends.

c. To determine the effect of inductive load on the power of the primary supply circuit, to investigate the phase-angle of the primary supply circuit, and the conditions under which it becomes a minimum, or the primary supply becomes non-inductive.

d. To redetermine the equations of the problem, while considering the power lost in the reactances, and apply these equations to a numerical example, plotting all the interesting values.

e. To investigate the effect of a change of frequency on the equations, more particularly on the constant-current regulation.

f. To investigate the effect of a distortion of wave-shape, that is, the existence of higher harmonics in the impressed e.m.f., and their suppression or reappearance in the secondary circuit.

g. To study the reversibility of the problem, that is, apply (*a.*) to (*f.*) to the reversed problem of transformation from constant current to constant potential.

Some of the transforming devices between constant potential and constant current are:

A. Single-phase.

a. The resonating circuit, or condensive and inductive reactances, of equal values, in series with each other in the constant-potential circuit, and the one reactance shunted by the constant-current circuit.

b. T-connection, as partially discussed in (A.).

c. The monocyclic square, as partially discussed in (B.).

d. The monocyclic triangle: a condensive reactance and an inductive reactance of equal values, in series with each other across the constant-potential circuit, the constant-current circuit connecting between the reactance neutral, or the common connection between the two (opposite) reactances, and the neutral of a compensator or auto-transformer connected across the constant-potential circuit. Instead of the compensator neutral, the constant-current circuit can be carried back to the neutral of the transformer connected to the constant-potential circuit.

B. Polyphase.

a. In the two-phase system the two phases of e.m.fs., e_0 and je_0, are connected in series with each other, giving the outside terminals, *A* and *B*, and the neutral or common connection, *C*. A condensive reactance and an inductive reactance of equal values, in series with each other and with their neutral or common connection, *D*, are connected either between *A* and *B*, and the constant-current circuit between *C* and *D*, or the reactances are connected between *A* and *C*, and the constant-current circuit between *B* and *D*. In either case, several arrangements are possible, of which only a few have a good apparatus economy.

b. In a three-phase system, a condensive reactance, an inductive reactance equal in value to that of the condensive reactance and the constant-current circuit, are connected in star connection between the three-phase, constant-potential terminals. Here also two arrangements are possible, of which one only gives good apparatus economy.

c. In a constant-potential three-phase system, each of the three terminals, A, B, C, connects with a condensive and an inductive reactance, and all these reactances are of equal value, and joined together in pairs to three terminals, a, b, c, so that each of these terminals, a, b, c, connects an inductive with a condensive reactance. a, b, c, then, are constant-current three-phase terminals, that is, the three currents at a, b, c are constant and independent of the load or the distribution of load, and displaced from each by one-third of a period. This arrangement is especially suitable for rectification of the constant alternating-current, to produce constant direct-current.

84. Some further problems are:

1. In a single-phase, constant-current transforming device, as the monocyclic square, the constant current, i, is in quadrature with the constant impressed e.m.f., e_0. By inserting a constant potential e.m.f., $\dot{E}_3$, into the constant-current circuit, the apparatus economy can be greatly increased, in the maximum can be doubled; that is, the e.m.f., $\dot{E}_3$, gives constant power output, and from no-load to half-load, the transformation is from constant current to constant potential, that is, a part of the power supply, $\dot{E}_3$, is transformed into the circuit, of e.m.f., e_0, that is, the circuit, e_0, receives power. Above half-load the circuit of e_0 transforms power from constant potential to constant current, into the circuit of e.m.f., $\dot{E}_3$.

Since i is in time quadrature with e_0, with non-inductive secondary load, that is, the secondary terminal voltage, $\dot{E}$, in time-phase with the secondary current, i, $\dot{E}_3$ should also be in phase with i, that is, $\dot{E}_3 = je_3$. With inductive secondary load, of phase angle, θ, $\dot{E}_3$ should be in phase with $\dot{E}$, that is, leading i by angle θ, or should be: $\dot{E}_3 = je_3 (1 - kj)$.

It is interesting, therefore, to investigate how the equation of the constant-potential, constant-current devices are changed by the introduction of such an e.m.f., $\dot{E}_3$, at non-inductive as

well as at inductive load, if $\dot{E}_s = je_s$, or $\dot{E}_s = j(e_s - je_s')$, in either case, and also to determine how such an e.m.f., $\dot{E}_s$, of the proper phase relation, can be derived directly or by transformation from a two-phase or three-phase system.

2. If in the constant-potential, constant-current transforming device one of the reactances is gradually changed, increased or decreased from its proper value, then in either case the regulation of the system is impaired. That is, the ratio of full-load current to no-load current falls off, but at the same time, the no-load current also changes.

With increase of load, the frequency of the system decreases, due to the decreasing speed of the prime mover, if the output of the system is an appreciable part of the rated output. If, therefore, the reactances are adjusted for equality of the frequency of full load, at the higher frequency of no-load, the inductive reactance is increased, and thereby the no-load current decreased below the value which it would have at constant reactance, and in this manner the increase of current from full load to no-load is reduced.

Such a drop of speed and therefore of frequency, s, can therefore be found, that the current at full load, with perfect equality between the reactances, equals the current at no-load, where the reactances are not quite equal. That is, the variation of frequency compensates for the incomplete regulation of the current, caused by the energy loss in the reactances. Furthermore, with a given variation of frequency, s, from no-load to full load, the reactances can be chosen so as to be slightly unequal at full load, and more unequal at no-load; the change of current caused hereby compensates for the incomplete current regulation, that is, with a given frequency variation, s (within certain limits), the current regulation can be made perfect from no-load to full load, by the proper degree of unequality of the reactances.

It is interesting to investigate this, and apply to an example, *a.*, to determine the proper s, for perfect equality of reactance at full load; *b.*, with a given value of $s = 0.04$, to determine the inequality of reactance required. Assuming $a = 0.03$; $b = 0.01$.

3. If one point of the constant-current circuit, either a terminal or an intermediate point, connects to a point of the constant-potential circuit, either a terminal or some inter-

mediate point (as inside of a transformer winding), the constant current is not changed hereby, that is, the regulation of the system is not impaired, and no current exists in the cross between the two circuits. The distribution of potential between the reactances, however, may be considerably changed, some reactances receiving a higher, others a lower voltage.

It follows herefrom, that a ground on a constant-current system does not act as a ground on the constant-potential system, but electrically the two systems, although connected with each other, are essentially independent, just as if separated from each other by a transformer. So, for instance, in the monocyclic square, one side may be short-circuited without change of current in the secondary, but with an increase of current in the other three sides. It is interesting to investigate how far this independence of the circuit extends.

In general, as an example, the following constants may be chosen: In the constant-potential circuit: $e_0 = 6600$ volts and $i_0' = 10$ amperes at full load.

In the constant-current circuit: $i = 7.5$ amperes, $e' = 7500$ volts at full load.

Or, especially in polyphase systems, e', respectively, i_0' corresponding to the maximum economy point,

and $$a = 0.03;\ b = 0.01.$$

CHAPTER XI.

RESISTANCE AND REACTANCE OF TRANSMISSION LINES.

85. In alternating-current circuits, e.m.f. is consumed in the feeders of distributing networks, and in the lines of long-distance transmissions, not only by the resistance, but also by the reactance, of the line. The e.m.f. consumed by the resistance is in phase, while the e.m.f. consumed by the reactance is in quadrature, with the current. Hence their influence upon the e.m.f. at the receiver circuit depends upon the difference of phase between the current and the e.m.f. in that circuit. As discussed before, the drop of potential due to the resistance is a maximum when the receiver current is in phase, a minimum when it is in quadrature, with the e.m.f. The change of potential due to line reactance is small if the current is in phase with the e.m.f., while a drop of potential is produced with a lagging, and a rise of potential with a leading, current in the receiver circuit.

Thus the change of potential due to a line of given resistance and reactance depends upon the phase difference in the receiver circuit, and can be varied and controlled by varying this phase difference; that is, by varying the admittance, $Y = g + jb$, of the receiver circuit.

The conductance, g, of the receiver circuit depends upon the consumption of power, — that is, upon the load on the circuit, — and thus cannot be varied for the purpose of regulation. Its susceptance, b, however, can be changed by shunting the circuit with a reactance, and will be increased by a shunted inductive reactance, and decreased by a shunted condensive reactance. Hence, for the purpose of investigation, the receiver circuit can be assumed to consist of two branches, a conductance, g, — the non-inductive part of the circuit, — shunted by a susceptance, b, which can be varied without expenditure of energy. The two components of current can thus be considered separately, the energy component as deter-

mined by the load on the circuit, and the wattless component, which can be varied for the purpose of regulation.

Obviously, in the same way, the e.m.f. at the receiver circuit may be considered as consisting of two components, the power component, in phase with the current, and the wattless component, in quadrature with the current. This will correspond to the case of a reactance connected in series to the non-inductive part of the circuit. Since the effect of either resolution into components is the same so far as the line is concerned, we need not make any assumption as to whether the wattless part of the receiver circuit is in shunt, or in series, to the power part.

Let

$Z_0 = r_0 - jx_0$ = impedance of the line;

$$z_0 = \sqrt{r_0^2 + x_0^2};$$

$Y = g + jb$ = admittance of receiver circuit;

$$y = \sqrt{g^2 + b^2};$$

$\dot{E}_0 = e_0 + je_0'$ = impressed e.m.f. at generator end of line;

$$E_0 = \sqrt{e_0^2 + e_0'^2};$$

$\dot{E} = e + je'$ = e.m.f. at receiver end of line;

$$E = \sqrt{e^2 + e'^2};$$

$\dot{I}_0 = i_0 + ji_0'$ = current in the line;

$$I_0 = \sqrt{i_0^2 + i_0'^2}.$$

The simplest condition is the non-inductive circuit.

(1.) *Non-inductive Receiver Circuit Supplied over an Inductive Line.*

86. In this case, the admittance of the receiver circuit is $Y = g$, since $b = 0$.

We have then

current, $\dot{I}_0 = \dot{E}g;$

impressed e.m.f. $\dot{E}_0 = \dot{E} + Z_0\dot{I}_0 = \dot{E}(1 + Z_0 g).$

Hence — e.m.f. at receiver circuit,

$$\dot{E} = \frac{\dot{E}_0}{1 + Z_0 g} = \frac{\dot{E}_0}{1 + gr_0 - jgx_0};$$

current,
$$\dot{I}_0 = \frac{\dot{E}_0 g}{1 + Z_0 g} = \frac{\dot{E}_0 g}{1 + gr_0 - jgx_0}.$$

Hence, in absolute values — e.m.f. at receiver circuit,

$$E = \frac{E_0}{\sqrt{(1 + gr_0)^2 + g^2 x_0^2}};$$

current,
$$I_0 = \frac{E_0 g}{\sqrt{(1 + gr_0)^2 + g^2 x_0^2}}.$$

The ratio of e.m.fs. at receiver circuit and at generator, or supply circuit, is

$$a = \frac{E}{E_0} = \frac{1}{\sqrt{(1 + gr_0)^2 + g^2 x_0^2}};$$

and the power delivered in the non-inductive receiver circuit, or

output,
$$P = I_0 E = \frac{E_0^2 g}{(1 + gr_0)^2 + g^2 x_0^2}.$$

As a function of g, and with a given E_0, r_0, and x_0, this power is a maximum, if

$$\frac{dP}{dg} = 0;$$

that is,

$$-1 + g^2 r_0^2 + g^2 x_0^2 = 0;$$

hence,
conductance of receiver circuit for maximum output,

$$g_m = \frac{1}{\sqrt{r_0^2 + x_0^2}} = \frac{1}{z_0}.$$

Resistance of receiver circuit, $r_m = \dfrac{1}{g_m} = z_0$;

and, substituting this in P,

Maximum output, $$P_m = \frac{E_0^2}{2\,(r_0 + z_0)} = \frac{E_0^2}{2\,\{r_0 + \sqrt{r_0^2 + x_0^2}\}};$$

and —

ratio of e.m.f. at receiver and at generator end of line,

$$a_m = \frac{E}{E_0} = \frac{1}{\sqrt{2\left(1 + \dfrac{r_0}{z_0}\right)}};$$

efficiency, $$\frac{r_m}{r_m + r_0} = \frac{z_0}{r_0 + z_0}.$$

That is, the output which can be transmitted over an inductive line of resistance, r_0, and reactance, x_0, — that is, of impedance, z_0, — into a non-inductive receiver circuit, is a maximum, if the resistance of the receiver circuit equals the impedance of the line, $r = z_0$, and is

$$P_m = \frac{E_0^2}{2\,(r_0 + z_0)}.$$

The output is transmitted at the efficiency of

$$\frac{z_0}{r_0 + z_0},$$

and with a ratio of e.m.fs. of

$$a_m = \frac{1}{\sqrt{2\left(1 + \dfrac{r_0}{z_0}\right)}}.$$

87. We see from this, that the maximum output which can be delivered over an inductive line is less than the output delivered over a non-inductive line of the same resistance — that is, which can be delivered by continuous currents with the same generator potential.

In Fig. 75 are shown, for the constants,

$E_0 = 1000$ volts,

$Z_0 = 2.5 - 6j$; that is, $r_0 = 2.5$ ohms, $x_0 = 6$ ohms, $z_0 = 6.5$ ohms,

with the current I_0 as abscissas, the values —

e.m.f. at receiver circuit, E,	(Curve I.);
output of transmission, P,	(Curve II.);
efficiency of transmission,	(Curve III.).

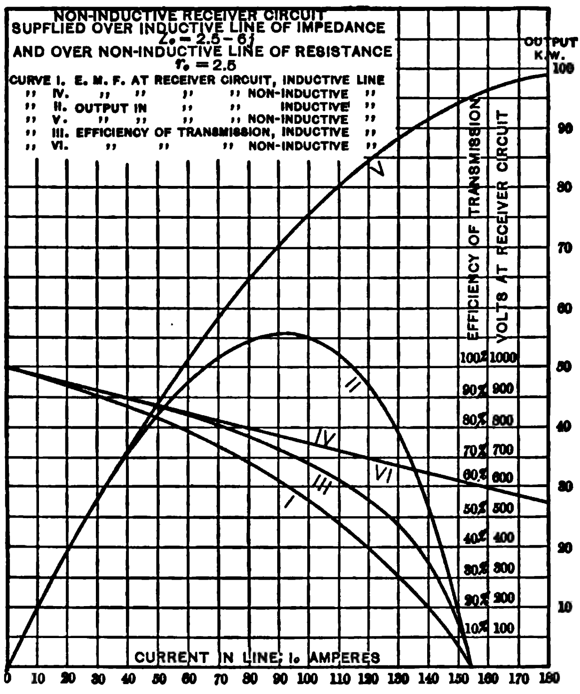


FIG. 75.—Non-inductive Receiver-Circuit supplied over Inductive Line.

The same quantities, E and P, for a non-inductive line of resistance, $r_0 = 2.5$ ohms, $x_0 = 0$, are shown in Curves IV., V., and VI

(2.) *Maximum Power Supplied over an Inductive Line.*

88. If the receiver circuit contains the susceptance, b, in addition to the conductance, g, its admittance can be written thus: —

$$Y = g + jb, \qquad y = \sqrt{g^2 + b^2}.$$

Then, current, $\dot{I}_0 = \dot{E}Y$;

Impressed e.m.f., $\dot{E}_0 = \dot{E} + \dot{I}_0 Z_0 = \dot{E}(1 + YZ_0)$.

Hence, e.m.f. at receiver terminals,

$$\dot{E} = \frac{\dot{E}_0}{1 + YZ_0} = \frac{\dot{E}_0}{(1 + r_0 g + x_0 b) - j(x_0 g - r_0 b)};$$

current,

$$\dot{I}_0 = \frac{\dot{E}_0 Y}{1 + YZ_0} = \frac{\dot{E}_0 (g + jb)}{(1 + r_0 g + x_0 b) - j(x_0 g - r_0 b)};$$

or, in absolute values, e.m.f. at receiver circuit,

$$E = \frac{E_0}{\sqrt{(1 + r_0 g + x_0 b)^2 + (x_0 g - r_0 b)^2}};$$

current,

$$I_0 = E_0 \sqrt{\frac{g^2 + b^2}{(1 + r_0 g + x_0 b)^2 + (x_0 g - r_0 b)^2}};$$

ratio of e.m.fs. at receiver circuit and at generator circuit,

$$a = \frac{E}{E_0} = \frac{1}{\sqrt{(1 + r_0 g + x_0 b)^2 + (x_0 g - r_0 b)^2}};$$

and the output in the receiver circuit is

$$P = E^2 g = E_0^2 a^2 g.$$

89. (*a.*) *Dependence of the output upon the susceptance of the receiver circuit.*

At a given conductance, g, of the receiver circuit, its output, $P = E_0^2 a^2 g$, is a maximum, if a^2 is a maximum; that is, when

$$f = \frac{1}{a^2} = (1 + r_0 g + x_0 b)^2 + (x_0 g - r_0 b)^2$$

is a minimum.

The condition necessary is

$$\frac{df}{db} = 0,$$

or, expanding,

$$x_0(1 + r_0 g + x_0 b) - r_0 (x_0 g - r_0 b) = 0.$$

Hence

Susceptance of receiver circuit,

$$b = - \frac{x_0}{r_0^2 + x_0^2} = - \frac{x_0}{z_0^2} = - b_0;$$

or $$b + b_0 = 0,$$

that is, if the sum of the susceptances of line and of receiver circuit equals zero.

Substituting this value, we get

ratio of e.m.fs. at maximum output,

$$a_1 = \frac{E}{E_0} = \frac{1}{z_0 (g + g_0)};$$

maximum output,

$$P_1 = \frac{E_0^2 g}{z_0^2 (g + g_0)^2};$$

current,

$$\dot{I}_0 = \frac{\dot{E}_0 Y}{1 + Z_0 Y} = \frac{\dot{E}_0(g - jb_0)}{1 + (r_0 - jx_0)(g - jb_0)}$$

$$= \frac{\dot{E}_0(g - jb_0)}{(1 + r_0 g - x_0 b_0) - j(r_0 b_0 + x_0 g)};$$

$$I_0 = E_0 \sqrt{\frac{g^2 + b_0^2}{(1 + r_0 g - x_0 b_0)^2 + (r_0 b_0 + x_0 g)^2}};$$

and, since,

$$b_0 = \frac{x_0}{r_0^2 + x_0^2}, \quad g_0 = \frac{r_0}{r_0^2 + x_0^2},$$

it is,

$$(1 + r_0 g - x_0 b_0)^2 + (r_0 b_0 + x_0 g)^2 = \left(r_0 g + 1 - \frac{x_0^2}{r_0^2 + b^2}\right)^2$$
$$+ \left(\frac{r_0 x_0}{r_0^2 + x_0^2} + x_0 g\right)^2$$
$$= \left(r_0 g + \frac{r_0^2}{r_0^2 + x_0^2}\right)^2 + \left(x_0 g + \frac{r_0 x_0}{r_0^2 + x_0^2}\right)^2$$
$$= r_0^2 (g + g_0)^2 + x_0^2 (g + g_0)^2$$
$$= z_0^2 (g + g_0)^2,$$

it is,

$$I_0 = \frac{E_0 \sqrt{g^2 + b_0^2}}{z_0 (g + g_0)};$$

phase difference in receiver circuit,

$$\tan \theta = \frac{b}{g} = - \frac{b_0}{g};$$

phase difference in generator circuit,

$$\tan \theta_0 = \frac{x + x_0}{r + r_0} = \frac{b_0 (y^2 - y_0^2)}{g_0 y^2 + g y_0^2}.$$

90. (*b.*) *Dependence of the output upon the conductance of the receiver circuit.*

At a given susceptance, b, of the receiver circuit, its output, $P = E_0^2 a^2 g$, is a maximum, if

$$\frac{dP}{dg} = 0, \quad \text{or} \quad \frac{d}{dg}\left(\frac{1}{P}\right) = 0,$$

$$\text{or,} \; \frac{d}{dg}\left(\frac{1}{a^2 g}\right) = \frac{d}{dg}\left(\frac{(1 + r_0 g + x_0 b)^2 + (x_0 g - r_0 b)^2}{g}\right) = 0;$$

that is, expanding,

$$(1 + r_0 g + x_0 b)^2 + (x_0 g - r_0 b)^2 - 2 g (r_0 + r_0^2 g + x_0^2 g) = 0;$$

or, expanding,

$$(b + b_0)^2 = g^2 - g_0^2; \quad g = \sqrt{g_0^2 + (b + b_0)^2}.$$

Substituting this value in the equation for α, § 88, we get — ratio of e.m.fs.,

$$\alpha_2 = \frac{1}{z_0\sqrt{2\{g_0^2 + (b + b_0)^2 + g_0\sqrt{g_0^2 + (b + b_0)^2}\}}}$$

$$= \frac{1}{z_0\sqrt{2g(g + g_0)}} = \frac{y_0}{\sqrt{2g(g + g_0)}};$$

power,

$$P_2 = \frac{E_0^2 y_0^2}{2(g + g_0)} = \frac{E_0^2 y_0^2}{2\{g_0 + \sqrt{g_0^2 + (b + b_0)^2}\}}$$

$$= \frac{E_0^2}{2\left\{r_0 + \sqrt{r_0^2 + \left(x_0 + x\frac{z_0^2}{z^2}\right)^2}\right\}}.$$

As a function of the susceptance, b, this power becomes a maximum for $\frac{dP_2}{db} = 0$, that is, according to § 89 if

$$b = -b_0.$$

Substituting this value, we get

$$b = -b_0,\ g = g_0,\ y = y_0, \text{ hence: } Y = g + jb = g_0 - jb_0;$$
$$x = -x_0,\ r = r_0,\ z = z_0, \qquad Z = r - jx = r_0 + jx_0;$$

substituting this value, we get —

ratio of e.m.fs., $\alpha_m = \frac{y_0}{2g_0} = \frac{z_0}{2r_0}$;

power, $P_m = \frac{E_0^2}{4r_0}$;

that is, the same as with a continuous-current circuit; or, in other words, the inductive reactance of the line and of the receiver circuit can be perfectly balanced in its effect upon the output.

91. As a summary, we thus have:

The output delivered over an inductive line of impedance, $Z_0 = r_0 - jx_0$, into a non-inductive receiver circuit, is a maximum for the resistance, $r = z_0$, or conductance, $g = y_0$, of the receiver circuit, and this maximum is

$$P = \frac{E_0^2}{2(r_0 + z_0)},$$

at the ratio of voltages,

$$a = \frac{1}{\sqrt{2\left(1 + \frac{r_0}{z_0}\right)}}.$$

With a receiver circuit of constant susceptance, b, the output, as a function of the conductance, g, is a maximum for the conductance,

$$g = \sqrt{g_0^2 + (b + b_0)^2},$$

and is

$$P = \frac{E_0^2 y_0^2}{2(g + g_0)},$$

at the ratio of voltages,

$$a = \frac{y_0}{\sqrt{2g(g + g_0)}}.$$

With a receiver circuit of constant conductance, g, the output, as a function of the susceptance, b, is a maximum for the susceptance $b = -b_0$, and is

$$P = \frac{E_0^2 g,}{z_0^2 (g + g_0)^2},$$

at the ratio of voltages,

$$a = \frac{1}{z_0 (g + g_0)}.$$

The maximum output which can be delivered over an inductive line, as a function of the admittance or impedance of the

receiver circuit, takes place when $Z = r_0 + jx_0$, or $Y = g_0 - jb_0$; that is, when the resistance or conductance of receiver circuit and line are equal, the reactance or susceptance of the receiver circuit and line are equal but of opposite sign, and is, $P = \frac{E_0^2}{4r_0}$, or independent of the reactances, but equal to the output of a continuous-current circuit of equal line resistance. The ratio of voltages is, in this case, $a = \frac{z_0}{2r_0}$, while in a continuous-current circuit it is equal to 0.5. The efficiency is equal to 50 per cent.

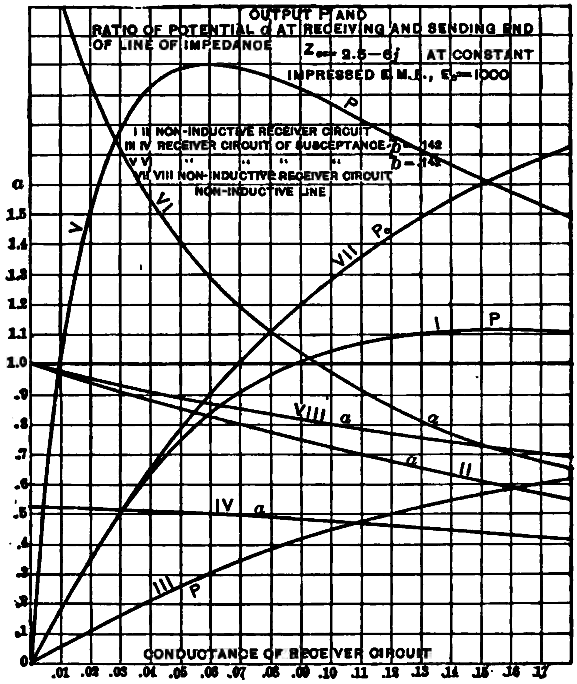


FIG. 76.—Variation of the Potential in Line at Different Loads.

92. As an example in Fig. 76 are shown, for the constants

$E_0 = 1000$ volts, and $Z_0 = 2.5 - 6j$; that is, for

$r_0 = 2.5$ ohms, $x_0 = 6$ ohms, $z_0 = 6.5$ ohms,

and with the variable conductances as abscissas, the values of the

output, in Curve I., Curve III., and Curve V.;
ratio of voltages, in Curve II., Curve IV., and Curve VI.;
Curves I. and II. refer to a non-inductive receiver circuit;
Curves III. and IV. refer to a receiver circuit of constant susceptance . . . . $b = 0.142$
Curves V. and VI. refer to a receiver circuit of constant susceptance . . . . $b = -0.142$
Curves VII. and VIII. refer to a non-inductive receiver circuit and non-inductive line.

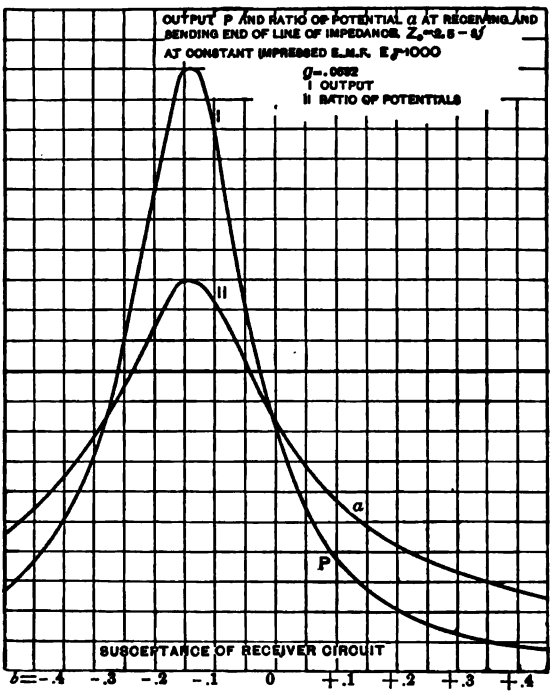


Fig. 77.— Variation of the Potential in Line at Various Loads.

In Fig. 77, the output is shown as Curve I., and the ratio of voltages as Curve II., for the same line constants, for a constant conductance, $g = 0.0592$ ohms, and for variable susceptances, b, of the receiver circuit.

(3.) *Maximum Efficiency.*

93. The output, for a given conductance, g, of a receiver circuit, is a maximum if $b = -b_0$. This, however, is generally not the condition of maximum efficiency.

The loss of power in the line is constant if the current is constant; the output of the generator for a given current and given generator e.m.f. is a maximum if the current is in time-phase with the e.m.f. at the generator terminals. Hence the condition of maximum output at given loss, or of maximum efficiency, is

$$\tan \theta_0 = 0.$$

The current is

$$\dot{I}_0 = \frac{\dot{E}_0}{Z + Z_0} = \frac{\dot{E}_0}{(r + r_0) - j(x + x_0)};$$

The current $\dot{I}_0$, is in phase with the e.m.f., E_0, if its quadrature component — that is, the imaginary term — disappears, or

$$x + x_0 = 0.$$

This, therefore, is the condition of maximum efficiency,

$$x = -x_0.$$

Hence, the condition of maximum efficiency is, that the reactance of the receiver circuit shall be equal, but of opposite sign, to the reactance of the line.

Substituting $x = -x_0$, we have,
ratio of e.m.fs.,

$$a = \frac{E}{E_0} = \frac{z}{(r + r_0)} = \frac{\sqrt{r^2 + x_0^2}}{(r + r_0)};$$

power, $$P = E_0^2 g a^2 = \frac{E_0^2 r}{(r + r_0)^2},$$

and depending upon the resistance only, and not upon the reactance.

This power is a maximum if $g = g_0$, as shown before; hence, substituting $g = g_0$, $r = r_0$,

maximum power at maximum efficiency, $P_m = \dfrac{E_0^2}{4 r_0}$,

at a ratio of potentials, $\alpha_m = \dfrac{z_0}{2 r_0}$,

or the same result as in § 90.

In Fig. 78 are shown, for the constants,

$E_0 = 1000$ volts,

$Z_0 = 2.5 - 6j$; $r_0 = 2.5$ ohms, $x_0 = 6$ ohms, $z_0 = 6.5$ ohms,

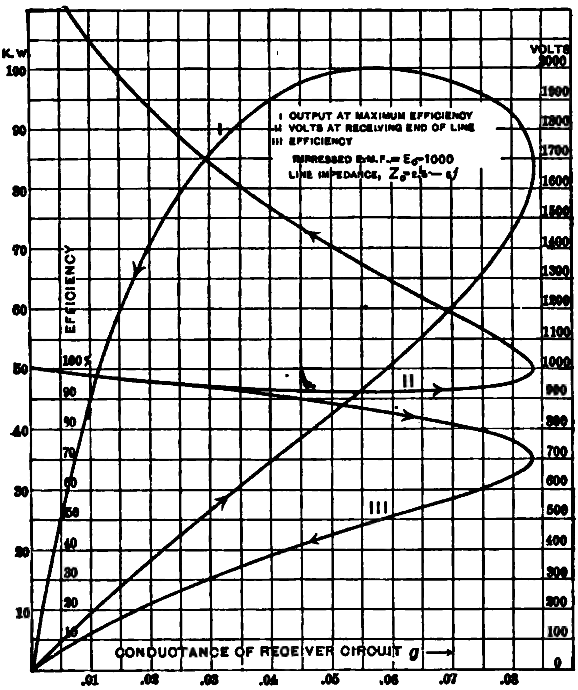


FIG. 78.—Load Characteristics of Transmission Lines.

and with the variable conductances, g, of the receiver circuit as abscissas, the

Output at maximum efficiency, (Curve I.);

Volts at receiving end of line, (Curve II.);

Efficiency $= \dfrac{r}{r + r_0}$, (Curve III.).

(4.) *Control of Receiver Voltage by Shunted Susceptance.*

94. By varying the susceptance of the receiver circuit, the potential at the receiver terminals is varied greatly. Therefore, since the susceptance of the receiver circuit can be varied at will, it is possible, at a constant generator e.m.f., to adjust the receiver susceptance so as to keep the potential constant at the receiver end of the line, or to vary it in any desired manner, and independently of the generator potential, within certain limits.

The ratio of e.m.fs. is

$$\alpha = \frac{E}{E_0} = \frac{1}{\sqrt{(1 + r_0 g + x_0 b)^2 + (x_0 g - r_0 b)^2}}.$$

If at constant generator potential E_0, the receiver potential E shall be constant,

$$\alpha = \text{constant};$$

hence,

$$(1 + r_0 g + x_0 b)^2 + (x_0 g - r_0 b)^2 = \frac{1}{\alpha^2};$$

or, expanding,

$$b = - b_0 + \sqrt{\left(\frac{y_0}{\alpha}\right)^2 - (g + g_0)^2},$$

which is the value of the susceptance, b, as a function of the receiver conductance, — that is, of the load, — which is required to yield constant potential, αE_0, at the receiver circuit.

For increasing g, that is, for increasing load, a point is reached where, in the expression

$$b = - b_0 + \sqrt{\left(\frac{y_0}{\alpha}\right)^2 - (g + g_0)^2},$$

the term under the root becomes imaginary, and it thus becomes impossible to maintain a constant potential, αE_0. Therefore the maximum output which can be transmitted at potential αE_0, is given by the expression

$$\sqrt{\left(\frac{y_0}{\alpha}\right)^2 - (g + g_0)^2} = 0;$$

hence the susceptance of receiver circuit is $b = -b_0$, and the conductance of receiver circuit is $g = -g_0 + \frac{y_0}{\alpha}$,

$$P = E_0^2 g \alpha^2 = \alpha^2 E_0^2 \left(\frac{y_0}{\alpha} - g_0\right), \text{ the output.}$$

95. If $\alpha = 1$, that is, if the voltage at the receiver circuit equals the generator potential

$$g = y_0 - g_0; \; P = E_0^2 (y_0 - g_0).$$

If $\alpha = 1$, when $g = 0$, $b = 0$

when $g > 0$, $b < 0$;

if $\alpha > 1$, when $g = 0$, or $g > 0$, $b < 0$,

that is, condensive reactance;

if $\alpha < 1$, when $g = 0$, $b > 0$,

when $g = -g_0 + \sqrt{\left(\frac{y_0}{\alpha}\right)^2 - b_0^2}$, $b = 0$;

when $g > -g_0 + \sqrt{\left(\frac{y_0}{\alpha}\right)^2 - b_0^2}$, $b < 0$,

or, in other words, if $\alpha < 1$, the phase difference in the main line must change from lag to lead with increasing load.

96. The value of α giving the maximum possible output in a receiver circuit is determined by $\frac{dP}{d\alpha} = 0$;

expanding $$2\alpha\left(\frac{y_0}{\alpha} - g_0\right) - \frac{\alpha^2 y^0}{\alpha^2} = 0;$$

hence, $$y_0 = 2\alpha g_0,$$

and $$\alpha = \frac{y_0}{2g_0} = \frac{1}{2\sqrt{g_0 r_0}} = \frac{z_0}{2r_0};$$

the maximum output is determined by

$$g = -g_0 + \frac{y_0}{\alpha} = g_0;$$

and is,
$$P = \frac{E_0^2}{4r}.$$

From
$$\alpha = \frac{y_0}{2g_0} = \frac{z_0}{2r_0},$$

the line reactance, x_0, can be found, which delivers a maximum output into the receiver circuit at the ratio of potentials, α,

as
$$z_0 = 2r_0\alpha,$$
$$x_0 = r_0\sqrt{4\alpha^2 - 1};$$

for $\alpha = 1$,
$$z_0 = 2r_0;$$
$$x_0 = r_0\sqrt{3}.$$

If, therefore, the line impedance equals 2α times the line resistance, the maximum output, $P = \frac{E_0^2}{4r_0}$, is transmitted into the receiver circuit at the ratio of potentials, α.

If $z_0 = 2r_0$, or $x_0 = r_0\sqrt{3}$, the maximum output, $P = \frac{E_0^2}{4r_0}$, can be supplied to the receiver circuit, without change of potential at the receiver terminals.

Obviously, in an analogous manner, the law of variation of the susceptance of the receiver circuit can be found which is required to increase the receiver voltage proportionally to the load; or, still more generally, — to cause any desired variation of the potential at the receiver circuit independently of any variation of the generator potential, as, for instance, to keep the potential of a receiver circuit constant, even if the generator potential fluctuates widely.

97. In Figs. 79, 80, and 81, are shown, with the output, $P = E_0^2 g\alpha^2$, as abscissas, and a constant impressed e.m.f.,

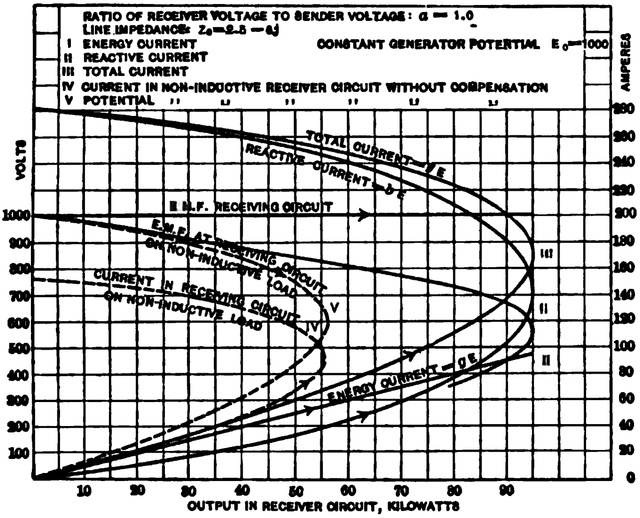


Fig. 79.—Variation of Voltage of Transmission Lines.

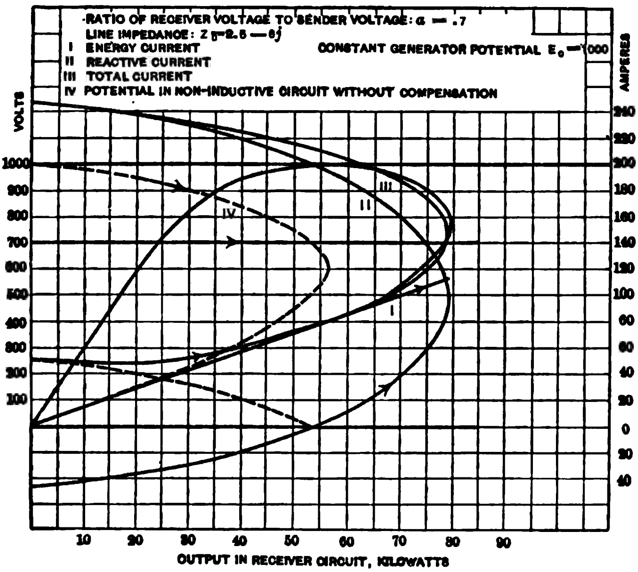


Fig. 80. —Variation of Voltage of Transmission Lines.

$E_0 = 1000$ volts, and a constant line impedance, $Z_0 = 2.5 - 6j$, or $r_0 = 2.5$ ohms, $x_0 = 6$ ohms, $z = 6.5$ ohms, the following values:

power component of current,	gE, (Curve I.);
reactive, or wattless component of current,	bE, (Curve II.);
total current,	yE, (Curve III.),

for the following conditions:

$\alpha = 1.0$ (Fig. 79); $\alpha = 0.7$ (Fig. 80); $\alpha = 1.3$ (Fig. 81).

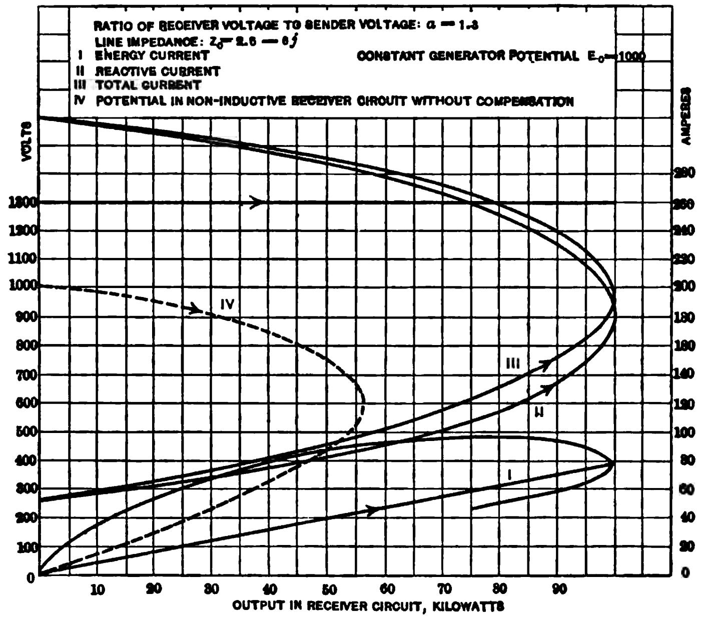


FIG. 81. — Variation of Voltage of Transmission Lines.

For the non-inductive receiver circuit (in dotted lines), the curve of e.m.f., E, and of the current, $I = gE$, are added in the three diagrams for comparison, as Curves IV. and V.

As shown, the output can be increased greatly, and the potential at the same time maintained constant, by the judicious use of shunted reactance, so that a much larger output can be transmitted over the line with no drop, or even with a rise, of potential.

(5.) *Maximum Rise of Potential at Receiver Circuit.*

98. Since, under certain circumstances, the potential at the receiver circuit may be higher than at the generator, it is of interest to determine what is the maximum value of potential, E, that can be produced at the receiver circuit with a given generator potential, E_0.

The condition is that

$$\alpha = \text{maximum or } \frac{1}{\alpha^2} = \text{minimum};$$

that is,

$$\frac{d\left(\frac{1}{\alpha^2}\right)}{dg} = 0, \quad \frac{d\left(\frac{1}{\alpha^2}\right)}{db} = 0;$$

substituting,

$$\frac{1}{\alpha^2} = (1 + r_0 g + x_0 b)^2 + (x_0 g - r_0 b)^2,$$

and expanding, we get,

$$\frac{d\left(\frac{1}{\alpha^2}\right)}{dg} = 0; \quad g = -\frac{r_0}{z_0^2};$$

— a value which is impossible, since neither r_0 nor g can be negative. The next possible value is $g = 0$, — a wattless circuit.

Substituting this value, we get,

$$\frac{1}{\alpha^2} = (1 + x_0 b)^2 + r_0^2 b^2;$$

and by substituting, in

$$\frac{d\left(\frac{1}{\alpha^2}\right)}{db} = 0, \quad b = -\frac{x_0}{z_0^2} = -b_0,$$

$$b + b_0 = 0;$$

that is, the sum of the susceptances $= 0$, or the condition of resonance is present.

Substituting

$$b = -b_0,$$

we have

$$a = \frac{1}{\sqrt{r_0 g_0}} = \frac{z_0}{r_0} = \frac{y_0}{g_0}.$$

The current in this case is

$$I = E_0 g_0 = \frac{E_0}{r_0},$$

or the same as if the line resistance were short-circuited without any inductive reactance.

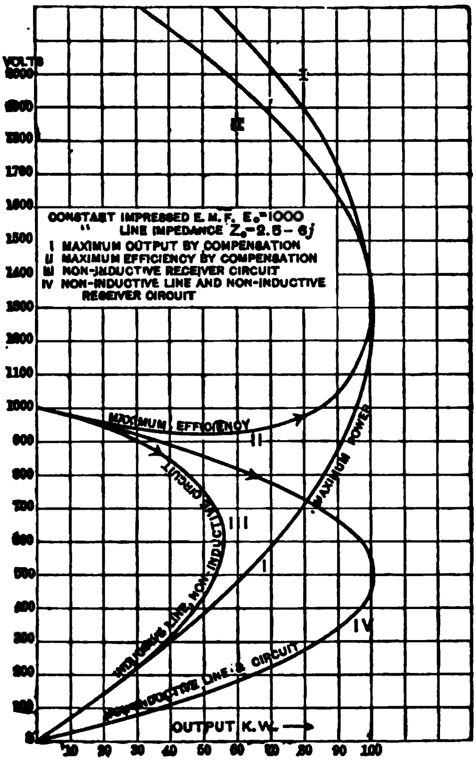


Fig. 82.—Efficiency and Output of Transmission Lines.

This is the condition of perfect resonance, with current and e.m.f. in phase.

99. As summary to this chapter, in Fig. 82 are plotted, for a constant generator e.m.f., $E_0 = 1000$ volts, and a line impedance, $Z_0 = 2.5 - 6j$, or, $r_0 = 2.5$ ohms, $x_0 = 6$ ohms, $z_0 = 6.5$ ohms; and with the receiver output as abscissas and the receiver voltages as ordinates, curves representing

the condition of maximum output, (Curve I.);

the condition of maximum efficiency, (Curve II.);

the condition $b = 0$, or a non-inductive receiver circuit, (Curve III.);

the condition $b = 0$, $b_0 = 0$, or a non-inductive line and non-inductive receiver circuit.

In conclusion, it may be remarked here that of the sources of susceptance, or reactance,

a choking coil or reactive coil corresponds to an inductive reactance;

a condenser corresponds to a condensive reactance;

a polarization cell corresponds to a condensive reactance;

a synchronous machine (motor, generator or converter) corresponds to an inductive or a condensive reactance at will;

an induction motor or generator corresponds to an inductive reactance.

The choking coil and the polarization cell are specially suited for series reactance, and the condenser and synchronous machine for shunted susceptance.

CHAPTER XII.

PHASE CONTROL.

100. At constant voltage, e_0, impressed upon a circuit, as a transmission line, resistance, r, inserted in series with the receiving circuit, causes the voltage, e, at the receiver circuit to decrease with increasing current, I, through the resistance. The decrease of the voltage, e, is greatest if the current, I, is in time-phase with the voltage, e, — less, if the current is not in time-phase. Inductive reactance in series with the receiving circuit, e, at constant impressed e.m.f., e_0, causes the voltage, e, to drop less with a unity power-factor current, I, but far more with a lagging current, and causes the voltage, e, to rise with a leading current.

While series resistance always causes a drop of voltage, series inductive reactance, x, may cause a drop of voltage or a rise of voltage, depending on whether the current is lagging or leading, and if the supply line contains resistance, r, as well as reactance, x, and the time-phase of the current, I, can be varied at will, by producing in the receiver circuit lagging or leading currents, the change of voltage, e, with a change of load in the circuit can be controlled; as for instance, by changing the current from lagging at no-load to lead at heavy load, the reactance, x, can be made to lower the voltage at light load, and raise it at overload, and so make up for the increasing drop of voltage with increasing load, caused by the resistance, r, that is, to maintain constant voltage, or even a voltage, e, which rises with the load on the receiving circuit, at constant voltage, e_0, at the generator side of the line. Or the wattless component of the current can be varied so as to maintain unity power-factor at the generator end of the line, e_0, etc.

This method of controlling a circuit supplied over an inductive line, by varying the phase relation of the current in the circuit, has been called "phase control," and is used to a great extent, especially in the transmission of three-phase power for

conversion to direct current by synchronous converters for railroading.

It requires a receiving circuit, in which, independent of the load, a lagging or leading component of current can be produced at will. Such is the case in synchronous motors or converters: in a synchronous motor, a lagging current can be produced by decreasing, a leading current by increasing the field excitation.

101. If in a direct-current motor, at constant impressed e.m.f., the field excitation and therefore the field magnetism is decreased, the motor speed increases, as the armature has to revolve faster, to consume the impressed e.m.f., and if the field excitation is increased, the motor slows down. A synchronous motor, however, cannot vary in speed, since it must keep in step with the impressed frequency, and if, therefore, at constant impressed voltage, the field excitation is decreased below that which gives a field magnetism, that at the synchronous speed consumes the impressed e.m.f., the field magnetism still must remain the same, and the armature current so changes in phase in such a manner as to magnetize the field, and make up for the deficiency in the field excitation. That is, the armature current becomes lagging. Inversely, if the field excitation of the synchronous motor is increased, the magnetic flux still must remain the same as to correspond to the impressed e.m.f. at synchronous speed, and the armature current so becomes demagnetizing — that is, leading.

By varying the field excitation of a synchronous motor or converter, quadrature components of current can be produced at will, proportional to the variation of the field excitation from the value that gives a magnetic flux, which at synchronous speed just consumes the impressed e.m.f. (after allowing for the impedance of the motor).

Phase control of transmission lines is especially suited for circuits supplying synchronous motors or converters; since such machines, in addition to their mechanical or electrical load, can with a moderate increase of capacity carry or produce considerable values of wattless current. For instance, a quadrature component of current equal to 50 per cent of the power component of current consumed by a synchronous motor, would increase the total current only to $\sqrt{1 + 0.5^2} = 1.118$, or 11.8

per cent, while a quadrature component of current equal to 30 per cent of the power component of the current would give an increase of 4.4 per cent only, that is, could be carried without any appreciable increase of the motor heating.

Phase control depends upon the inductive reactance of the line or circuit between generating and receiving voltage, e_0 and e, and where the inductive reactance of the transmission line is not sufficient, additional reactance may be inserted in the form of reactive coils. This is usually the case in railway transmissions to synchronous converters.

While, therefore, the resistance, r, of the line is fixed, as it would not be economical to increase it, the reactance, x, can be increased beyond that given by line and transformer, by the insertion of reactive coils; and therefore can be adjusted so as to give best results in phase control, which are usually obtained when the quadrature component of the current is a minimum.

102. Let then

e = voltage at receiving circuit, chosen as zero vector.

$\dot{I} = i + ji'$ = current in receiving circuit, comprising a power component, i, which depends upon the load in the receiving circuit, and a quadrature component, i', which can be varied to suit the requirements of regulation, and is considered positive when lagging, negative when leading.

$\dot{E}_0 = e_0' + je_0''$ = voltage impressed upon the system at the generator end, or supply voltage, and the absolute value is

$$e_0 = \sqrt{e_0'^2 + e_0''^2}.$$

$Z = r - jx$ = impedance of the circuit between voltage e and voltage e_0, and the absolute value is $z = \sqrt{r^2 + x^2}$.

If e = terminal voltage of receiving station, e_0 = terminal voltage of generating station, Z = impedance of transmission line; if e = nominal induced e.m.f. of receiving synchronous machine, that is, voltage corresponding to its field excitation, and e_0 = nominal induced e.m.f. of generator, Z also includes the synchronous impedance of both machines, and of step-up and step-down transformers, where used,

It is

$$\dot{E}_0 = e + Z\dot{I},$$

or:

$$\dot{E}_0 = (e + ri + xi') + j\,(ri' - xi), \tag{1}$$

and in absolute value we have

$$e_0^2 = (e + ri + xi')^2 + (ri' - xi)^2. \tag{2}$$

This is the fundamental equation of phase control, giving the relation of the two voltages, e and e_0, with the two components of current, i and i', and the circuit constants r and x.

From equation (2), follows:

$$e = \sqrt{e_0^2 - (ri' - xi)^2} - (ri + xi'), \tag{3}$$

expressing the receiver voltage, e, as a function of e_0 and I.
and:

$$i' = \pm\sqrt{\frac{e_0^2}{z^2} - \left(\frac{er}{z^2} + i\right)^2} - \frac{ex}{z^2}. \tag{4}$$

Denoting

$$\tan\,\theta = \frac{x}{r}, \tag{5}$$

where θ is the phase angle of the line impedance, we have

$$r = z\cos\theta \quad \text{and} \quad x = z\sin\theta \tag{6}$$

and

$$i' = \pm\sqrt{\frac{e_0^2}{z^2} - \left(\frac{e\cos\theta}{z} + i\right)^2} - \frac{e\sin\theta}{z}, \tag{7}$$

gives the reactive component of the current, i', required by the power component of the current, i, at the voltages, e and e_0.

103. The phase angle of the impressed e.m.f., $\dot{E}_0$, is, from (1),

$$\tan\,\theta_0 = \frac{ri' - xi}{e + ri + xi'}, \tag{8}$$

the phase angle of the current

$$\tan\,\theta_1 = \frac{i'}{i}, \tag{9}$$

hence, to bring the current, $\dot{I}$, into phase with the impressed e.m.f., $\dot{E}_0$, or produce unity power-factor at the generator terminal, e_0, it must be

$$\theta_0 = \theta_1;$$

hence,

$$\frac{ri' - xi'}{e + ri + xi'} = \frac{i'}{i},$$

and herefrom follows:

$$i' = \frac{\pm\sqrt{e^2 - 4\, x^2 i^2} - e}{2\, x}, \tag{10}$$

hence always negative, or leading, but $i' = o$ for $i = o$, or at no load.

From equation (10) follows, that i' becomes imaginary, if the term under the square root, $(e^2 - 4\, x^2 i^2)$, becomes negative, that is, if

$$i > \frac{e}{2\, x},$$

that is, the maximum load, or power component of current, at which unity power-factor can still be maintained at the supply voltage, e_0, is given by

$$i_m = \frac{e}{2\, x}, \tag{11}$$

and the leading quadrature component of current, required to compensate for the line reactance x at maximum current i_m, is from equation (10),

$$i_m' = \frac{e}{2\, x}, \tag{12}$$

that is, in this case of the maximum load which can be delivered at e, with unity power-factor at e_0, the total current, $\dot{I}$, leads the receiver voltage, e, by 45 time-degrees.

Substituting the value, i', of equation (10), which compensates for the line reactance, x, and so gives unity power-factor at e_0, into equation (2), gives as required supply voltage e_0

$$e_0^2 = \frac{e^2 z^2}{2\, x^2} + \frac{(x - r)\,(e - 2\, ix)\sqrt{e^2 - 4\, i^2 x^2}}{2\, x}. \tag{13}$$

As illustration are shown, in Fig. 83, with the load current, i, as abscissas, the values of leading quadrature component of current, i', and of generator voltage, e_0, for the constants

$$e = 400 \text{ volts}; \; r = 0.05 \text{ ohms, and } x = 0.10 \text{ ohms.}$$

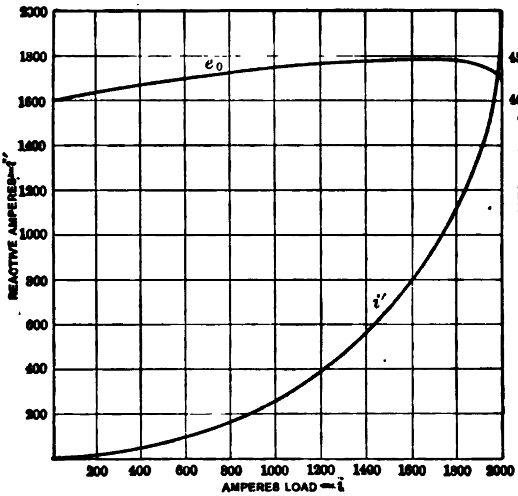


Fig. 83.

104. More frequently than for controlling the power-factor, phase control is used for controlling the voltage, that is, to maintain the receiver voltage, e, constant, or raise it with increasing load, i, at constant generator-voltage, e_0.

In this case, equation (4) gives the quadrature component of current, i', required by current, i, at constant receiver-voltage, e, and constant generator-voltage, e_0.

Since the equation (4) of i' contains a square root, the maximum value of i, that is, the maximum load which can be carried at constant voltage, e and e_0, is given by equating the term under the square root to zero

$$\frac{e_0^2}{z^2} - \left(\frac{er}{z^2} + i\right)^2 = 0;$$

as

$$i_m = \frac{e_0 z - er}{z^2} = \frac{e_0 - e \cos \theta}{z}, \qquad (14)$$

and the corresponding quadrature component of current, by (4), is

$$i_m' = -\frac{ex}{z^2} = -\frac{e \sin \theta}{z}, \tag{15}$$

that is, leading.

From equation (14) follows as the impedance, z, which, at constant line-resistance, r, gives the maximum value of i_m

$$\frac{di_m}{dz} = 0;$$

hence,

$$z_m = 2r\frac{e}{e_0}, \tag{16}$$

and for this value of impedance z_m, substituting in (14) and (15)

$$i_{mm} = \frac{e_0^2}{4r}, \quad \text{and} \quad i_{mm}' = \frac{e_0^2}{4r}. \tag{17}$$

The maximum load, i, which can be delivered at constant voltage, e, therefore depends upon the line impedance, and the voltages, e and e_0.

Since e_0 and e are not very different from each other, the ratio, $\frac{e}{e_0}$ in equation (16) is approximately unity, and the impedance, z, which permits maximum load to be transmitted, is approximately twice the line resistance, r, or rather slightly less.

$$z \leqq 2r,$$

gives

$$x \leqq r\sqrt{3}.$$

A relatively low line-reactance, x, so gives maximum output. In practice, a far higher reactance, x, is used, since it gives sufficient output, and a lesser quadrature component of current.

By substituting $i = 0$ in equation (4), the value of the quadrature component of current at no-load is found as

$$\left.\begin{aligned} i_0' &= \frac{\sqrt{e_0^2 z^2 - e^2 r^2} - ex}{z^2} \\ &= \frac{\sqrt{e_0^2 - e^2 \cos^2\theta} - e \sin\theta}{z} \end{aligned}\right\} \tag{18}$$

This can be written in the form

$$i_0' = \frac{\sqrt{(e_0^2 - e^2) + e^2 \sin^2\theta} - e \sin\theta}{z},$$

and then shows, that for $e = e_0$, $i_0' = 0$, or no quadrature component of current exists at no-load; for $e > e_0$, $i_0' < 0$ or negative, that is, the quadrature component of current is already leading at no-load. For: $e < e_0$, $i_0' > 0$ or lagging, that is, the quadrature component of current i_0' is lagging at no-load, becomes zero at some load, and leading at still higher loads.

The latter arrangement, $e < e_0$, is generally used, as the quadrature component of current passes through zero at some intermediate load, and so is less over the range of required load than it would be, if i_0' were 0 or negative.

From (18) follows that the larger z, and at constant resistance r, also x, the smaller the quadrature component of current. That is, increase of the line reactance, x, reduces the quadrature current at no-load, i_0', and in the same way at load, that is, improves the power-factor of the circuit, and so is desirable, and the insertion of reactive coils in the line for this reason customary.

Increase of reactance, however, reduces the maximum output i_m, and too large a reactance is for this reason objectionable.

Let

$$i = i_1$$

be the load at which the quadrature component of current vanishes, $i' = 0$, that is, the receiver circuit has unity power-factor.

Substituting $i = i_1$; $i' = 0$ into equation (2), gives

$$e_0^2 = (e + ri_1)^2 + x^2 i_1^2 \tag{19}$$

and, substituting (19) in (4), (18), (14), gives reactive component of current

$$i' = \sqrt{\frac{e^2 \sin^2\theta}{z^2} + \frac{2\, e \cos\theta}{z}(i_1 - i) + (i_1^2 - i^2)} - \frac{e \sin\theta}{z}, \tag{20}$$

and at no-load

$$i_0' = \sqrt{\frac{e^2 \sin^2\theta}{z^2} + \frac{2\, e i_1 \cos\theta}{z} + i_1^2} - \frac{e \sin\theta}{z}, \tag{21}$$

Maximum output current

$$i_m = \sqrt{\frac{e^2}{z^2} + \frac{2\, e i_1 \cos\theta}{z} + i_1^2} - \frac{e \cos\theta}{z}. \tag{22}$$

105. Of importance in phase control for constant voltage, e, at constant, e_0, are the three currents

i_1, the power component of current at which the quadrature component of current vanishes: $i' = 0$.

i_m, the maximum load which can be transmitted at constant voltage, e.

i_0', the reactive component of current at no-load.

The equation of phase control, (2), however, contains only two quantities, which can be chosen: The reactance, x, which can be increased by inserting reactive coils, and the generator voltage, e_0, which can be made anything desired; even with an existing generating station, since between e and e_0 practically always transformers are interposed, and their ratio can be chosen so as to correspond to any desired generator voltage, e_0, as they usually are supplied with several voltage steps.

Of the three quantities, i_1, i_m and i_0', only two can be chosen, and the constants, x and e_0, derived therefrom. The third current then also follows, and if the value found for it does not suit the requirements of the problem, other values have to be tried. For instance, choosing i_1, as corresponding to three-fourths load, and i_0' fairly small, gives very good power-factors over the whole range of load, but a relatively low value of i_m, and where very great overload capacities are required, i_m may not be sufficient, and i_1 may have to be chosen corresponding to full-load, and a higher value of i_0' permitted, that is, some sacrifice made in the power-factor, in favor of overload capacity.

So, for instance, the values may be chosen

i_1, corresponding to full-load,

and required that i_0' does not exceed half of full-load current;

$$i_0' < .5\, i_1,$$

and that the synchronous converter or motor can carry at least 100 per cent overload, that is,

$$i_m > 2\,i_1.$$

We then can put, $i_m = 2\,i_1\,c$ and $i_0' = \dfrac{0.5\,i_1}{c}$, (23)

and substitute (23) in (19), (21), (22) and determine x, e_0, c.

106. The variation of the reactive current, i', with the load, i, equation (4), is brought about by varying the field excitation of the receiving synchronous machine. Where the load on the synchronous machine is direct-current output, as in a motor generator and especially a converter, the most convenient way of varying the field excitation with the load is automatically, by a series field-coil traversed by the direct-current output. The field windings of converters intended for phase control — as for the supply of power to electric railways, from substations fed by a high-potential alternating-current transmission line — are compound-wound, and the shunt field is adjusted for under-excitation, so as to produce at no-load the lagging current, i_0', and the series field adjusted so as to make the reactive component of current, i', disappear at the desired load, i_1.

In this case, however, the variation of the field excitation by the series field is directly proportional to the load, as is also the variation of i', that is, it varies from $i' = i_0'$ for $i = 0$, to $i' = 0$ for $i = i_1$, and can be expressed by the equation

$$\left.\begin{aligned} i' &= i_0'\left(1 - \frac{i}{i_1}\right) \\ &= q\,(i_1 - i) \end{aligned}\right\} \qquad (24)$$

where

$$q = \frac{i_0'}{i_1} \qquad (25)$$

is the ratio of (reactive) no-load current, i_0', to (effective) non-inductive load current, i_1.

To maintain constant voltage, e, at constant, e_0, the required variation of i' is not quite linear, and with a linear variation of

i', as given by a compound field-winding on the synchronous machine, the receiver-voltage, e, at constant impressed voltage does not remain perfectly constant, but when adjusted for the same value at no-load and at full load, e is slightly high at intermediate loads, low at higher loads. It is, however, sufficiently constant for all practical purposes.

Choosing then the full-load current, i_1, and the no-load current, $i_0' = qi_1$, and let the reactive component of current, i', by a compound field-winding vary as a linear function of the load, i:

$$i' = q\,(i_1 - i).$$

Then, substituting i_1 and $i_0' = qi_1$ in the equations (2) for phase control:

No-load:
$$i = 0, \qquad i' = qi_1;$$
$$e_0^2 = (e + qxi_1)^2 + qri_1^2. \tag{26}$$

Full load:
$$i_r = i_1, \qquad i' = 0;$$
$$e_0^2 = (e + ri_1)^2 + xi_1^2. \tag{27}$$

From these equations (26) and (27) then calculate the required reactance, x, and the generator voltage, e_0, as:

$$x = \frac{\frac{qe}{i_1} \pm \sqrt{\frac{e^2}{i_1^2}(1 + q^2) - \left[\frac{e}{i_1} + r\,(1 - q^2)\right]^2}}{1 - q^2}, \tag{28}$$

and from (27) or (26) the voltage, e_0.

The terminal voltage at the receiving circuit then is, by equation (3):

$$e = \sqrt{e_0^2 - [qri_1 - (qr + x)\,i]^2} - [(r - qx)\,i + qxi_1]. \tag{29}$$

As an example is shown, in Fig. 84, the curve of receiving voltage, e, with the load, i, as abscissas, for the values:

$e = 400$ volts at no-load and at full load,
$i_1 = 500$ amp. at full load, power component of current,
$i_0' = 200$ amp., lagging reactive or quadrature component of current at no-load,

hence $q = 0.4$,
$i' = 200 - .4\,i$,
and $r = 0.05$ ohm.

From equation (28) then follows:

$$x = 0.381 \pm 0.165 \text{ ohm}.$$

Choosing the lower value:

$$x = 0.216 \text{ ohm}$$

gives, from equation (27):

$$e_0 = 443.4 \text{ volts};$$

hence

$$e = \sqrt{196420 + 5.76\,i - .0576\,i^2} - (43.2 - .0264\,i).$$

For comparison is shown, in Fig. 84, the receiving voltage, e', at the same supply voltage, $e_0 = 443.4$ volts, but without phase control, that is, with a non-inductive receiver-circuit.

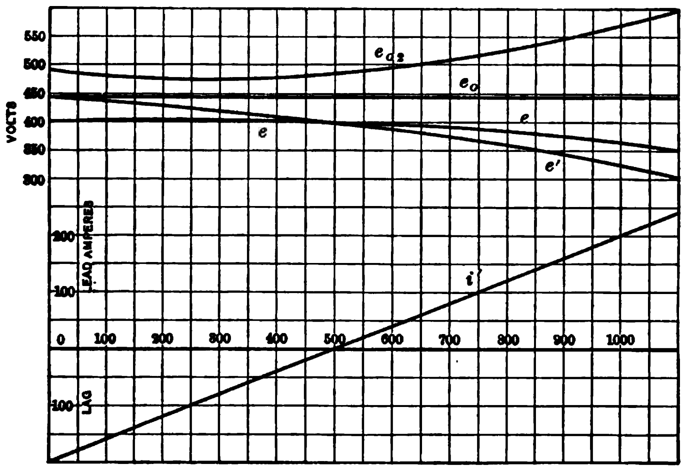


Fig. 84.

107. Equation (28) shows that there are two values of x: x_1 and x_2; and corresponding thereto two values of e_0: e_{01} and e_{02}, which as constant-supply voltage give the same receiver-voltage, e, at no-load and at full load, and so approximately constant receiver-voltage throughout.

One of the two reactances, x_2, is much larger than the other, x_1, and the corresponding voltage, e_{02}, accordingly larger than e_{01}.

In addition to the terminal voltage, e, at the receiver-circuit, there are therefore two further points of constant voltage in the system: e_{01}, distant from e by the resistance, r, and reactance, x_1, and: e_{02}, distant from e_{01} by the reactance $x_0 = x_2 - x_1$.

That is, by the proper choice of the reactances, x_1 and x_0, three points of the system can be maintained automatically at approximately constant voltage, by phase control: e, e_{01} and e_{02}.

Such *multiple-phase control* can advantageously be employed by using:

e as the terminal voltage of the receiving circuit,

e_{01} as the generator terminal voltage, and

e_{02} as the nominal induced e.m.f. of the generator, that is, the voltage corresponding to the field-excitation. Constancy of e_{02} accordingly means constant field-excitation.

That is, with constant field-excitation of the generator, the voltage remains approximately constant, by multiple-phase control, at the generator busbars as well as at the terminals of the receiving circuit, at the end of the transmission line of resistance, r.

In this case:

x_1 = reactance of transmission line plus reactive coils inserted in the line (usually at the receiving station).

$x_0 = x_2 - x_1$ = synchronous reactance of the generator plus reactive coils inserted between generator and generator busbars, where necessary.

Since the generator also contains a small resistance, r_0, the two values of reactance, x_1 and $x_2 = x_1 + x_0$, are given by the equation (28) as:

$$x_1 = \frac{\frac{qe}{i_1} - \sqrt{\frac{e^2}{i_1^{\,2}}(1+q^2) - \left[\frac{e}{i_1} + r(1-q^2)\right]^2}}{1-q^2},$$

and

$$x_2 = \frac{\frac{qe}{i_1} + \sqrt{\frac{e^2}{i_1^{\,2}}(1+q^2) - \left[\frac{e}{i_1} + (r+r_0)(1-q^2)\right]^2}}{1-q^2}.$$

Assuming in above example:

$$r_0 = 0.01 \text{ ohm}$$

gives

$$x_2 = 0.440 \text{ ohm};$$

hence,

$$x_0 = 0.224 \text{ ohm}.$$

The curve of nominal generated e.m.f., e_{02}, of the generator is shown in Fig. 84 as e_{02}.

That is, at constant field-excitation, corresponding to a nominal generated e.m.f.,

$$e_{02} = 488.2 \text{ volts}.$$

The generator of synchronous impedance,

$$Z_0 = r_0 - jx_0 = 0.01 - .224\, j \text{ ohms},$$

maintains constant voltage at its own terminals, or at the generator busbars,

$$e_0 = 443.4 \text{ volts},$$

and at the same time maintains constant voltage,

$$e = 400 \text{ volts},$$

at the end of a transmission line of impedance,

$$Z = r - jx_1 = 0.06 - 0.216\, j \text{ ohms approximately},$$

if by phase control in the receiving circuit, by compounded converter, the reactive or quadrature component of current, i', is varied with the load or power component of current, i, and proportional thereto, that is:

$$\begin{aligned} i' &= q\,(i_1 - i) \\ &= 200 - 0.4\, i. \end{aligned}$$

108. To adjust a circuit experimentally for phase control for constant voltage, by overcompounded synchronous converter: at constant-supply voltage and no-load on the converter

— with the transmission line with its transformers, reactances, etc., or an impedance equal thereto, in the circuit between converter and supply voltage — the shunt field of the converter is adjusted by the field rheostat so as to give the desired direct-current voltage at the converter brushes. Then load is put on the converter, and, without changing the supply voltage or the adjustment of the shunt field, the rheostat or shunt across the series field of the converter is adjusted so as to give the desired direct-current voltage.

If the supply voltage can be varied, as is usually provided for by different voltage taps on the transformer, then, before adjusting the converter fields as described above, first the proper supply voltage is found. This is done by loading the converter with the current, at which unity power-factor at the converter is desired — for instant full load — and then varying the converter shunt field so as to get minimum alternating-current input, and varying the supply voltage so as to get — at minimum alternating-current input — the desired direct-current voltage. Where the supply voltage can only be varied in definite steps: at some voltage step, the converter field — at the desired non-inductive load — is adjusted for minimum alternating-current input; if then the direct-current voltage is too low, the transformer connections are changed to the next higher supply voltage step; if the direct-current voltage is too high, the change is made to the next lower supply voltage step, until that supply voltage step is found, which, at the adjustment of the converter field for minimum alternating-current input, brings the direct-current voltage nearest to that desired. Then for this supply voltage step, the converter field circuits are adjusted for phase control, as above described.

CHAPTER XIII.

EFFECTIVE RESISTANCE AND REACTANCE.

109. The resistance of an electric circuit is determined:

(1) By direct comparison with a known resistance (Wheatstone bridge method, etc.).

This method gives what may be called the true ohmic resistance of the circuit.

(2) By the ratio:

$$\frac{\text{Volts consumed in circuit}}{\text{Amperes in circuit}}.$$

In an alternating-current circuit, this method gives, not the resistance of the circuit, but the impedance,

$$z = \sqrt{r^2 + x^2}.$$

(3) By the ratio:

$$r = \frac{\text{Power consumed}}{(\text{Current})^2};$$

where, however, the "power" does not include the work done by the circuit, and the counter e.m.fs. representing it, as, for instance, in the case of the counter e.m.f. of a motor.

In alternating-current circuits, this value of resistance is the power coefficient of the e.m.f.,

$$r = \frac{\text{Power component of e.m.f.}}{\text{Total current}}.$$

It is called the *effective resistance* of the circuit, since it represents the effect, or power, expended by the circuit. The power coefficient of current,

$$g = \frac{\text{Power component of current}}{\text{Total e.m.f.}},$$

is called the *effective conductance* of the circuit.

In the same way, the value,

$$x = \frac{\text{Wattless component of e.m.f.}}{\text{Total current}},$$

is the *effective reactance*, and

$$b = \frac{\text{Wattless component of current}}{\text{Total e.m.f.}},$$

is the *effective susceptance* of the circuit.

While the true ohmic resistance represents the expenditure of power as heat inside of the electric conductor by a current of uniform density, the effective resistance represents the total expenditure of power.

Since in an alternating-current circuit, in general power is expended not only in the conductor, but also outside of it, through hysteresis, secondary currents, etc., the effective resistance frequently differs from the true ohmic resistance in such way as to represent a larger expenditure of power.

In dealing with alternating-current circuits, it is necessary, therefore, to substitute everywhere the values "effective resistance," "effective reactance," "effective conductance," and "effective susceptance," to make the calculation applicable to general alternating-current circuits, such as inductive reactances containing iron, etc.

While the true ohmic resistance is a constant of the circuit, depending only upon the temperature, but not upon the e.m.f., etc., the effective resistance and effective reactance are, in general, not constants, but depend upon the e.m.f., current, etc. This dependence is the cause of most of the difficulties met in dealing analytically with alternating-current circuits containing iron.

110. The foremost sources of energy loss in alternating-current circuits, outside of the true ohmic resistance loss, are as follows:

(1) Molecular friction, as,

(*a*) Magnetic hysteresis;

(*b*) Dielectric hysteresis.

(2) Primary electric currents, as,

(a) Leakage or escape of current through the insulation, brush discharge;

(b) Eddy currents in the conductor or unequal current distribution.

(3) Secondary or induced currents, as,

(a) Eddy or Foucault currents in surrounding magnetic materials;

(b) Eddy or Foucault currents in surrounding conducting materials;

(c) Secondary currents of mutual inductance in neighboring circuits.

(4) Induced electric charges, electrostatic induction or influence.

While all these losses can be included in the terms effective resistance, etc., only the magnetic hysteresis and the eddy currents in the iron will form the subject of what follows, since they are the most frequent and important sources of energy loss.

Magnetic Hysteresis.

111. In an alternating-current circuit surrounded by iron or other magnetic material, energy is expended outside of the conductor in the iron, by a kind of molecular friction, which, when the energy is supplied electrically, appears as magnetic hysteresis, and is caused by the cyclic reversals of magnetic flux in the iron in the alternating magnetic field.

To examine this phenomenon, first a circuit may be considered, of very high inductive reactance, but negligible true ohmic resistance; that is, a circuit entirely surrounded by iron, as, for instance, the primary circuit of an alternating-current transformer with open secondary circuit.

The wave of current produces in the iron an alternating magnetic flux which generates in the electric circuit an e.m.f., — the counter e.m.f. of self-induction. If the ohmic resist-

ance is negligible, that is, practically no e.m.f. consumed by the resistance, all the impressed e.m.f. must be consumed by the counter e.m.f. of self-induction, that is, the counter e.m.f. equals the impressed e.m.f.; hence, if the impressed e.m.f. is a sine wave, the counter e.m.f., and, therefore, the magnetic flux which generates the counter e.m.f., must follow a sine wave also. The alternating wave of current is not a sine wave in this case, but is distorted by hysteresis. It is possible, however, to plot the current wave in this case from the hysteretic cycle of magnetic flux.

From the number of turns, n, of the electric circuit, the effective counter e.m.f., E, and the frequency, f, of the current, the maximum magnetic flux, Φ, is found by the formula:

$$E = \sqrt{2}\,\pi n f\, \Phi\, 10^{-8};$$

hence,

$$\Phi = \frac{E\, 10^{8}}{\sqrt{2}\,\pi n f}.$$

A maximum flux, Φ, and magnetic cross-section, A, give the maximum magnetic induction, $\mathfrak{B} = \frac{\Phi}{A}$.

If the magnetic induction varies periodically between $+\mathfrak{B}$ and $-\mathfrak{B}$, the m.m.f. varies between the corresponding values $+\mathfrak{F}$ and $-\mathfrak{F}$, and describes a looped curve, the cycle of hysteresis.

If the ordinates are given in lines of magnetic force, the abscissas in tens of ampere-turns, then the area of the loop equals the energy consumed by hysteresis in ergs per cycle.

From the hysteretic loop the instantaneous value of m.m.f. is found, corresponding to an instantaneous value of magnetic flux, that is, of generated e.m.f.; and from the m.m.f., $\mathfrak{F}$, in ampere-turns per unit length of magnetic circuit, the length, l, of the magnetic circuit, and the number of turns, n, of the electric circuit, are found the instantaneous values of current, i, corresponding to a m.m.f., $\mathfrak{F}$; that is, magnetic induction, $\mathfrak{B}$, and thus generated e.m.f., e, as:

$$i = \frac{\mathfrak{F}\, l}{n}.$$

112. In Fig. 85, four magnetic cycles are plotted, with maximum values of magnetic induction, $\mathfrak{B} = 2{,}000$, 6,000, 10,000, and 16,000, and corresponding maximum m.m.fs., $\mathfrak{F} = 1.8$, 2.8, 4.3, 20.0. They show the well-known hysteretic loop, which becomes pointed when magnetic saturation is approached.

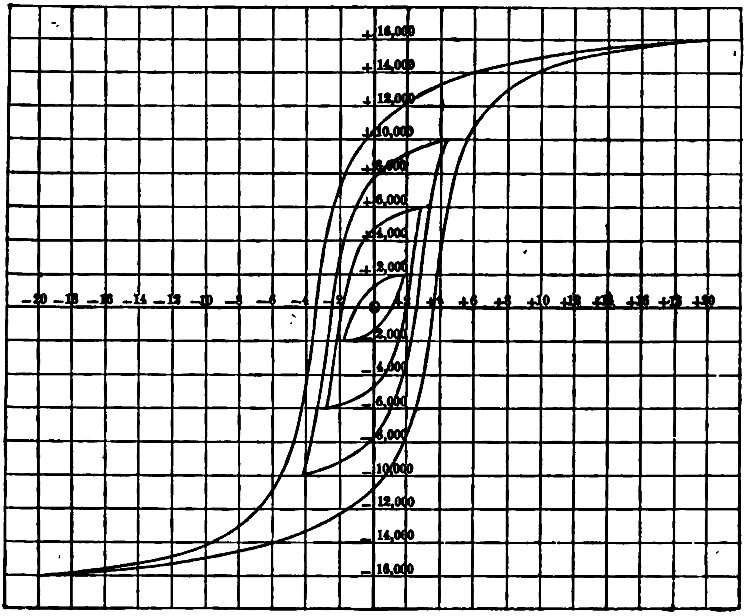


Fig. 85. — Hysteretic Cycle of Sheet Iron.

These magnetic cycles correspond to sheet iron or sheet steel, of a hysteretic coefficient, $\eta = 0.0033$, and are given with ampere-turns per centimeter as abscissas, and kilolines of magnetic force as ordinates.

In Figs. 86 and 87, the curve of magnetic induction as derived from the generated e.m.f. is a sine wave. For the different values of magnetic induction of this sine curve, the corresponding values of m.m.f., hence of current, are taken from Fig. 85, and plotted, giving thus the exciting current required to produce the sine wave of magnetism; that is, the wave of current which a sine wave of impressed e.m.f. will establish in the circuit.

As shown in Figs. 86 and 87, these waves of alternating current are not sine waves, but are distorted by the superposition of higher harmonics, and are complex harmonic waves. They reach their maximum value at the same time with the maximum of magnetism, that is, 90 time-degrees

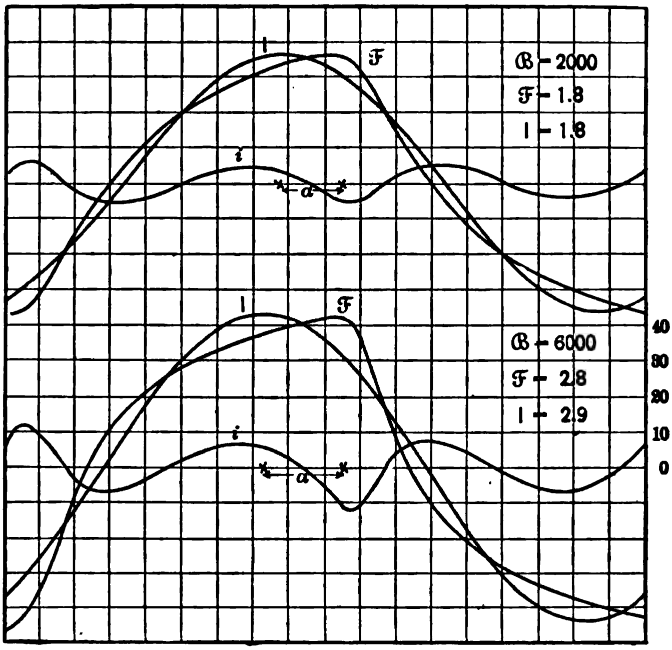


FIG. 86. — Distortion of Current Wave by Hysteresis.

ahead of the maximum generated e.m.f., and hence about 90 time-degrees behind the maximum impressed e.m.f., but pass the zero line considerably ahead of the zero value of magnetism, or 42°, 52°, 50°, and 41°, respectively.

The general character of these current waves is, that the maximum point of the wave coincides in time with the maximum point of the sine wave of magnetism; but the current wave is bulged out greatly at the rising, and hollowed in at the decreasing, side. With increasing magnetization, the maximum of the current wave becomes more pointed, as shown by the curves of Fig. 87, for $\mathfrak{B} = 10{,}000$; and at still higher saturation a peak is formed at the maximum point, as in the curve for $\mathfrak{B} = 16{,}000$.

This is the case when the curve of magnetization reaches within the range of magnetic saturation, since in the proximity of saturation the current near the maximum point of magnetization has to rise abnormally to cause even a small increase of magnetization. The four curves, Figs. 86 and 87 are not drawn to the same scale. The maximum values of m.m.f., corresponding to

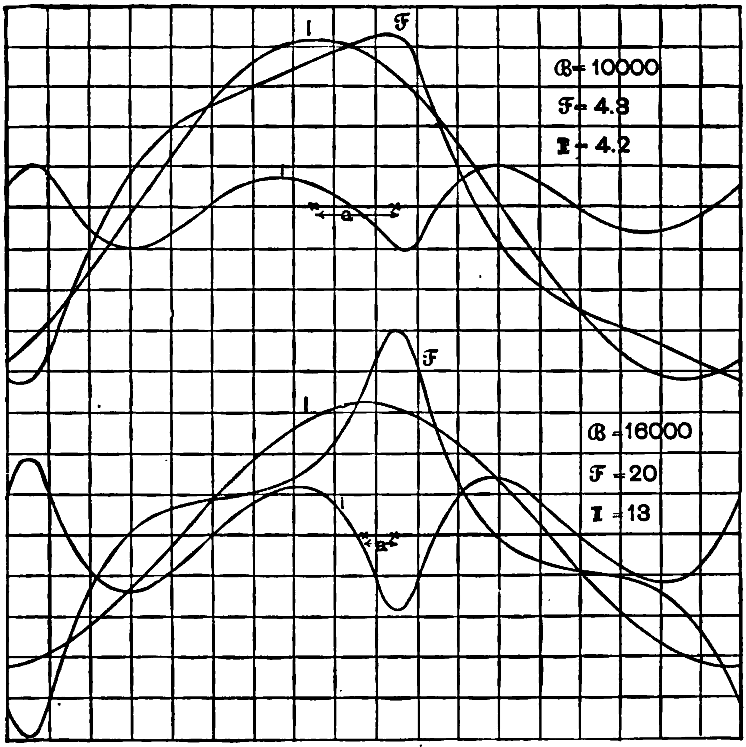


FIG. 87. — Distortion of Current Wave by Hysteresis.

the maximum values of magnetic induction, $\mathfrak{B} = 2{,}000$, $6{,}000$, $10{,}000$, and $16{,}000$ lines of force per square centimeter, are $\mathfrak{F} = 1.8$, 2.8, 4.3, and 20.0 ampere-turns per centimeter. In the different diagrams these are represented in the ratio of $8:6:4:1$, in order to bring the current curves to approximately the same height. The m.m.f., in c.g.s. units, is

$$\mathfrak{H} = 4\,\pi/10\,\mathfrak{F} = 1.257\,\mathfrak{F}.$$

113. The distortion of the current waves, $\mathfrak{F}$, in Figs. 86 and 87, is almost entirely due to the magnetizing current, and is caused by the disproportionality between magnetic induction,

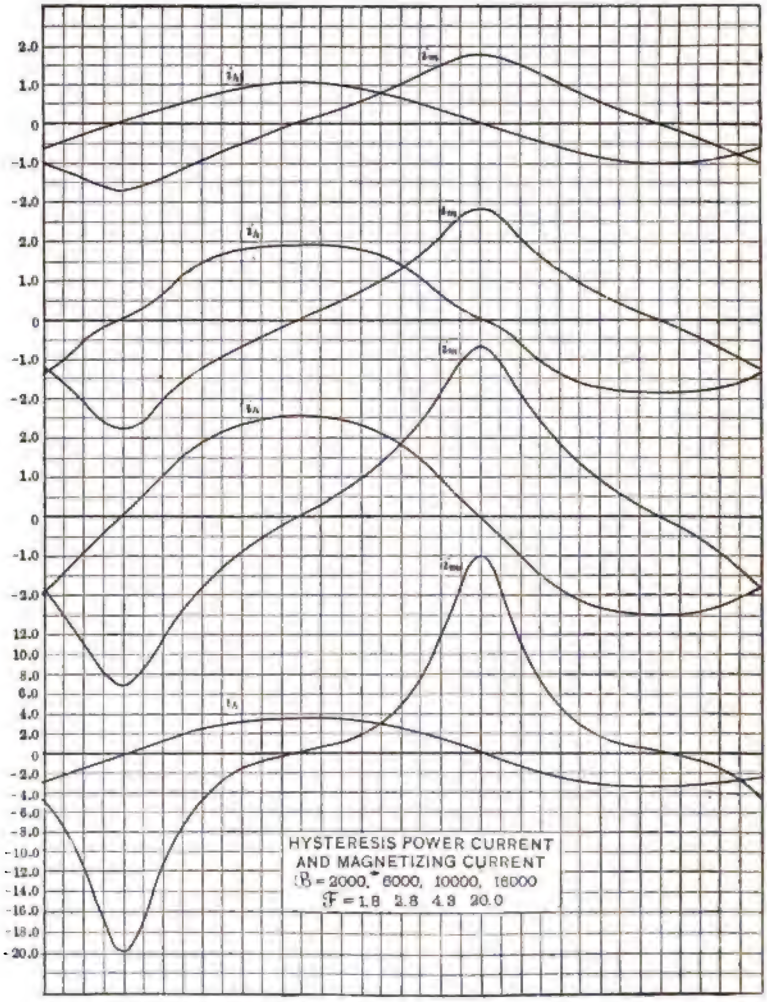


FIG. 88.

$\mathfrak{B}$, and magnetizing force, $\mathfrak{F}$, as exhibited by the magnetic characteristic or saturation curve, and is very little due to hysteresis.

Resolving these curves, $\mathcal{F}$, of Figs. 86 and 87 into two components, one in phase with the magnetic induction, $\mathcal{B}$, or symmetrical thereto, hence in quadrature with the induced e.m.f., and therefore wattless: the magnetizing current, i_m; and the other, in time quadrature with the magnetic induction, $\mathcal{B}$, hence in time-phase, or symmetrical, with the generated e.m.f., that is, representing power: the hysteresis power-current, i_h. Then we see that the hysteresis power-current, i_h, is practically a sine wave, while the magnetizing current, i_m, differs considerably from a sine wave, and tends towards peakedness — the more, the higher the magnetic induction, $\mathcal{B}$, that is, the more magnetic saturation is approached, so that for $\mathcal{B} = 16{,}000$ a very high peak is shown, and the wave of magnetizing current, i_m, does not resemble a sine wave at all, but at the maximum value is nearly four times higher than a sine wave of the same instantaneous values near zero induction would have.

These curves of hysteresis power-current, i_h, and magnetizing current, i_m, derived by resolving the distorted current curves, $\mathcal{F}$, of Figs. 86 and 87, are plotted in Fig. 88, the last one, corresponding to $\mathcal{B} = 16{,}000$, with one-quarter the ordinates of the first three.

As curves, symmetrical with regard to the maximum value of $\mathcal{B} - i_m -$, and to the zero value of $\mathcal{B} - i_h -$, these curves are constructed thus:

Let

$$b = \mathcal{B} \sin \phi = \text{sine wave of magnetic induction,}$$

then

$$i_m = \tfrac{1}{2} (\mathcal{F}_\phi + \mathcal{F}_{180-\phi}),$$

$$i_h = \tfrac{1}{2} (\mathcal{F}_\phi + \mathcal{F}_{-\phi}).$$

That is, i_m is the average value of $\mathcal{F}$ for an angle, ϕ, and its supplementary angle $180 - \phi$, i_h the average value of $\mathcal{F}$ for an angle ϕ and its negative angle $- \phi$.

114. The distortion of the wave of magnetizing current is as large as shown here only in an iron-closed magnetic circuit expending power by hysteresis only, as in an ironclad transformer on open secondary circuit. As soon as the circuit expends power in any other way, as in resistance, or by mutual

inductance, or if an air-gap is introduced in the magnetic circuit the distortion of the current wave rapidly decreases and practically disappears, and the current becomes more sinusoidal. That is, while the distorting component remains the same, the sinusoidal component of the current greatly increases, and obscures the distortion. For example, in Fig. 89, two waves are shown, corresponding in magnetization to the last curve of Fig. 86, as the one most distorted. The first curve in Fig. 89 is the current wave of a transformer at 0.1 load. At higher loads the distortion is correspondingly still less, except where the magnetic flux of self-induction, that is, flux passing between primary and secondary, and increasing in proportion to the load, is so large as to reach saturation, in which case a distortion appears again and increases with increasing load. The second curve of Fig. 89 is the exciting current of a magnetic circuit containing an air-gap whose length equals $\frac{1}{100}$ the length of the magnetic circuit. These two curves are drawn to one-third the size of the curve in Fig. 86. As shown, both curves are practically sine waves. The sine curves of magnetic flux are shown dotted as ϕ.

115. The distorted wave of current can be resolved into two components: *A true sine wave of equal effective intensity and equal power to the distorted wave*, called the *equivalent sine wave*, and a *wattless higher harmonic*, consisting chiefly of a term of triple frequency.

In Figs. 86, 87 and 89 are shown, as I, the equivalent sine waves, and as i, the difference between the equivalent sine wave and the real distorted wave, which consists of wattless complex higher harmonics. The equivalent sine wave of m.m.f. or of current, in Figs. 86 and 87, leads the magnetism in time phase by 34°, 44°, 38°, and 15°.5, respectively. In Fig. 89 the equivalent sine wave almost coincides with the distorted curve, and leads the magnetism by only 9 time-degrees.

It is interesting to note, that even in the greatly distorted curves of Figs. 86 and 87, the maximum value of the equivalent sine wave is nearly the same as the maximum value of the original distorted wave of m.m.f., so long as magnetic saturation is not approached, being 1.8, 2.9, and 4.2, respectively, against 1.8, 2.8, and 4.3, the maximum values of the distorted curve.

Since, by the definition, the effective value of the equivalent sine wave is the same as that of the distorted wave, it follows, that this distorted wave of exciting current shares with the sine wave the feature, that the maximum value and the effective value

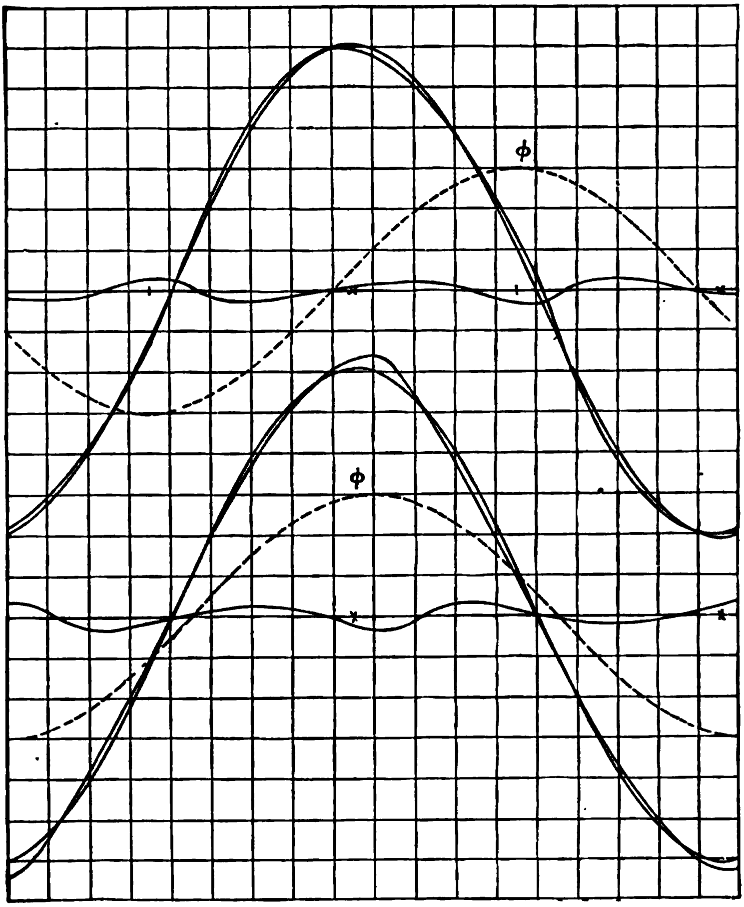


FIG. 89. — Distortion of Current Wave by Hysteresis.

have the ratio of $\sqrt{2} \div 1$. Hence, below saturation, the maximum value of the distorted curve can be calculated from the effective value — which is given by the reading of an electro-dynamometer — by using the same ratio that applies to a true

sine wave, and the magnetic characteristic can thus be determined by means of alternating currents, with sufficient exactness, by the electrodynamometer method, in the range below saturation.

116. In Fig. 90 is shown the true magnetic characteristic of a sample of average sheet iron, as found by the method of slow reversals with the magnetometer; for comparison there is shown in dotted lines the same characteristic, as determined with alternating currents by the electrodynamometer, with ampere-turns per centimeter as ordinates, and magnetic inductions as abscissas. As represented, the two curves practically coincide up to a value of $\mathfrak{B} = 13{,}000$; that is, up to the highest inductions practicable in most alternating-current apparatus. For higher saturations, the curves rapidly diverge, and the electro-dynamometer curve shows comparatively small m.m.f.'s producing apparently very high magnetizations.

The same Fig. 90 gives the curve of hysteretic loss, in ergs per cubic centimeter and cycle, as ordinates, and magnetic inductions as abscissas.

The electrodynamometer method of determining the magnetic characteristic is preferable for use with alternating-current apparatus, since it is not affected by the phenomenon of magnetic "creeping," which, especially at low densities, may in the magnetometer tests bring the magnetism very much higher, or the m.m.f. lower, than found in practice in alternating-current apparatus.

So far as current strength and power consumption are concerned, the distorted wave can be replaced by the equivalent sine wave and the higher harmonics neglected.

All the measurements of alternating currents, with the single exception of instantaneous readings, yield the equivalent sine wave only, since all measuring instruments give either the mean square of the current wave, or the mean product of instantaneous values of current and e.m.f., which, by definition, are the same in the equivalent sine wave as in the distorted wave.

Hence, in all practical applications, it is permissible to neglect the higher harmonic altogether, and replace the distorted wave by its equivalent sine wave, keeping in mind,

however, the existence of a higher harmonic as a possible disturbing factor which may become noticeable in those cases where the frequency of the higher harmonic is near the fre-

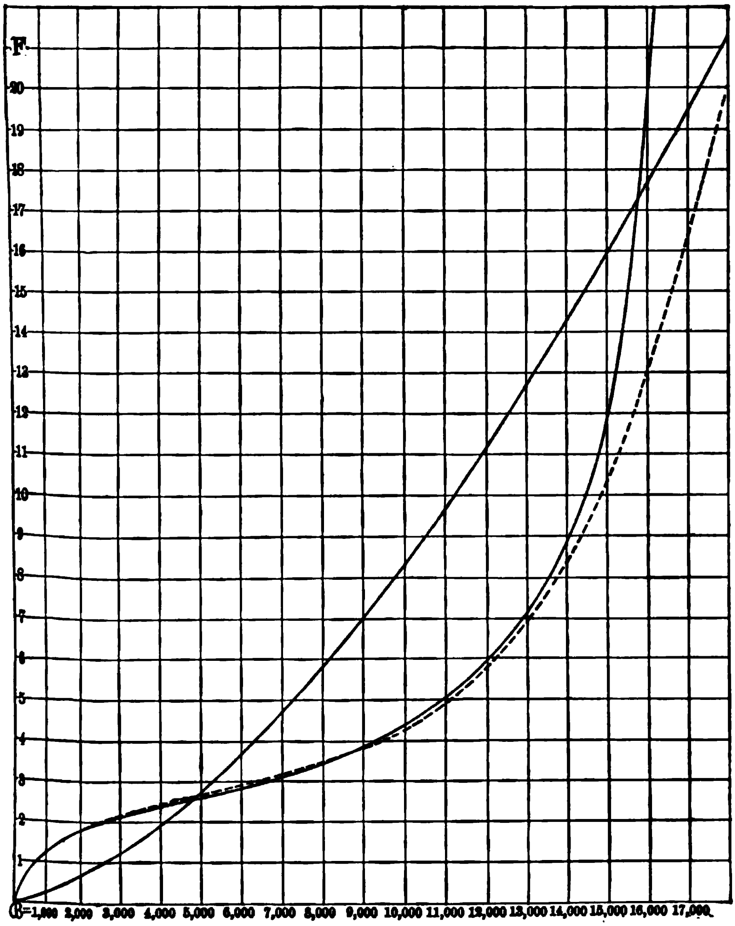


FIG. 90. — Magnetization and Hysteresis Curve.

quency of resonance of the circuit, that is, in circuits containing condensive as well as inductive reactance, or in those circuits in which the higher harmonic of current is suppressed, and thereby the voltage is distorted, as discussed in Chapter XXIX.

117. The equivalent sine wave of exciting current leads the sine wave of magnetism by an angle α, which is called the *angle of hysteretic advance of phase.* Hence the current lags behind the e.m.f. by the time angle $(90° - \alpha)$, and the power is therefore,

$$P = IE \cos (90° - \alpha) = IE \sin \alpha.$$

Thus the exciting current, I, consists of a power component, $I \sin \alpha$, called the *hysteretic or magnetic power component*, and a wattless component, $I \cos \alpha$, which is called the *magnetizing current.* Or, conversely, the e.m.f. consists of a power component, $E \sin \alpha$, the *hysteretic power component,* and a wattless component, $E \cos \alpha$, the e.m.f. consumed by *self-induction.*

Denoting the absolute value of the impedance of the circuit, $\frac{E}{I}$, by z, — where z is determined by the magnetic characteristic of the iron, and the shape of the magnetic and electric circuits, — the impedance is represented, in phase and intensity, by the symbolic expression,

$$Z = r - jx = z \sin \alpha - jz \cos \alpha;$$

and the admittance by,

$$Y = g + jb = \frac{1}{z} \sin \alpha + j \frac{1}{z} \cos \alpha = y \sin \alpha + jy \cos \alpha.$$

The quantities, z, r, x, and y, g, b, are, however, not constants as in the case of the circuit without iron, but depend upon the intensity of magnetization, $\mathfrak{B}$, — that is, upon the e.m.f. This dependence complicates the investigation of circuits containing iron.

In a circuit entirely inclosed by iron, α is quite considerable, ranging from 30° to 50° for values below saturation. Hence, even with negligible true ohmic resistance, no great lag can be produced in ironclad alternating-current circuits.

118. The loss of energy by hysteresis due to molecular friction is, with sufficient exactness, proportional to the 1.6^{th} power

of magnetic induction, $\mathfrak{B}$. Hence it can be expressed by the formula:

$$W_H = \eta \mathfrak{B}^{1.6}$$

where —

W_H = loss of energy per cycle, in ergs or (c.g.s.) units (= 10^{-7} joules) per cubic centimeter,

$\mathfrak{B}$ = maximum magnetic induction, in lines of force per sq. cm., and η = the *coefficient of hysteresis*.

This I found to vary in iron from .001 to .0055. As a safe mean, 0.0033* can be accepted for average annealed sheet iron or sheet steel. In gray cast iron, η averages 0.013; it varies from 0.0032 to 0.028 in cast steel, according to the chemical or physical constitution; and reaches values as high as 0.08 in hardened steel (tungsten and manganese steel). Soft nickel and cobalt have about the same coefficient of hysteresis as gray cast iron; in magnetite I found η = 0.023.

In the curves of Figs. 85 to 90, η = 0.0033.

At the frequency, f, the loss of power in the volume, V, is, by this formula,

$$P = \eta f V \mathfrak{B}^{1.6} 10^{-7} \text{ watts}$$

$$= \eta f V \left(\frac{\Phi}{A}\right)^{1.6} 10^{-7} \text{ watts},$$

where A is the cross-section of the total magnetic flux, Φ.

The maximum magnetic flux, Φ, depends upon the counter e.m.f. of self-induction,

$$E = \sqrt{2}\, \pi f n \Phi\, 10^{-8},$$

or

$$\Phi = \frac{E\, 10^8}{2\, \pi f n},$$

where n = number of turns of the electric circuit.

* At present, with the improvements in the production and selection of sheet steel for alternating apparatus, 0.002 can be considered a fair average in selected materials, and values considerably below 0.001 have been reached in silicon steel (1908).

Substituting this in the value of the power, P, and canceling, we get,

$$P = \eta \frac{E^{1.6}}{f^{0.6}} \frac{V\, 10^{5.8}}{2^{.8}\, \pi^{1.6} A^{1.6} n^{1.6}} = 58\, \eta \frac{E^{1.6}}{f^{.6}} \frac{V\, 10^{3}}{A^{1.6} n^{1.6}};$$

or $$P = \frac{KE^{1.6}}{f^{0.6}}, \text{ where } K = \eta \frac{V\, 10^{5.8}}{2^{.8} \pi^{1.6} A^{1.6} n^{1.6}} = 58\, \eta \frac{V\, 10^{3}}{A^{1.6} n^{1.6}};$$

or, substituting $\eta = .0033$, we have $K = 191.4 \dfrac{V}{A^{1.6} n^{1.6}}$;

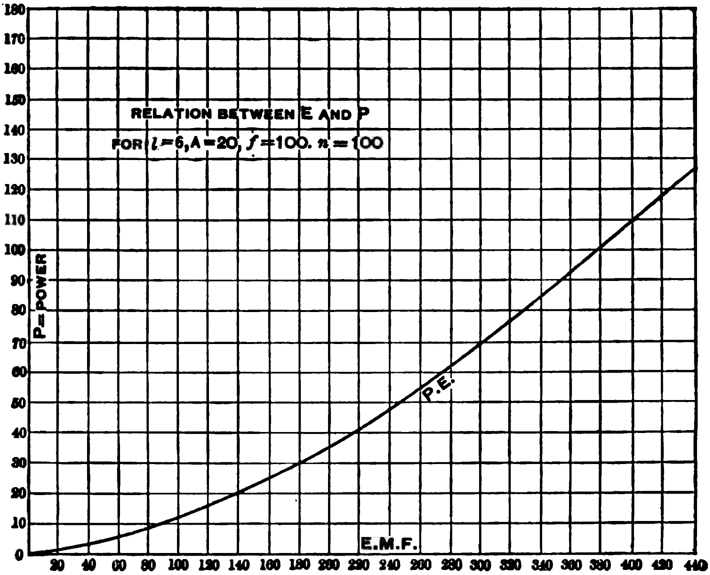


FIG. 91. — Hysteresis Loss as Function of E.M.F.

or, substituting $V = Al$, where l = length of magnetic circuit,

$$K = \frac{\eta l\, 10^{5.8}}{2^{.8} \pi^{1.6} A^{.6} n^{1.6}} = \frac{58\, \eta l\, 10^{3}}{A^{.6} n^{1.6}} = 191.4 \frac{l}{A^{.6} n^{1.6}};$$

and $$P = \frac{58\, \eta E^{1.6} l\, 10^{3}}{f^{.6} A^{.6} n^{1.6}} = \frac{191.4\, E^{1.6} l}{f^{.6} A^{.6} n^{1.6}}.$$

In Figs. 91, 92, and 93, is shown a curve of hysteretic loss, with the loss of power as ordinates, and

in curve 91, with the e.m.f., E, as abscissas,

for $l = 6$, $A = 20$, $f = 100$, and $n = 100$;

in curve 92, with the number of turns as abscissas, for

$l = 6$, $A = 20$, $f = 100$, and $E = 100$;

in curve 93, with the frequency, f, or the cross-section, A, as

abscissas, for $l = 6$, $n = 100$, and $E = 100$.

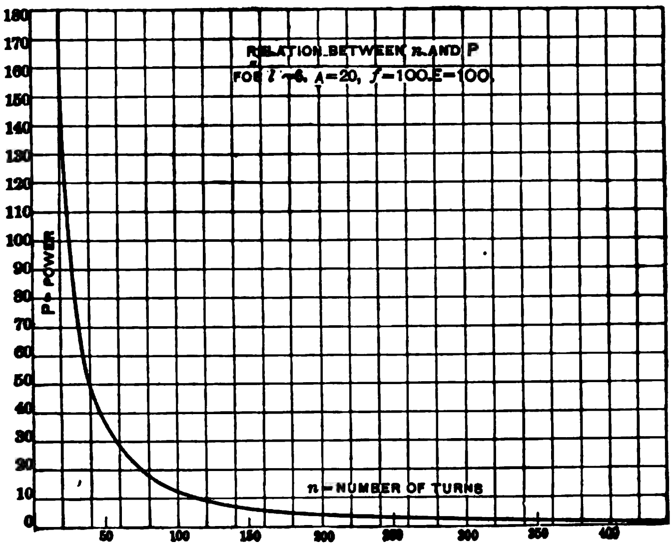


FIG. 92. — Hysteresis Loss as Function of Number of Turns.

As shown, the hysteretic loss is proportional to the 1.6th power of the e.m.f., inversely proportional to the 1.6th power of the number of turns, and inversely proportional to the 0.6th power of the frequency, and of the cross-section.

119. If g = effective conductance, the power component of a current is $I = Eg$, and the power consumed in a conductance, g, is $P = IE = E^2g$.

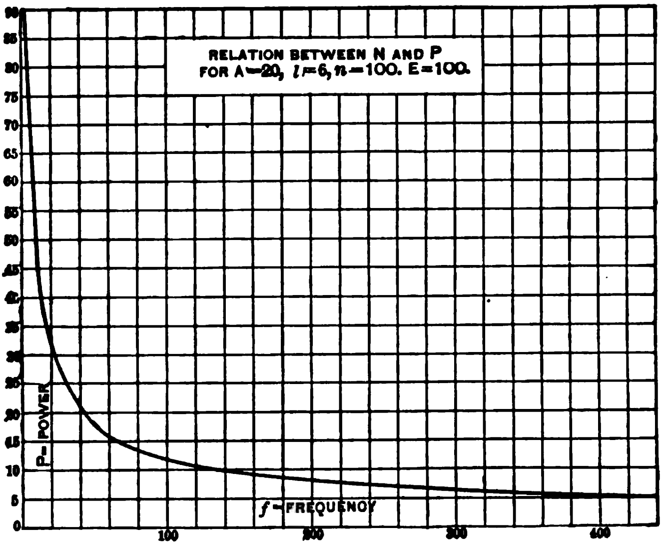


Fig. 93. — Hysteresis Loss as Function of Cycles.

Since, however,

$$P = K \frac{E^{1.6}}{f^{.6}}, \text{ we have } K \frac{E^{1.6}}{f^{.6}} = E^2 g;$$

or

$$g = \frac{K}{f^{.6}E^{.4}} = \frac{58\,\eta l\, 10^3}{E^{.4}f^{.6}A^{.6}n^{1.6}} = 191.4 \frac{l}{E^{.4}f^{.6}A^{.6}n^{1.6}}.$$

From this we have the following deduction:

The effective conductance due to magnetic hysteresis is proportional to the coefficient of hysteresis, η, and to the length of the magnetic circuit, l, and inversely proportional to the 0.4*th power of the e.m.f., to the* 0.6*th power of the frequency, f, and of the cross-section of the magnetic circuit, A, and to the* 1.6*th power of the number of turns, n.*

Hence, the effective hysteretic conductance increases with decreasing e.m.f., and decreases with increasing e.m.f.; it varies, however, much slower than the e.m.f., so that, if the hysteretic conductance represents only a part of the total power consumption, it can, within a limited range of variation — as, for instance, in constant-potential transformers — be assumed as constant without serious error.

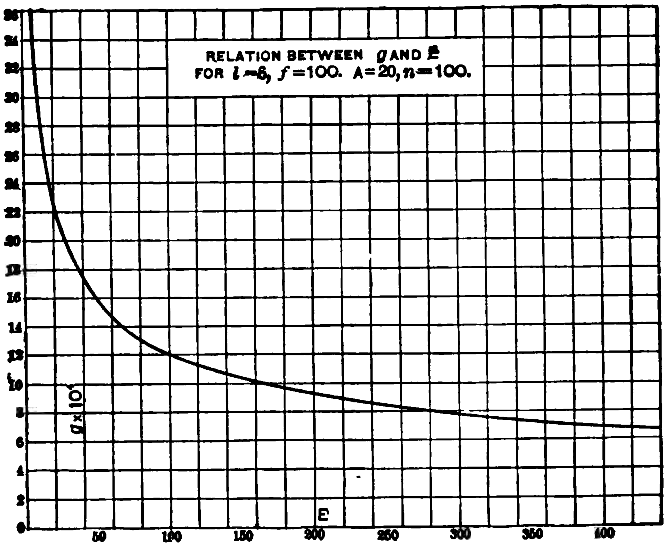


Fig. 94. — Hysteresis Conductance as Function of E.M.F.

In Figs. 94, 95, and 96, the hysteretic conductance, g, is plotted, for $l = 6$, $E = 100$, $f = 100$, $A = 20$ and $n = 100$, respectively, with the conductance, g, as ordinates, and with

E as abscissas in Curve 94.

f as abscissas in Curve 95.

n as abscissas in Curve 96.

As shown, a variation in the e.m.f. of 50 per cent causes a variation in g of only 14 per cent, while a variation in f or A by 50 per cent causes a variation in g of 21 per cent.

If $\mathcal{R}$ = magnetic reluctance of a circuit, $\mathfrak{F}_A$ = maximum m.m.f., I = effective current, since $I\sqrt{2}$ = maximum current, the magnetic flux,

$$\Phi = \frac{\mathfrak{F}_A}{\mathcal{R}} = \frac{nI\sqrt{2}}{\mathcal{R}}.$$

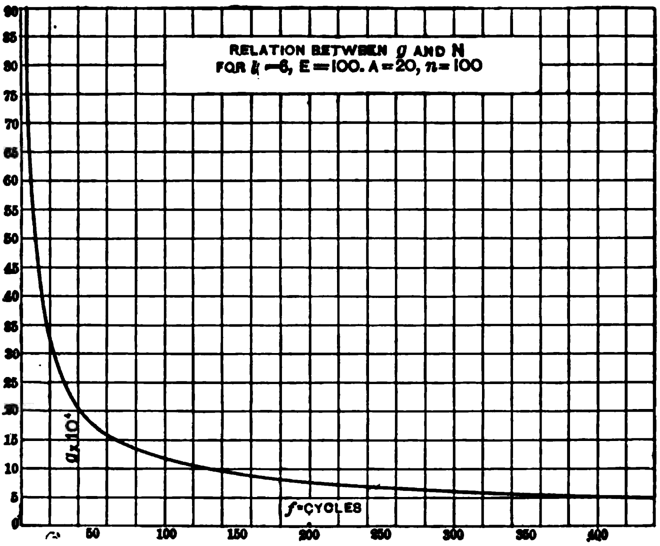


FIG. 95. — Hysteresis Conductance as Function of Cycles.

Substituting this in the equation of the counter e.m.f. of self-induction,

$$E = \sqrt{2}\,\pi f n \Phi\, 10^{-8},$$

we have

$$E = \frac{2\,\pi n^2 f I\, 10^{-8}}{\mathcal{R}};$$

hence, the absolute admittance of the circuit is

$$y = \sqrt{g^2 + b^2} = \frac{I}{E} = \frac{\mathcal{R}\, 10^8}{2\,\pi n^2 f} = \frac{a\mathcal{R}}{f},$$

where

$$a = \frac{10^8}{2\,\pi n^2}, \text{ a constant.}$$

Therefore, the absolute admittance, y, of a circuit of negligible resistance is proportional to the magnetic reluctance, $\mathcal{R}$, and inversely proportional to the frequency, f, and to the square of the number of turns, n.

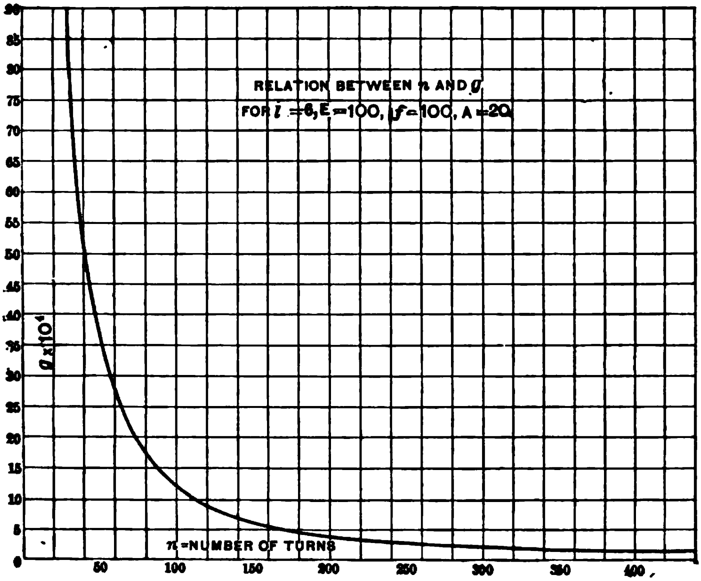


FIG. 96. — Hysteresis Conductance as Function of Number of Turns.

120. In a circuit containing iron, the reluctance, $\mathcal{R}$, varies with the magnetization; that is, with the e.m.f. Hence the admittance of such a circuit is not a constant, but is also variable.

In an ironclad electric circuit, — that is, a circuit whose magnetic field exists entirely within iron, such as the magnetic circuit of a well-designed alternating-current transformer, — $\mathcal{R}$ is the reluctance of the iron circuit. Hence, if μ = permeability, since

$$\mathcal{R} = \frac{\mathcal{F}_A}{\Phi},$$

and

$$\mathcal{F}_A = lF = \frac{10}{4\pi} l\mathcal{H} = \text{m.m.f.},$$

$$\Phi = A\mathcal{B} = \mu A\mathcal{H} = \text{magnetic flux},$$

and

$$\mathcal{R} = \frac{10\, l}{4\, \pi \mu A};$$

substituting this value in the equation of the admittance,

$$y = \frac{\mathfrak{B}\, 10^8}{2\,\pi n^2 f},$$

we have
$$y = \frac{l\, 10^9}{8\,\pi^2 n^2 \mu A f} = \frac{c}{f\mu},$$

where
$$c = \frac{l\, 10^9}{8\,\pi^2 n^2 A} = \frac{127\, l\, 10^{\cancel{7}\,5}}{n^2 A}.$$

Therefore, in an ironclad circuit, the absolute admittance, y, is inversely proportional to the frequency, f, to the permeability, μ, to the cross-section, A, and to the square of the number of turns, n; and directly proportional to the length of the magnetic circuit, l.

The conductance is
$$g = \frac{K}{f^{.6} E^{.4}};$$

and the admittance,
$$y = \frac{c}{f\mu};$$

hence, the angle of hysteretic advance is

$$\sin\alpha = \frac{g}{y} = \frac{K\mu f^{.4}}{cE^{.4}};$$

or, substituting for A and c (§ 119),

$$\sin\alpha = \mu\,\frac{f^{.4}}{E^{.4}}\;\frac{\eta l\, 10^{5.8}}{2^{.8}\pi^{1.6} A^{.6} n^{1.6}}\;\frac{8\,\pi^2 n^2 A}{l\, 10^9}$$
$$= \frac{\mu\eta f^{.4} n^{.4} A^{.4} \pi^{.4}\, 2^{2.2}}{E^{.4}\, 10^{3.2}};$$

or, substituting
$$E = 2^{.5}\,\pi f n A \mathfrak{B}\, 10^{-8},$$

we have
$$\sin\alpha = \frac{4\,\mu\eta}{\mathfrak{B}^{.4}},$$

which is independent of frequency, number of turns, and shape and size of the magnetic and electric circuit.

Therefore, in an ironclad inductance, the angle of hysteretic advance, α, depends upon the magnetic constants, permeability

and coefficient of hysteresis, and upon the maximum magnetic induction, but is entirely independent of the frequency, of the shape and other conditions of the magnetic and electric circuit; and, therefore, all ironclad magnetic circuits constructed of the same quality of iron and using the same magnetic density, give the same angle of hysteretic advance.

The angle of hysteretic advance, α, in a closed circuit transformer, depends upon the quality of the iron, and upon the magnetic density only.

The sine of the angle of hysteretic advance equals 4 times the product of the permeability and coefficient of hysteresis, divided by the 0.4^{th} power of the magnetic density.

121. If the magnetic circuit is not entirely ironclad, and the magnetic structure contains air-gaps, the total reluctance is the sum of the iron reluctance and of the air reluctance, or

$$\mathcal{R} = \mathcal{R}_i + \mathcal{R}_a;$$

hence the admittance is

$$y = \sqrt{g^2 + b^2} = \frac{a}{f}(\mathcal{R}_i + \mathcal{R}_a).$$

Therefore, in a circuit containing iron, the admittance is the sum of the admittance due to the iron part of the circuit, $y_i = \frac{a\mathcal{R}_i}{f}$, and of the admittance due to the air part of the circuit, $y_a = \frac{a\mathcal{R}_a}{f}$, if the iron and the air are in series in the magnetic circuit.

The conductance, g, represents the loss of power in the iron, and, since air has no magnetic hysteresis, is not changed by the introduction of an air-gap. Hence the angle of hysteretic advance of phase is

$$\sin \alpha = \frac{g}{y} = \frac{g}{y_i + y_a} = \frac{g}{y_i} \frac{\mathcal{R}_i}{\mathcal{R}_i + \mathcal{R}_a},$$

and a maximum, $\frac{g}{y_i}$, for the ironclad circuit, but decreases with increasing width of the air-gap. The introduction of the air-gap of reluctance, $\mathcal{R}_a$, decreases $\sin \alpha$ in the ratio,

$$\frac{\mathcal{R}_i}{\mathcal{R}_i + \mathcal{R}_a}.$$

In the range of practical application, from $\mathfrak{B} = 2{,}000$ to $\mathfrak{B} = 12{,}000$, the permeability of iron varies between 900 and 2,000 approximately, while sin α in an ironclad circuit varies in this range from 0.51 to 0.69. In air, $\mu = 1$.

If, consequently, one per cent of the length of the iron consists of an air-gap, the total reluctance only varies through the above range of densities in the proportion of $1\frac{1}{9}$ to $1\frac{1}{20}$, or about 6 per cent, that is, remains practically constant; while the angle of hysteretic advance varies from $\sin \alpha = 0.035$ to $\sin \alpha = 0.064$. Thus g is negligible compared with b, and b is practically equal to y.

Therefore, in an electric circuit containing iron, but forming an open magnetic circuit whose air-gap is not less than $\frac{1}{100}$ the length of the iron, the susceptance is practically constant and equal to the admittance, so long as saturation is not yet approached, or,

$$b = \frac{\mathfrak{R}_a}{f}, \text{ or: } x = \frac{f}{\mathfrak{R}_a}.$$

The angle of hysteretic advance is small, below 4°, and the hysteretic conductance is

$$g = \frac{K}{E^{.4} f^{.6}}.$$

The current wave is practically a sine wave.

As an example, in Fig. 89, Curve II, the current curve of a circuit is shown, containing an air-gap of only $\frac{1}{400}$ of the length of the iron, giving a current wave much resembling the sine shape, with an hysteretic advance of 9°.

122. To determine the electric constants of a circuit containing iron, we shall proceed in the following way:

Let

$$E = \text{counter e.m.f. of self-induction;}$$

then from the equation,

$$E = \sqrt{2}\, \pi n f \Phi 10^{-8},$$

where f = frequency, n = number of turns,
we get the magnetism, Φ, and by means of the magnetic cross-section, A, the maximum magnetic induction: $\mathfrak{B} = \frac{\Phi}{A}$.

From $\mathfrak{B}$, we get, by means of the magnetic characteristic of the iron, the m.m.f., $= \mathfrak{F}$ ampere-turns per centimeter length, where

$$\mathfrak{F} = \frac{10}{4\pi}\mathfrak{H},$$

if $\mathfrak{H}$ = m.m.f. in c.g.s. units.

Hence,

if l_i = length of iron circuit, $\mathfrak{F}_i = l_i \mathfrak{F}$ = ampere-turns required in the iron;

if l_a = length of air circuit, $\mathfrak{F}_a = \frac{10\, l_a \mathfrak{B}}{4\pi}$ = ampere-turns required in the air;

hence, $\mathfrak{F} = \mathfrak{F}_i + \mathfrak{F}_a$ = total ampere-turns, maximum value, and $\frac{\mathfrak{F}}{\sqrt{2}}$ = effective value. The exciting current is

$$I = \frac{\mathfrak{F}}{n\sqrt{2}},$$

and the absolute admittance,

$$y = \sqrt{g^2 + b^2} = \frac{I}{E}.$$

If $\mathfrak{F}_i$ is not negligible as compared with $\mathfrak{F}_a$, this admittance, y, is variable with the e.m.f., E.

If V = volume of iron, η = coefficient of hysteresis, the loss of power by hysteresis due to molecular magnetic friction is

$$P = \eta f V \mathfrak{B}^{1.6};$$

hence the hysteretic conductance is $g = \frac{P}{E^2}$, and variable with the e.m.f., E.

The angle of hysteretic advance is

$$\sin \alpha = \frac{g}{y};$$

the susceptance, $b = \sqrt{y^2 - g^2};$

the effective resistance, $r = \frac{g}{y^2};$

and the reactance, $x = \frac{b}{y^2}.$

123. As conclusions, we derive from this chapter the following:

(1) In an alternating-current circuit surrounded by iron, the current produced by a sine wave of e.m.f. is not a true sine wave, but is distorted by hysteresis, and inversely, a sine wave of current requires waves of magnetism and e.m.f. differing from sine shape.

(2) This distortion is excessive only with a closed magnetic circuit transferring no energy into a secondary circuit by mutual inductance.

(3) The distorted wave of current can be replaced by the equivalent sine wave, — that is, a sine wave of equal effective intensity and equal power, — and the superposed higher harmonic, consisting mainly of a term of triple frequency, may be neglected except in resonating circuits.

(4) Below saturation, the distorted curve of current and its equivalent sine wave have approximately the same maximum value.

(5) The angle of hysteretic advance — that is, the phase difference between the magnetic flux and equivalent sine wave of m.m.f. — is a maximum for the closed magnetic circuit, and depends there only upon the magnetic constants of the iron, upon the permeability, μ, the coefficient of hysteresis, η, and the maximum magnetic induction, as shown in the equation,

$$\sin \alpha = \frac{4 \mu \eta}{\mathfrak{B}^{.4}}.$$

(6) The effect of hysteresis can be represented by an admittance, $Y = g + jb$, or an impedance, $Z = r - jx$.

(7) The hysteretic admittance, or impedance, varies with the magnetic induction; that is, with the e.m.f., etc.

(8) The hysteretic conductance, g, is proportional to the coefficient of hysteresis, η, and to the length of the magnetic circuit, l, inversely proportional to the 0.4^{th} power of the e.m.f., E, to the 0.6^{th} power of frequency, f, and of the cross-section of the magnetic circuit, A, and to the 1.6^{th} power of the number of turns of the electric circuit, n, as expressed in the equation,

$$g = \frac{58\, \eta l\, 10^3}{E^{.4} f^{.6} A^{.6} n^{1.6}}.$$

(9) The absolute value of hysteretic admittance,

$$y = \sqrt{g^2 + b^2},$$

is proportional to the magnetic reluctance: $\mathcal{R} = \mathcal{R}_i + \mathcal{R}_a$, and inversely proportional to the frequency, f, and to the square of the number of turns, n, as expressed in the equation,

$$y = \frac{(\mathcal{R}_i + \mathcal{R}_a)\, 10^8}{2\, \pi f n^2}.$$

(10) In an ironclad circuit, the absolute value of admittance is proportional to the length of the magnetic circuit, and inversely proportional to cross-section, A, frequency, f, permeability, μ, and square of the number of turns, n, or

$$y_i = \frac{127\, l\, 10^{6}}{n^2 A f \mu}.$$

(11) In an open magnetic circuit, the conductance, g, is the same as in a closed magnetic circuit of the same iron part.

(12) In an open magnetic circuit, the admittance, y, is practically constant, if the length of the air-gap is at least $\frac{1}{100}$ of the length of the magnetic circuit, and saturation be not approached.

(13) In a closed magnetic circuit, conductance, susceptance, and admittance can be assumed as constant through a limited range only.

(14) From the shape and the dimensions of the circuits, and the magnetic constants of the iron, all the electric constants, g, b, y; r, x, z, can be calculated.

CHAPTER XIV.

FOUCAULT OR EDDY CURRENTS.

124. While magnetic hysteresis due to molecular friction is a magnetic phenomenon, eddy currents are rather an electrical phenomenon. When iron passes through a magnetic field, a loss of energy is caused by hysteresis, which loss, however, does not react magnetically upon the field. When cutting an electric conductor, the magnetic field produces a current therein. The m.m.f. of this current reacts upon and affects the magnetic field, more or less; consequently, an alternating magnetic field cannot penetrate deeply into a solid conductor, but a kind of screening effect is produced, which makes solid masses of iron unsuitable for alternating fields, and necessitates the use of laminated iron or iron wire as the carrier of magnetic flux.

Eddy currents are true electric currents, though existing in minute circuits; and they follow all the laws of electric circuits.

Their e.m.f. is proportional to the intensity of magnetization, $\mathfrak{B}$, and to the frequency, f.

Eddy currents are thus proportional to the magnetization, $\mathfrak{B}$, the frequency, f, and to the electric conductivity, λ, of the iron; hence, can be expressed by

$$i = b\lambda\mathfrak{B}f.$$

The power consumed by eddy currents is proportional to their square, and inversely proportional to the electric conductivity, and can be expressed by

$$P = b\lambda\mathfrak{B}^2f^2;$$

or, since $\mathfrak{B}f$ is proportional to the generated e.m.f., E, in the equation

$$E = \sqrt{2}\,\pi A n f \mathfrak{B}\, 10^{-8},$$

it follows that, *The loss of power by eddy currents is proportional to the square of the e.m.f., and proportional to the electric conductivity of the iron*; or,

$$P = aE^2\lambda.$$

Hence, that component of the effective conductance which is due to eddy currents is

$$g = \frac{P}{E^2} = a\lambda;$$

that is, *The equivalent conductance due to eddy currents in the iron is a constant of the magnetic circuit; it is independent of e.m.f., frequency, etc., but proportional to the electric conductivity of the iron,* λ.

125. Eddy currents, like magnetic hysteresis, cause an advance of phase of the current by an *angle of advance*, β; but, unlike hysteresis, eddy currents in general do not distort the current wave.

The angle of advance of phase due to eddy currents is

$$\sin \beta = \frac{g}{y},$$

where y = absolute admittance of the circuit, g = eddy current conductance.

While the equivalent conductance, g, due to eddy currents, is a constant of the circuit, and independent of e.m.f., frequency, etc., the loss of power by eddy currents is proportional to the square of the e.m.f. of self-induction, and therefore proportional to the square of the frequency and to the square of the magnetization.

Only the power component, gE, of eddy currents, is of interest, since the wattless component is identical with the wattless component of hysteresis, discussed in the preceding chapter.

126. To calculate the loss of power by eddy currents,

Let V = volume of iron;
$\mathfrak{B}$ = maximum magnetic induction;
f = frequency;
λ = electric conductivity of iron;
ϵ = coefficient of eddy currents.

The loss of energy per cubic centimeter, in ergs per cycle, is

$$w = \epsilon\lambda f \mathcal{B}^2;$$

hence, the total loss of power by eddy currents is

$$P = \epsilon\lambda V f^2 \mathcal{B}^2\, 10^{-7} \text{ watts},$$

and the equivalent conductance due to eddy currents is

$$g = \frac{P}{E^2} = \frac{10\,\epsilon\lambda l}{2\pi^2 A n^2} = \frac{0.507\,\epsilon\lambda l}{A n^2},$$

where

l = length of magnetic circuit,
A = section of magnetic circuit,
n = number of turns of electric circuit.

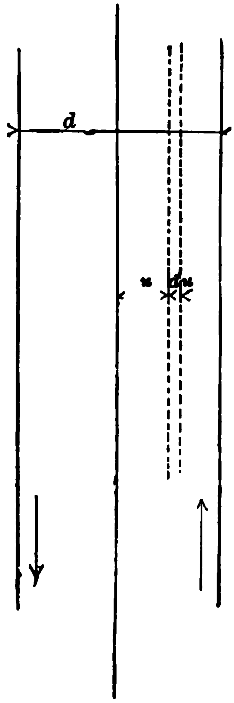


FIG. 97.

The coefficient of eddy currents, ϵ, depends merely upon the shape of the constituent parts of the magnetic circuit; that is, whether of iron plates or wire, and the thickness of plates or the diameter of wire, etc.

The two most important cases are:

(a) Laminated iron.
(b) Iron wire.

127. (a) *Laminated Iron.*

Let, in Fig. 97,

d = thickness of the iron plates;
$\mathcal{B}$ = maximum magnetic induction;
f = frequency;
λ = electric conductivity of the iron.

Then, if u is the distance of a zone, du, from the center of the sheet, the conductance of a zone of thickness, du, and of one centimeter length and width is λdu; and the magnetic flux cut by this zone is $\mathcal{B}u$. Hence, the e.m.f. induced in this zone is

$$\delta E = \sqrt{2}\,\pi f \mathcal{B} u, \text{ in c.g.s. units.}$$

This e.m.f. produces the current, $dI = \delta E\,\lambda du = \sqrt{2}\,\pi f \mathcal{B}\,u du$, in c.g.s. units, provided the thickness of the plate is negligible as compared with the length, in order that the current may be assumed as parallel to the sheet, and in opposite directions on opposite sides of the sheet.

The power consumed by the current in this zone, du, is

$$dP = \delta E dI = 2\,\pi^2 f^2 \mathcal{B}^2 \lambda u^2 du,$$

in c.g.s. units or ergs per second, and, consequently, the total power consumed in one square centimeter of the sheet of thickness, d, is

$$\delta P = \int_{-\frac{d}{2}}^{+\frac{d}{2}} dP = 2\,\pi^2 f^2 \mathcal{B}^2 \lambda \int_{-\frac{d}{2}}^{+\frac{d}{2}} u^2 du$$

$$= \frac{\pi^2 f^2 \quad {}^2\lambda d^3}{6}, \text{ in c.g.s. units;}$$

the power consumed per cubic centimeter of iron is, therefore,

$$p = \frac{\delta P}{d} = \frac{\pi^2 f^2 \mathcal{B}^2 \lambda\, d^2}{6}, \text{ in c.g.s. units or erg-seconds,}$$

and the energy consumed per cycle and per cubic centimeter of iron is

$$w = \frac{p}{f} = \frac{\pi^2 \lambda\, d^2 f \mathcal{B}^2}{6} \text{ ergs.}$$

The coefficient of eddy currents for laminated iron is, therefore,

$$\epsilon = \frac{\pi^2 d^2}{6} = 1.645\, d^2,$$

where λ is expressed in c.g.s. units. Hence, if λ is expressed in practical units or 10^{-9} c.g.s. units,

$$\epsilon = \frac{\pi^2\, d^2\, 10^{-9}}{6} = 1.645\, d^2\, 10^{-9}.$$

Substituting for the conductivity of sheet iron the approximate value,

$$\lambda = 10^5,^*$$

we get as the coefficient of eddy currents for laminated iron,

$$\epsilon = \frac{\pi^2}{6} d^2\, 10^{-9} = 1.645\, d^2\, 10^{-9};$$

loss of energy per cubic centimeter and cycle,

$$W = \epsilon \lambda f \mathfrak{B}^2 = \frac{\pi^2}{6} d^2 \lambda f \mathfrak{B}^2\, 10^{-9} = 1.645\, d^2 \lambda f \mathfrak{B}^2\, 10^{-9} \text{ ergs}$$

$$= 1.645\, d^2 f \mathfrak{B}^2\, 10^{-4} \text{ ergs};$$

or, $W = \epsilon \lambda f \mathfrak{B}^2\, 10^{-7} = 1.645\, d^2 f \mathfrak{B}^2\, 10^{-11}$ joules.

The loss of power per cubic centimeter at frequency, f, is

$$p = fW = \epsilon \lambda f^2 \mathfrak{B}^2\, 10^{-7} = 1.645\, d^2 f^2 \mathfrak{B}^2\, 10^{-11} \text{ watts};$$

the total loss of power in volume, V, is.

$$P = Vp = 1.645\, V d^2 f^2 \mathfrak{B}^2\, 10^{-11} \text{ watts}.$$

As an example,

$d = 1$ mm. $= 0.1$ cm.; $f = 100$; $\mathfrak{B} = 5{,}000$; $V = 1{,}000$ cu. cm.;
$\epsilon = 1{,}645 \times 10^{-11}$;
$W = 4{,}110$ ergs
$= 0.000411$ joules;
$p = 0.0411$ watts:
$P = 41.1$ watts.

128. (*b*) *Iron Wire.*

Let, in Fig. 98, d = diameter of a piece of iron wire; then if u is the radius of a circular zone of thickness, du, and one cm. in length, the conductance of this zone is $\frac{\lambda\, du}{2\, \pi u}$, and the magnetic flux inclosed by the zone is $\mathfrak{B} u^2 \pi$.

* In some of the modern silicon steels used for transformer iron, λ reaches values as low as 2×10^4, and even lower; and the eddy current losses are reduced in the same proportion (1908).

Hence, the e.m.f. generated in this zone is

$$\delta E = \sqrt{2}\,\pi^2 f \mathfrak{B} u^2 \text{ in c.g.s. units,}$$

and the current produced thereby is

$$dI = \frac{\lambda\, du}{2\,\pi u} \times \sqrt{2}\,\pi^2 f \mathfrak{B} u^2$$

$$= \frac{\sqrt{2}\,\pi}{2} \lambda f \mathfrak{B} u\, du, \text{ in c.g.s. units.}$$

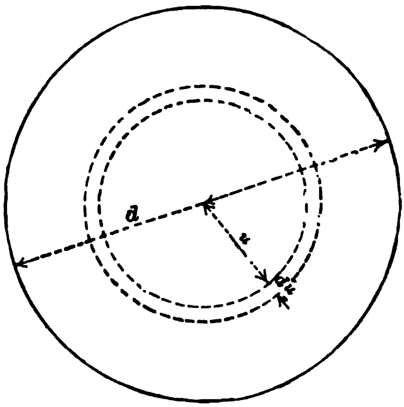


FIG. 98.

The power consumed in this zone is, therefore,

$$dP = \delta E dI = \pi^3 \lambda f^2 \mathfrak{B}^2 u^3 du, \text{ in c.g.s. units;}$$

consequently, the total power consumed in one centimeter length of wire is

$$\delta P = \int_0^{\frac{d}{2}} dW = \pi^3 \lambda f^2 \mathfrak{B}^2 \int_0^{\frac{d}{2}} u^3 du$$

$$= \frac{\pi^3}{64} \lambda f^2 \mathfrak{B}^2 d^4, \text{ in c.g.s. units.}$$

Since the volume of one centimeter length of wire is

$$v = \frac{d^2\pi}{4},$$

the power consumed in one cubic centimeter of iron is

$$p = \frac{\delta P}{v} = \frac{\pi^2}{16}\lambda f^2 \mathfrak{B}^2 d^2, \text{ in c.g.s. units or erg-seconds,}$$

and the energy consumed per cycle and cubic centimeter of iron is

$$W = \frac{p}{f} = \frac{\pi^2}{16}\lambda f \mathfrak{B}^2 d^2 \text{ ergs.}$$

Therefore, the coefficient of eddy currents for iron wire is

$$\varepsilon = \frac{\pi^2}{16} d^2 = 0.617\, d^2;$$

or, if λ is expressed in practical units, or 10^{-9} c.g.s. units,

$$\varepsilon = \frac{\pi^2}{16} d^2\, 10^{-9} = 0.617\, d^2\, 10^{-9}.$$

Substituting

$$\lambda = 10^5,$$

we get as the coefficient of eddy currents for iron wire,

$$\varepsilon = \frac{\pi^2}{16} d^2\, 10^{-9} = 0.617\, d^2\, 10^{-9}.$$

The loss of energy per cubic centimeter of iron, and per cycle becomes

$$\begin{aligned} W = \varepsilon\lambda f \mathfrak{B}^2 &= \frac{\pi^2}{16} d^2 \lambda f \mathfrak{B}^2\, 10^9 = 0.617\, d^2 \lambda f \mathfrak{B}^2\, 10^{-9} \\ &= 0.617\, d^2 f \mathfrak{B}^2\, 10^{-4} \text{ ergs,} \\ &= \varepsilon\lambda f \mathfrak{B}^2\, 10^{-7} = 0.617\, d^2 f \mathfrak{B}^2\, 10^{-11} \text{ joules;} \end{aligned}$$

loss of power per cubic centimeter at frequency, f,

$$p = fW = \varepsilon\lambda N^2 \mathfrak{B}^2\, 10^{-7} = 0.617\, d^2 N^2 \mathfrak{B}^2\, 10^{-11} \text{ watts;}$$

total loss of power in volume, V,

$$P = Vp = 0.617\, V d^2 f^2 \mathfrak{B}^2\, 10^{-11} \text{ watts.}$$

As an example,

$d = 1$ mm., $= 0.1$ cm.; $f = 100$; $\mathfrak{B}^2 = 5{,}000$; $V = 1{,}000$ cu. cm.

Then,

$$\begin{aligned} \epsilon &= 0.617 \times 10^{-11}, \\ W &= 1{,}540 \text{ ergs} = 0.000154 \text{ joules}, \\ p &= 0.0154 \text{ watts}, \\ P &= 15.4 \text{ watts}, \end{aligned}$$

hence very much less than in sheet iron of equal thickness.

129. *Comparison of sheet iron and iron wire.*

If

d_1 = thickness of lamination of sheet iron, and
d_2 = diameter of iron wire,

the eddy current coefficient of sheet iron being

$$\epsilon_1 = \frac{\pi^2}{6} d_1^2 10^{-9},$$

and the eddy current coefficient of iron wire

$$\epsilon_2 = \frac{\pi^2}{16} d_2^2 10^{-9},$$

the loss of power is equal in both — other things being equal - if $\epsilon_1 = \epsilon_2$; that is, if

$$d_2^2 = \tfrac{8}{3} d_1^2, \text{ or } d_2 = 1.63\, d_1.$$

It follows that the diameter of iron wire can be 1.63 times, or, roughly, $1\frac{2}{3}$ as large as the thickness of laminated iron, to give the same loss of power through eddy currents, as shown in Fig. 99.

130. *Demagnetizing, or screening effect of eddy currents.*

The formulas derived for the coefficient of eddy currents in laminated iron and in iron wire, hold only when the eddy currents are small enough to neglect their magnetizing force. Other-

wise the phenomenon becomes more complicated; the magnetic flux in the interior of the lamina, or the wire, is not in phase with the flux at the surface, but lags behind it. The magnetic flux at the surface is due to the impressed m.m.f., while the flux in the interior is due to the resultant of the impressed m.m.f. and to the m.m.f. of eddy currents; since the eddy currents lag 90 time-degrees behind the flux producing them, their resultant with the impressed m.m.f., and therefore the magnetism in the interior, is made lagging. Thus, progressing from the surface

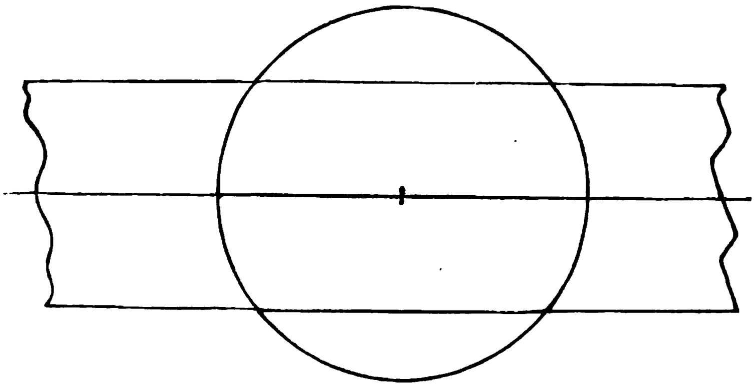


FIG. 99.

towards the interior, the magnetic flux gradually lags more and more in phase, and at the same time decreases in intensity. While the complete analytical solution of this phenomenon is beyond the scope of this book, a determination of the magnitude of this demagnetization, or screening effect, sufficient to determine whether it is negligible, or whether the subdivision of the iron has to be increased to make it negligible, can be made by calculating the maximum magnetizing effect, which cannot be exceeded by the eddys.

Assuming the magnetic density as uniform over the whole cross-section, and therefore all the eddy currents in phase with each other, their total m.m.f. represents the maximum possible value, since by the phase difference and the lesser magnetic density in the center the resultant m.m.f. is reduced.

In laminated iron of thickness d, the current in a zone of thickness du, at distance u from center of sheet, is

$$dI = \sqrt{2}\,\pi f \mathfrak{B} \lambda u\, du \text{ units (c.g.s.)}$$
$$= \sqrt{2}\,\pi f \mathfrak{B} \lambda u\, du\, 10^{-8} \text{ amperes;}$$

hence the total current in the sheet is

$$I = \int_0^{\frac{d}{2}} dI = \sqrt{2}\,\pi f \mathfrak{B} \lambda\, 10^{-8} \int_0^{\frac{d}{2}} u\, du$$
$$= \frac{\sqrt{2}\,\pi}{8} f \mathfrak{B} \lambda d^2\, 10^{-8} \text{ amperes.}$$

Hence, the maximum possible demagnetizing ampere-turns, acting upon the center of the lamina, are

$$I = \frac{\sqrt{2}\,\pi}{8} f \mathfrak{B} \lambda d^2\, 10^{-8} = 0.555\, f \mathfrak{B} \lambda d^2\, 10^{-8},$$
$$= 0.555\, f \mathfrak{B} \lambda d^2\, 10^{-8} \text{ ampere-turns per cm.}$$

Example: $d = 0.1$ cm., $f = 100$, $\mathfrak{B} = 5{,}000$, $\lambda = 10^5$,

or $I = 2.775$ ampere-turns per cm.

131. In iron wire of diameter, d, the current in a tubular zone of du thickness and u radius is

$$dI = \frac{\sqrt{2}}{2} \pi f \mathfrak{B} \lambda u\, du 10^{-8} \text{ amperes;}$$

hence, the total current is

$$I = \int_0^{\frac{d}{2}} dI = \frac{\sqrt{2}}{2} \pi f \mathfrak{B} \lambda 10^{-8} \int_0^{\frac{d}{2}} u\, dx$$
$$= \frac{\sqrt{2}}{16} \pi f \mathfrak{B} \lambda d^2\, 10^{-8} \text{ amperes.}$$

Hence, the maximum possible demagnetizing ampere-turns, acting upon the center of the wire, are

$$I = \frac{\sqrt{2}\,\pi}{16} f \mathfrak{B} \lambda d^2\, 10^{-8} = 0.2775\, f \mathfrak{B} \lambda d^2\, 10^{-8}$$
$$= 0.2775\, f \mathfrak{B} \lambda d^2\, 10^{-8} \text{ ampere-turns per cm.}$$

For example, if $d = 0.1$ cm., $f = 100$, $\mathfrak{B} = 5{,}000$, $\lambda = 10^5$, then $I = 1{,}338$ ampere-turns per cm.; that is, half as much as in a lamina of the thickness, d.

For a more complete investigation of the screening effect of eddy currents in laminated iron, see Section III of "Theory and Calculation of Transient Electric Phenomena and Oscillations."

132. Besides the eddy, or Foucault, currents proper, which exist as parasitic currents in the interior of the iron lamina or wire, under certain circumstances eddy currents also exist in larger orbits from lamina to lamina through the whole magnetic structure. Obviously a calculation of these eddy currents is possible only in a particular structure. They are mostly surface currents, due to short circuits existing between the laminæ at the surface of the magnetic structure.

Furthermore, eddy currents are produced outside of the magnetic iron circuit proper, by the magnetic stray field cutting electric conductors in the neighborhood, especially when drawn towards them by iron masses behind, in electric conductors passing through the iron of an alternating field, etc. All these phenomena can be calculated only in particular cases, and are of less interest, since they can and should be avoided.

The power consumed by such large eddy currents frequently increases more than proportional to the square of the voltage, when approaching magnetic saturation, by the magnetic stray field reaching unlaminated conductors, and so, while negligible at normal voltage, this power may become large at over normal voltage.

Eddy Currents in Conductor, and Unequal Current Distribution.

133. If the electric conductor has a considerable size, the alternating magnetic field, in cutting the conductor, may set up differences of potential between the different parts thereof, thus giving rise to local or eddy currents in the copper. This phenomenon can obviously be studied only with reference to a particular case, where the shape of the conductor and the distribution of the magnetic field are known.

Only in the case where the magnetic field is produced by the current in the conductor can a general solution be given. The

alternating current in the conductor produces a magnetic field, not only outside of the conductor, but inside of it also; and the lines of magnetic force which close themselves inside of the conductor generate e.m.fs. in their interior only. Thus the counter e.m.f. of self-induction is largest at the axis of the conductor, and least at its surface; consequently, the current density at the surface will be larger than at the axis, or, in extreme cases, the current may not penetrate at all to the center, or a reversed current may exist there. Hence it follows that only the exterior part of the conductor may be used for the conduction of electricity, thereby causing an increase of the ohmic resistance due to unequal current distribution.

The general discussion of this problem, as applicable to the distribution of alternating current in very large conductors, as the iron rails of the return circuit of alternating-current railways, is given in Section III of " Theory and Calculation of Transient Electric Phenomena and Oscillations."

In practice, this phenomenon is observed mainly with very high-frequency currents, as lightning discharges; in power distribution circuits it has to be avoided by either keeping the frequency sufficiently low, or having a shape of conductor such that unequal current-distribution does not take place, as by using a tubular or a flat conductor, or several conductors in parallel.

134. It will, therefore, here be sufficient to determine the largest size of round conductor, or the highest frequency, where this phenomenon is still negligible.

In the interior of the conductor, the current density is not only less than at the surface, but the current lags in time-phase behind the current at the surface, due to the increased effect of self-induction. This time-lag of the current causes the magnetic fluxes in the conductor to be out of phase with each other, making their resultant less than their sum, while the lesser current density in the center reduces the total flux inside of the conductor. Thus, by assuming, as a basis for calculation, a uniform current density and no difference of phase between the currents in the different layers of the conductor, the unequal distribution is found larger than it is in reality. Hence this assumption brings us on the safe side, and at the same time

greatly simplifies the calculation; however, it is permissible only where the current density is still fairly uniform.

Let Fig. 100 represent a cross-section of a conductor of radius, R, and a uniform current density,

$$i = \frac{I}{R^2\pi},$$

where I = total current in conductor.

The magnetic reluctance of a tubular zone of unit length and thickness du, of radius u, is

$$\mathcal{R}_u = \frac{2u\pi}{du}.$$

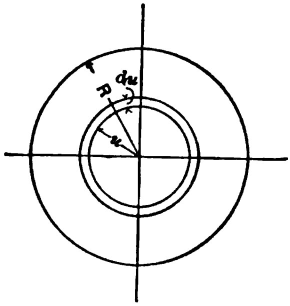


FIG. 100.

The current inclosed by this zone is $I_u = iu^2\pi$, and therefore, the m.m.f. acting upon this zone is

$$\mathcal{F}_u = 0.4\,\pi I_u = 0.4\,\pi^2 iu^2,$$

and the magnetic flux in this zone is

$$d\Phi = \frac{\mathcal{F}_u}{\mathcal{R}_u} = 0.2\,\pi iu\,du.$$

Hence, the total magnetic flux inside the conductor is

$$\Phi = \int_0^R d\Phi = \frac{2\,\pi}{10}\,i\int_0^R u\,du = \frac{\pi i R^2}{10} = \frac{I}{10}.$$

From this we get, as the excess of counter e.m.f. at the axis of the conductor over that at the surface,

$$\Delta E = \sqrt{2}\,\pi f \Phi\, 10^{-8} = \sqrt{2}\,\pi f I\, 10^{-9}, \text{ per unit length,}$$
$$= \sqrt{2}\,\pi^2 f i R^2\, 10^{-9};$$

and the reactivity, or specific reactance at the center of the conductor, becomes $k = \dfrac{\Delta E}{i} = \sqrt{2}\,\pi^2 f R^2\, 10^{-9}$.

Let ρ = resistivity, or specific resistance, of the material of the conductor.

We have then, $$\frac{k}{\rho} = \frac{\sqrt{2}\,\pi^2 f R^2\, 10^{-9}}{\rho};$$

and $$\frac{\rho}{\sqrt{k^2 + \rho^2}},$$

the ratio of current densities at center and at periphery.

For example, if, in copper, $\rho = 1.7 \times 10^{-6}$, and the percentage decrease of current density at center shall not exceed 5 per cent, that is,

$$\rho \div \sqrt{k^2 + \rho^2} = 0.95 \div 1,$$

we have $$k = 0.51 \times 10^{-6};$$

hence $$0.51 \times 10^{-6} = \sqrt{2}\,\pi^2 f R^2\, 10^{-9},$$

or $$fR^2 = 36.6;$$

hence, when	$f =$	125	100	60	25
	$R =$	0.541	0.605	0.781	1.21 cm.
	$D = 2\,R =$	1.08	1.21	1.56	2.42 cm.

Hence, even at a frequency of 125 cycles, the effect of unequal current distribution is still negligible at one centimeter diameter of the conductor. Conductors of this size are, however, excluded from use at this frequency by the external self-induction, which is several times larger than the resistance. We thus see that unequal current distribution is usually negligible in practice.

The above calculation was made under the assumption that the conductor consists of unmagnetic material. If this is not the case, but the conductor of iron of permeability μ, then $d\Phi = \frac{\mu \mathfrak{F}_u}{\mathfrak{R}_u}$; and thus ultimately, $k = \sqrt{2}\,\pi^2 f \mu R^2\, 10^{-9}$, and $\frac{k}{\rho} = \sqrt{2}\,\pi^2 \frac{f\mu R^2\, 10^{-9}}{\rho}$. Thus, for instance, for iron wire at $\rho = 10 \times 10^{-6}$, $\mu = 500$, it is, permitting 5 per cent difference between center and outside of wire, $k = 3.2 \times 10^{-6}$, and $fR^2 = 0.46$;

hence, when	$f =$	125	100	60	25
	$R =$	0.061	0.068	0.088	0.136 cm.;

thus the effect is noticeable even with relatively small iron wire.

Mutual Induction.

135. When an alternating magnetic field of force includes a secondary electric conductor, it generates therein an e.m.f. which produces a current, and thereby consumes energy if the circuit of the secondary conductor is closed.

Particular cases of such secondary currents are the eddy or Foucault currents previously discussed.

Another important case is the generation of secondary e.m.fs. in neighboring circuits; that is, the interference of circuits running parallel with each other.

In general, it is preferable to consider this phenomenon of mutual induction as not merely producing a power component and a wattless component of e.m.f. in the primary conductor, but to consider explicitly both the secondary and the primary circuit, as will be done in the chapter on the alternating-current transformer.

Only in cases where the energy transferred into the secondary circuit constitutes a small part of the total primary energy, as in the discussion of the disturbance caused by one circuit upon a parallel circuit, may the effect on the primary circuit be considered analogously as in the chapter on eddy currents by the introduction of a power component, representing the loss of power, and a wattless component, representing the decrease of self-induction.

Let

$x = 2\pi f L$ = reactance of main circuit; that is, L = total number of interlinkages with the main conductor, of the lines of magnetic force produced by unit current in that conductor;

$x_1 = 2\pi f L_1$ = reactance of secondary circuit; that is, L_1 = total number of interlinkages with the secondary conductor, of the lines of magnetic force produced by unit current in that conductor;

$x_m = 2\pi f L_m$ = mutual inductive reactance of the circuits; that is, L_m = total number of interlinkages with the secondary conductor, of the lines of magnetic force produced by unit current in the main conductor, or total number of interlinkages

with the main conductor of the lines of magnetic force produced by unit current in the secondary conductor.

Obviously: $$x_m^2 \leqq x x_1.*$$

Let r_1 = resistance of secondary circuit. Then the impedance of secondary circuit is

$$Z_1 = r_1 - j x_1, \qquad z_1 = \sqrt{r_1^2 + x_1^2};$$

$$\text{e.m.f. generated in the secondary circuit, } \dot{E}_1 = j x_m \dot{I},$$

where $\dot{I}$ = primary current. Hence, the secondary current is

$$\dot{I}_1 = \frac{\dot{E}_1}{z_1} = \frac{j x_m}{r_1 - j x_1} \dot{I};$$

and the e.m.f. generated in the primary circuit by the secondary current, $\dot{I}_1$, is

$$\dot{E} = j x_m \dot{I}_1 = \frac{-x_m^2}{r_1 - j x_1} \dot{I};$$

or, expanded,

$$\dot{E} = \left\{ \frac{-x_m^2 r_1}{r_1^2 + x_1^2} - \frac{j x_m^2 x_1}{r_1^2 + x_2^1} \right\} \dot{I}.$$

* As self-inductance L, L_1, the total flux surrounding the conductor is here meant. Usually in the discussion of inductive apparatus, especially of transformers, that part of the magnetic flux which surrounds one circuit is denoted as the self-inductance of this circuit, but not the other circuit; that is, it is the flux which passes between both circuits. Hence, the total self-inductance, L, is in this case equal to the sum of the self-inductance, L_1, and mutual inductance, L_m.

The object of this distinction is to separate the wattless part, L_1, of the total self-inductance, L, from that part, L_m, which represents the transfer of e.m.f. into the secondary circuit, since the action of these two components is essentially different.

Thus, in alternating-current transformers it is customary — and will be done later in this book — to denote as the self-inductance, L, of each circuit only that part of the magnetic flux produced by the circuit which passes between both circuits, and thus acts in "choking" only, but not in transforming; while the flux surrounding both circuits is called the mutual inductance, or useful magnetic flux.

With this denotation, in transformers the mutual inductance, L_m, is usually very much greater than the self-inductance, L', and L_1', while, if the self-inductance, L and L_1, represent the total flux, their product is larger than the square of the mutual inductance, L_m; or

$$L L_1 \geqq L_m^2; \qquad (L' + L_m)(L_1' + L_m) \geqq L_m^2.$$

Hence, the e.m.f. consumed thereby,

$$\dot{E}' = \left\{ \frac{x_m^2 r_1}{r_1^2 + x_1^2} + \frac{j x_m^2 x_1}{r_1^2 + x_1^2} \right\} \dot{I} = (r - jx)\, \dot{I}.$$

$$r = \frac{x_m^2 r_1}{r_1^2 + x_1^2} = \text{effective resistance of mutual inductance;}$$

$$x = \frac{-x_m^2 x_1}{r_1^2 + x_1^2} = \text{effective reactance of mutual inductance.}$$

The susceptance of mutual inductance is negative, or of opposite sign from the reactance of self-inductance. Or,

Mutual inductance consumes energy and decreases the self-inductance.

For the calculation of the mutual inductance between circuits L_m, see "Theoretical Elements of Electrical Engineering," 3rd Ed.

Dielectric and Electrostatic Phenomena.

136. While magnetic hysteresis and eddy currents can be considered as the power component of inductive reactance, condensive reactance has a power component also; namely, dielectric hysteresis. In an alternating magnetic field, energy is consumed in hysteresis due to molecular friction; and similarly, energy is also consumed in an alternating electrostatic field in the dielectric medium, in what is called electrostatic or dielectric hysteresis.

While the laws of the loss of energy by magnetic hysteresis are fairly well understood, and the magnitude of the effect known, the phenomenon of dielectric hysteresis is still almost entirely unknown as concerns its laws and the magnitude of the effect.

It is quite probable that the loss of power in the dielectric in an alternating electrostatic field consists of two distinctly different components, of which the one is directly proportional to the frequency, — analogous to magnetic hysteresis, and thus a constant loss of energy per cycle, independent of the frequency; while the other component is proportional to the square of the

frequency, — analogous to the loss of power by eddy currents in the iron, and thus a loss of energy per cycle proportional to the frequency.

The existence of a loss of power in the dielectric, proportional to the square of the frequency and voltage, I observed some years ago in paraffined paper in a high electrostatic field and at high frequency, by the electrodynamometer method.

Power-factor measurements of paraffined paper condensers made for frequencies of 25 to 125 cycles, and over a wide range of voltages, above and below the safe operating voltage of the dielectric, show no marked variation of the power-factor with the voltage or the frequency, that is, point towards a power consumption proportional to the square of voltage and of frequency, of a magnitude of between ½ per cent and 1 per cent of the volt-amperes absorbed by the condenser.

Arno of Turin found at low frequencies and low field strength in a large number of dielectrics, a loss of energy per cycle independent of the frequency, but proportional to the 1.6th power of the field strength, — that is, following the same law as the magnetic hysteresis,

$$W_{\mathfrak{H}} = \eta \mathcal{B}^{1.6}.$$

This loss, probably true dielectric static hysteresis, was observed under conditions such that a loss proportional to the square of density and frequency must be small, while at high densities and frequencies, as in condensers, the true dielectric hysteresis may be entirely obscured by a viscous loss, represented by $W_{\mathfrak{H}} = \epsilon f \mathcal{B}^2$.

137. If the loss of power by electrostatic hysteresis is proportional to the square of the frequency and of the field intensity, — as it probably nearly is under the working conditions of alternating-current condensers, — then it is proportional to the square of the e.m.f., that is, the effective conductance, g, due to dielectric hysteresis is a constant; and, since the condenser susceptance, $-b = b'$, is a constant also, — unlike the magnetic inductive reactance, — the ratio of conductance and susceptance, that is, the angle of difference of phase due to dielectric hysteresis, is a constant. This I proved by ex-

periment. This would mean that the dielectric hysteretic admittance of a condenser,

$$Y = g + jb = g - jb',$$

where g = hysteretic conductance, b' = hysteretic susceptance; and the dielectric hysteretic impedance of a condenser,

$$Z = r - jx = r + jx_c,$$

where r = hysteretic resistance, x_c = hysteretic reactance; and the angle of dielectric hysteretic lag, $\tan \alpha = \frac{b'}{g} = \frac{x_c}{r}$, are constants of the circuit, independent of e.m.f. and frequency. The e.m.f. is obviously inversely proportional to the frequency.

The true static dielectric hysteresis, observed by Arno as proportional to the 1.6^{th} power of the density, will enter the admittance and the impedance as a term variable and dependent upon e.m.f. and frequency, in the same manner as discussed in the chapter on magnetic hysteresis.

To the magnetic hysteresis corresponds, in the electrostatic field, the static component of dielectric hysteresis, following, probably, the same law of 1.6^{th} power.

To the eddy currents in the iron corresponds, in the electrostatic field, the viscous component of dielectric hysteresis, following the square law.

As a rule, however, these hysteresis losses in the alternating electrostatic field of a condenser are very much smaller than the losses in an alternating magnetic field, so that while the latter exert a very marked effect on the design of apparatus, representing frequently the largest of all the losses of energy, the dielectric losses are so small as to be very difficult to observe. In alternating-current condensers the losses frequently are only a fraction of 1 per cent of the volt-ampere input, from 25 to 125 cycles, and therefore very difficult to observe.

To the phenomenon of mutual induction corresponds, in the electrostatic field, the electrostatic induction, or influence.

138. The alternating electrostatic field of force of an electric circuit induces, in conductors within the field of force, electrostatic charges by what is called electrostatic influence. These

charges are proportional to the field strength; that is, to the e.m.f. in the main circuit.

If a current is produced by the induced charges, energy is consumed proportional to the square of the charge; that is, to the square of the e.m.f.

These induced charges, reacting upon the main conductor, influence therein charges of equal but opposite phase, and hence lagging behind the main e.m.f. by the angle of lag between induced charge and inducing field. They require the expenditure of a charging current in the main conductor in quadrature with the induced charge thereon; that is, nearly in quadrature with the e.m.f., and hence consisting of a power component in phase with the e.m.f. — representing the power consumed by electrostatic influence — and a wattless component, which increases the capacity of the conductor, or, in other words, reduces its condensive reactance.

Thus, the electrostatic influence introduces an effective conductance, g, and an effective susceptance, b, — of the same sign with condenser susceptance, — into the equations of the electric circuit.

While theoretically g and b should be constants of the circuit, frequently they are very far from such, due to disruptive phenomena beginning to appear at high electrostatic stresses.

Even the condensive reactance changes at very high potentials; escape of electricity into the air and over the surfaces of the supporting insulators by brush discharge or electrostatic glow takes place. As far as this electrostatic corona reaches, the space is in electric connection with the conductor, and thus the capacity of the circuit is determined, not by the surface of the metallic conductor, but by the exterior surface of the electrostatic glow surrounding the conductor. This means that with increasing potential, the capacity increases as soon as the electrostatic corona appears; hence, the condensive reactance decreases, and at the same time a power component appears, representing the loss of power in the corona.

This phenomenon thus shows some analogy with the decrease of magnetic inductive reactance due to saturation.

At moderate potentials, the condensive reactance due to capacity can be considered as a constant, consisting of a watt-

less component, the condensive reactance proper, and a power component, the dielectric hysteresis.

The condensive reactance of a polarization cell, however, begins to decrease already at very low potentials, as soon as the counter e.m.f. of chemical dissociation is approached, at about 1.4 volts.

The condensive reactance of the aluminum cell, that is, a pair of aluminum plates in an electrolyte which does not attack aluminum, with increasing voltage, first increases and then decreases again, at a value of from 100 to 600 volts, and very greatly depends upon the previous history of the cell.

The condensive reactance of a synchronized alternator is of the nature of a variable quantity; that is, the effective reactance changes gradually, according to the relation of impressed and of counter e.m.f., from inductive over zero to condensive reactance.

Besides the phenomena discussed in the foregoing as terms of the power components and the wattless components of current and of e.m.f., the electric leakage is to be considered as a further power component; that is, the direct escape of electricity from conductor to return conductor through the surrounding medium, due to imperfect insulating qualities. This leakage current represents an effective conductance, g, theoretically independent of the e.m.f., but in reality frequently increasing greatly with the e.m.f., owing to the decrease of the insulating strength of the medium upon approaching the limits of its disruptive strength.

139. In the foregoing, the phenomena causing loss of energy in an alternating-current circuit have been discussed; and it has been shown that the mutual relation between current and e.m.f. can be expressed by two of the four constants:

power component of e.m.f., in phase with current, and = current $\times$ effective resistance, or r;

wattless component of e.m.f., in quadrature with current, and = current $\times$ effective reactance, or x;

power component of current, in phase with e.m.f., and = e.m.f. $\times$ effective conductance, or g;

wattless component of current, in quadrature with e.m.f., and = e.m.f. $\times$ effective susceptance, or b.

In many cases the exact calculation of the quantities, r, x, g, b, is not possible in the present state of the art.

In general, r, x, g, b, are not constants of the circuit, but depend — besides upon the frequency — more or less upon e.m.f., current, etc. Thus, in each particular case it becomes necessary to discuss the variation of r, x, g, b, or to determine whether, and through what range, they can be assumed as constant.

In what follows, the quantities r, x, g, b, will always be considered as the coefficients of the power and wattless components of current and e.m.f., — that is, as the *effective* quantities, — so that the results are directly applicable to the general electric circuit containing iron and dielectric losses.

Introducing now, in Chapters VIII. to XII., instead of "ohmic resistance," the term "effective resistance," etc., as discussed in the preceding chapter, the results apply also — within the range discussed in the preceding chapter — to circuits containing iron and other materials producing energy losses outside of the electric conductor.

CHAPTER XV.

POWER, AND DOUBLE-FREQUENCY QUANTITIES IN GENERAL.

140. Graphically, alternating currents and e.m.fs. are represented by vectors, of which the length represents the intensity, the direction the phase of the alternating wave. The vectors generally issue from the center of coordinates.

In the topographical method, however, which is more convenient for complex networks, as interlinked polyphase circuits, the alternating wave is represented by the straight line between two points, these points representing the absolute values of potential (with regard to any reference point chosen as coordinate center), and their connection the difference of potential in phase and intensity.

Algebraically these vectors are represented by complex quantities. The impedance, admittance, etc., of the circuit is a complex quantity also, in symbolic denotation.

Thus current, e.m.f., impedance, and admittance are related by multiplication and division of complex quantities in the same way as current, e.m.f., resistance, and conductance are related by Ohm's law in direct-current circuits.

In direct-current circuits, power is the product of current into e.m.f. In alternating-current circuits, if

$$\dot{E} = e^1 + je^{11},$$

$$\dot{I} = i^1 + ji^{11}.$$

The product,

$$P_0 = \dot{E}\dot{I} = (e^1 i^1 - e^{11} i^{11}) + j\,(e^{11} i^1 + e^1 i^{11}),$$

is not the power; that is, multiplication and division, which are correct in the inter-relation of current, e.m.f., impedance, do not give a correct result in the inter-relation of e.m.f., current, power. The reason is, that $\dot{E}$ and $\dot{I}$ are vectors of the same fre-

quency, and Z a constant numerical factor which thus does not change the frequency.

The power, P, however, is of double frequency compared with $\dot{E}$ and $\dot{I}$, that is, makes a complete wave for every half wave of $\dot{E}$ or $\dot{I}$, and thus cannot be represented by a vector in the same diagram with $\dot{E}$ and $\dot{I}$.

$P_0 = \dot{E}\dot{I}$ is a quantity of the same frequency with $\dot{E}$ and $\dot{I}$, and thus cannot represent the power.

141. Since the power is a quantity of double frequency of $\dot{E}$ and $\dot{I}$, and thus a phase angle, θ, in $\dot{E}$ and $\dot{I}$ corresponds to a phase angle, $2\,\theta$, in the power, it is of interest to investigate the product, $\dot{E}\dot{I}$, formed by doubling the phase angle.

Algebraically it is,

$$\begin{aligned} \dot{P} &= \dot{E}\dot{I} = (e^1 + je^{11})(i^1 + ji^{11}) \\ &= (e^1i^1 + j^2e^{11}i^{11}) + (je^{11}i^1 + e^1ji^{11}). \end{aligned}$$

Since $j^2 = -1$, that is, 180° rotation for $\dot{E}$ and $\dot{I}$, for the double-frequency vector, P, $j^2 = +1$, or 360° rotation, and

$$\begin{aligned} j \times 1 &= j, \\ 1 \div j &= -j. \end{aligned}$$

That is, multiplication with j reverses the sign, since it denotes a rotation by 180° for the power, corresponding to a rotation of 90° for $\dot{E}$ and $\dot{I}$.

Hence, substituting these values, we have

$$\dot{P} = [\dot{E}\dot{I}] = (e^1i^1 + e^{11}i^{11}) + j\,(e^{11}i^1 - e^1i^{11}).$$

The symbol $[\dot{E}\dot{I}]$ here denotes the transfer from the frequency of $\dot{E}$ and $\dot{I}$ to the double frequency of $\dot{P}$.

The product, $\dot{P} = [\dot{E}\dot{I}]$, consists of two components: the real component,

$$P^1 = [\dot{E}\dot{I}]^1 = (e^1i^1 + e^{11}i^{11});$$

and the imaginary component,

$$jP^j = j\,[\dot{E}\dot{I}]^j = j\,(e^{11}i^1 - e^1i^{11}).$$

The component,

$$P^1 = [\dot{E}\dot{I}]^1 = (e^1i^1 + e^{11}i^{11}),$$

is the true or "effective" power of the circuit, $= EI \cos(\dot{E}\dot{I})$.

The component,

$$P^j = [\dot{E}\dot{I}]^j = (e^{11}i^1 - e^1i^{11}),$$

is what may be called the "reactive power," or the wattless or quadrature volt-amperes of the circuit, $= EI \sin(\dot{E}\dot{I})$.

The real component will be distinguished by the index 1; the imaginary or wattless component by the index, j.

By introducing this symbolism, the power of an alternating circuit can be represented in the same way as in the direct-current circuit, as the symbolic product of current and e.m.f.

Just as the symbolic expression of current and e.m.f. as complex quantity does not only give the mere intensity, but also the phase,

$$\dot{E} = e^1 + je^{11}$$

$$E = \sqrt{e^{1^2} + e^{11^2}}$$

$$\tan\theta = \frac{e^{11}}{e^1},$$

so the double-frequency vector product $P = [\dot{E}\dot{I}]$ denotes more than the mere power, by giving with its two components, $P^1 = [\dot{E}\dot{I}]^1$ and $P^j = [\dot{E}\dot{I}]^j$, the true power volt-ampere, or "effective power," and the wattless volt-amperes, or "reactive power."

If

$$\dot{E} = e^1 + je^{11},$$
$$\dot{I} = i^1 + ji^{11};$$

then

$$E = \sqrt{e^{1^2} + e^{11^2}},$$

$$I = \sqrt{i^{1^2} + i^{11^2}},$$

and

$$P^1 = [\dot{E}\dot{I}]^1 = (e^1i^1 + e^{11}i^{11}),$$
$$P^j = [\dot{E}\dot{I}]^j = (e^{11}i^1 - e^1i^{11}),$$

or

$$P^{1^2} + P^{j^2} = e^{1^2}i^{1^2} + e^{11^2}i^{11^2} + e^{11^2}i^{1^2} + e^{1^2}i^{11^2}$$
$$= (e^{1^2} + e^{11^2})(i^{1^2} + i^{11^2}) = (EI)^2 = P_a^2$$

where P_a = total volt-amperes of circuit. That is,

The effective power, P^1, and the reactive power, P^j, are the two rectangular components of the total apparent power, P_a, of the circuit.

Consequently,

In symbolic representation as double-frequency vector products, powers can be combined and resolved by the parallelogram of vectors just as currents and e.m.fs. in graphical or symbolic representation.

The graphical methods of treatment of alternating-current phenomena are here extended to include double-frequency quantities, as power, torque, etc.

$$p = \frac{P^1}{P_a} = \cos\theta = \text{power-factor.}$$

$$q = \frac{P^j}{P_a} = \sin\theta = \text{induction factor}$$

of the circuit, and the general expression of power is

$$\dot{P} = P_a\,(p + jq) = P_a\,(\cos\theta + j\sin\theta).$$

142. The introduction of the double-frequency vector product, $\dot{P} = [\dot{E}\dot{I}]$, brings us outside of the limits of algebra, however, and the commutative principle of algebra, $a \times b = b \times a$, does not apply any more, but we have

$$[\dot{E}\dot{I}] \text{ unlike } [\dot{I}\dot{E}]$$

since

$$[\dot{E}\dot{I}] = [\dot{E}\dot{I}]^1 + j[\dot{E}\dot{I}]^j$$

$$[\dot{I}\dot{E}] = [\dot{I}\dot{E}]^1 + j[\dot{I}\dot{E}]^j = [\dot{E}\dot{I}]^1 - j[\dot{E}\dot{I}]^j,$$

we have

$$[\dot{E}\dot{I}]^1 = [\dot{I}\dot{E}]^1$$

$$[\dot{E}\dot{I}]^j = -[\dot{I}\dot{E}]^j$$

that is, the imaginary component reverses its sign by the interchange of factors.

The physical meaning is, that if the reactive power, $[\dot{E}\dot{I}]^j$, is lagging with regard to $\dot{E}$, it is leading with regard to $\dot{I}$.

The reactive component of power is absent, or the total apparent power is effective power, if

$$[\underset{\cdot}{E}\underset{\cdot}{I}]^j = (e^{11}i^1 - e^1 i^{11}) = 0;$$

that is,

$$\frac{e^{11}}{e^1} = \frac{i^{11}}{i^1}$$

or,

$$\tan(\underset{\cdot}{E}) = \tan(\underset{\cdot}{I});$$

that is, $\underset{\cdot}{E}$ and $\underset{\cdot}{I}$ are in time phase or in opposition.

The effective power is absent, or the total apparent power reactive, if

$$[\underset{\cdot}{E}\underset{\cdot}{I}]^1 = (e^1 i^1 + e^{11}i^{11}) = 0;$$

that is,

$$\frac{e^{11}}{e^1} = -\frac{i^1}{i^{11}}$$

or,

$$\tan \underset{\cdot}{E} = -\cot \underset{\cdot}{I};$$

that is, $\underset{\cdot}{E}$ and $\underset{\cdot}{I}$ are in quadrature.

The reactive component of power is lagging (with regard to $\underset{\cdot}{E}$ or leading with regard to $\underset{\cdot}{I}$) if

$$[\underset{\cdot}{E}\underset{\cdot}{I}]^j > 0,$$

and leading if

$$[\underset{\cdot}{E}\underset{\cdot}{I}]^j < 0.$$

The effective power is negative, that is, power returns, if

$$[\underset{\cdot}{E}\underset{\cdot}{I}]^1 < 0.$$

We have,

$$[\underset{\cdot}{E}, -\underset{\cdot}{I}] = [-\underset{\cdot}{E}, \underset{\cdot}{I}] = -[\underset{\cdot}{E}\underset{\cdot}{I}]$$
$$[-\underset{\cdot}{E}, -\underset{\cdot}{I}] = +[\underset{\cdot}{E}\underset{\cdot}{I}]$$

that is, when representing the power of a circuit or a part of a circuit, current and e.m.f. must be considered in their proper *relative* phases, but their phase relation with the remaining part of the circuit is immaterial.

We have further,

$$[\underset{\cdot}{E}, j\underset{\cdot}{I}] = -j[\underset{\cdot}{E}, \underset{\cdot}{I}] = [\underset{\cdot}{E}, \underset{\cdot}{I}]^j - j[\underset{\cdot}{E}, \underset{\cdot}{I}]^1$$
$$[j\underset{\cdot}{E}, \underset{\cdot}{I}] = j[\underset{\cdot}{E}, \underset{\cdot}{I}] = -[\underset{\cdot}{E}, \underset{\cdot}{I}]^j + j[\underset{\cdot}{E}, \underset{\cdot}{I}]^1$$
$$[j\underset{\cdot}{E}, j\underset{\cdot}{I}] = [\underset{\cdot}{E}, \underset{\cdot}{I}] = [\underset{\cdot}{E}\underset{\cdot}{I}]^1 + j[\underset{\cdot}{E}, \underset{\cdot}{I}]^j$$

143. If $\underset{.}{P}_1 = [\underset{.}{E}_1 \underset{.}{I}_1]$, $\underset{.}{P}_2 = [\underset{.}{E}_2 \underset{.}{I}_2] \ldots \underset{.}{P}_n = [\underset{.}{E}_n \underset{.}{I}_n]$

are the symbolic expressions of the power of the different parts of a circuit or network of circuits, the total power of the whole circuit or network of circuits is

$$\underset{.}{P} = \underset{.}{P}_1 + \underset{.}{P}_2 + \ldots + \underset{.}{P}_n,$$
$$P^1 = P_1^1 + P_2^1 + \ldots + P_n^1,$$
$$P^j = P_1^j + P_2^j \ldots + P_n^j.$$

In other words, the total power in symbolic expression (effective as well as reactive) of a circuit or system is the sum of the powers of its individual components in symbolic expression.

The first equation is obviously directly a result from the law of conservation of energy.

One result derived herefrom is, for instance:

If in a generator supplying power to a system the current is out of phase with the e.m.f. so as to give the reactive power P^j, the current can be brought into phase with the generator e.m.f. or the load on the generator made non-inductive by inserting anywhere in the circuit an apparatus producing the reactive power $-P^j$; that is, compensation for wattless currents in a system takes place regardless of the location of the compensating device.

Obviously, wattless currents exist between the compensating device and the source of wattless currents to be compensated for, and for this reason it may be advisable to bring the compensator as near as possible to the circuit to be compensated.

144. Like power, torque in alternating apparatus is a double-frequency vector product also, of magnetism and m.m.f. or current, and thus can be treated in the same way.

In an induction motor, for instance, the torque is the product of the magnetic flux in one direction into the component of secondary current in phase with the magnetic flux in time, but in quadrature position therewith in space, times the number of turns of this current, or since the generated e.m.f. is in quadrature and proportional to the magnetic flux and the number of turns, the torque of the induction motor is the product of

the generated e.m.f. into the component of secondary current in quadrature therewith in time and space, or the product of the secondary current into the component of generated e.m.f. in quadrature therewith in time and space.

Thus, if

$\dot{E}^1 = e^1 + je^{11}$ = generated e.m.f. in one direction in space,

$\dot{I}_2 = i^1 + ji^{11}$ = secondary current in the quadrature direction in space,

the torque is

$$D = [\dot{E}\dot{I}]^j = e^{11}i^1 - e^1i^{11}.$$

By this equation the torque is given in watts, the meaning being that $D = [\dot{E}\dot{I}]^j$ is the power which would be exerted by the torque at synchronous speed, or the torque in synchronous watts.

The torque proper is then

$$D_0 = \frac{D}{2\,\pi f p},$$

where

$$p = \text{number of pairs of poles of the motor.}$$

In the polyphase induction motor, if $\dot{I}_2 = i^1 + ji^{11}$ is the secondary current in quadrature position, in space, to e.m.f. $\dot{E}_1$, the current in the same direction in space as $\dot{E}_1$ is $\dot{I}_1 = j\dot{I}_2 = -i^{11} + ji^1$; thus the torque can also be expressed as

$$D = [\dot{E}_1\dot{I}_1]^1 = e^{11}i^1 - e^1i^{11}.$$

It is interesting to note that the expression of torque,

$$D = [\dot{E}\dot{I}]^j,$$

and the expression of power,

$$P = [\dot{E}\dot{I}]^1,$$

are the same in character, but the former is the imaginary, the latter the real component. Mathematically, torque, in synchronous watts, can so be considered as imaginary power, and

inversely. Physically, "imaginary" means quadrature component; torque is defined as force times leverage, that is, force times length in quadrature position with force; while energy is defined as force times length in the direction of the force. Expressing quadrature position by "imaginary," thus gives torque of the dimension of imaginary energy; and "synchronous watts," which is torque times frequency, or torque divided by time, thus becomes of the dimension of imaginary power. Thus, in its complex imaginary form, the vector product of force and length contains two quadrature components, of which the one is energy, the other is torque:

$$P = [f, l] = [f, l]^1 + j[f, l]^j$$

and

$$[f, l]^1 = \text{energy}$$
$$[f, l]^j = \text{torque.}$$

CHAPTER XVI.

DISTRIBUTED CAPACITY, INDUCTANCE, RESISTANCE, AND LEAKAGE.

145. As far as capacity has been considered in the foregoing chapters, the assumption has been made that the condenser or other source of negative reactance is shunted across the circuit at a definite point. In many cases, however, the condensive reactance is distributed over the whole length of the conductor, so that the circuit can be considered as shunted by an infinite number of infinitely small condensers infinitely near together, as diagrammatically shown in Fig. 101.

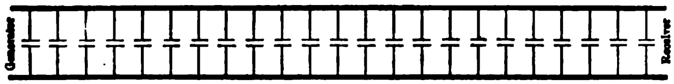


FIG. 101.

In this case the intensity as well as phase of the current, and consequently of the counter e.m.f. of inductive reactance and resistance, vary from point to point; and it is no longer possible to treat the circuit in the usual manner by the vector diagram.

This phenomenon is especially noticeable in long-distance lines, in underground cables, and to a certain degree in the high-potential coils of alternating-current transformers for very high voltage. It has the effect that not only the e.m.fs., but also the currents, at the beginning, end, and different points of the conductor, are different in intensity and in phase.

Where the capacity effect of the line is small, it may with sufficient approximation be represented by one condenser of the same capacity as the line, shunted across the line. Frequently it makes no difference either, whether this condenser is considered as connected across the line at the generator end, or at the receiver end, or at the middle.

The best approximation is to consider the line as shunted

at the generator and at the motor end, by two condensers of one-sixth the line capacity each, and in the middle by a condenser of two-thirds the line capacity. This approximation, based on Simpson's rule, assumes the variation of the electric quantities in the line as parabolic. If, however, the capacity of the line is considerable, and the condenser current is of the same magnitude as the main current, such an approximation is not permissible, but each line element has to be considered as an infinitely small condenser, and the differential equations based thereon integrated. Or the phenomena occurring in the circuit can be investigated graphically by the method given in Chapter VII., § 43, by dividing the circuit into a sufficiently large number of sections or line elements, and then passing from line element to line element, to construct the topographic circuit characteristics.

146. It is thus desirable to first investigate the limits of applicability of the approximate representation of the line by one or by three condensers.

Assuming, for instance, that the line conductors are of 1 cm. diameter, and at a distance from each other of 50 cm., and that the length of transmission is 50 km., we get the capacity of the transmission line from the formula —

$$C = 1.11 \times 10^{-6} \kappa l \div 4 \log_e 2 \frac{d}{\delta} \text{ microfarads,}$$

where

κ = dielectric constant of the surrounding medium = 1 in air;
l = length of conductor = 5×10^6 cm.;
d = distance of conductors from each other = 50 cm.;
δ = diameter of conductor = 1 cm.

Since $C = 0.3$ microfarad,

the condensive reactance is $x = \dfrac{10^6}{2 \pi f C}$ ohms,

where f = frequency; hence at $f = 60$ cycles,

$$x = 8{,}900 \text{ ohms;}$$

and the charging current of the line, at $E = 20{,}000$ volts, becomes,

$$i_0 = \frac{E}{x} = 2.25 \text{ amperes.}$$

The resistance of 100 km. of line of 1 cm. diameter is 22 ohms; therefore, at 10 per cent = 2,000 volts loss in the line, the main current transmitted over the line is

$$I = \frac{2,000}{22} = 91 \text{ amperes,}$$

representing about 1,800 kw.

In this case, the condenser current thus amounts to less than 2.5 per cent, and hence can still be represented by the approximation of one condenser shunted across the line.

If the length of transmission is 150 km., and the voltage, 30,000,

condensive reactance at 60 cycles,	$x = 2,970$ ohms;
charging current,	$i_0 = 10.1$ amperes;
line resistance,	$r = 66$ ohms;
main current at 10 per cent loss,	$I = 45.5$ amperes.

The condenser current is thus about 22 per cent of the main current, and the approximate calculation of the effect of line capacity still fairly accurate.

At 300 km. length of transmission it will, at 10 per cent loss and with the same size of conductor, rise to nearly 90 per cent of the main current, thus making a more explicit investigation of the phenomena in the line necessary.

In many cases of practical engineering, however, the capacity effect is small enough to be represented by the approximation of one; or, three condensers shunted across the line.

147. (*A*) *Line capacity represented by one condenser shunted across middle of line.*

Let

$$Y = g + jb = \text{admittance of receiving circuit;}$$

$$Z = r - jx = \text{impedance of line;}$$

$$b_c = \text{condenser susceptance of line.}$$

Denoting in Fig. 102,

the e.m.f., and current in receiving circuit by $\dot{E}$, $\dot{I}$,
the e.m.f. at middle of line by $\dot{E}'$,
the e.m.f., and current at generator by $\dot{E}_0$, $\dot{I}_0$;

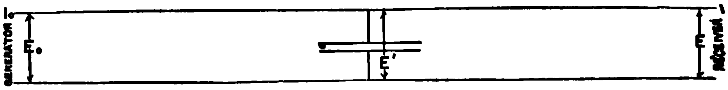


FIG. 102.

we have,

$$\dot{I} = \dot{E}\,(g + jb);$$

$$\dot{E}' = \dot{E} + \frac{r - jx}{2}\,\dot{I}$$

$$= \dot{E}\left\{1 + \frac{(r - jx)\,(g + jb)}{2}\right\};$$

$$\dot{I}_0 = \dot{I} - jb_c\dot{E}'$$

$$= \dot{E}\left\{g + jb - jb_c\left[1 + \frac{(r - jx)\,(g + jb)}{2}\right]\right\};$$

$$\dot{E}_0 = \dot{E}' + \frac{r - jx}{2}\,\dot{I}_0$$

$$= \dot{E}\left\{1 + \frac{(r - jx)\,(g + jb)}{2} + \frac{(r - jx)\,(g + jb)}{2} - \frac{jb_c\,(r - jx)}{2} - jb_c\,\frac{(r - jx)^2\,(g + jb)}{4}\right\};$$

or, expanding,

$$\dot{I}_0 = \dot{E}\{[g + b_c\,(rb - xg)] + j\,[(b - b_c) - b_c\,(rg + xb)]\};$$

$$\dot{E}_0 = \dot{E}\left\{1 + (r - jx)\,(g + jb) - \frac{jb_c}{2}\,(r - jx)\left[1 + \frac{(r - jx)\,(g + jb)}{2}\right]\right\}$$

$$= \dot{E}\left\{1 + (r - jx)\left(g + jb - \frac{jb_c}{2}\right) - \frac{jb_c}{4}\,(r - jx)^2\,(g + jb)\right\}.$$

148. (*B*) *Line capacity represented by three condensers, in the middle and at the ends of the line.*

Denoting, in Fig. 103,

the e.m.f., and current in receiving circuit by E, I,
the e.m.f. at middle of line by E',
the current on receiving side of line by I',
the current on generator side of line by I'',
the e.m.f., and current at generator by E_0, I_0,

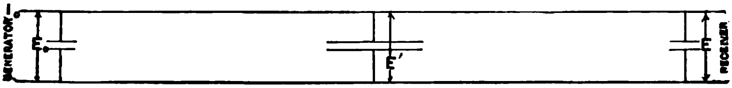


FIG. 103.

otherwise retaining the same denotations as in (A), Fig. 103, we have,

$$I = E\,(g + jb);$$

$$I' = I - \frac{jb_c}{6}E$$

$$= E\left\{g + jb - \frac{jb_c}{6}\right\};$$

$$E' = E + \frac{r - jx}{2}I'$$

$$= E\left\{1 + \frac{r - jx}{2}\left(g + jb - \frac{jb_c}{6}\right)\right\};$$

$$I'' = I' - \frac{4\,jb_c}{6}E'$$

$$= E\left\{g + jb - \frac{5\,jb_c}{6} - \frac{jb_c}{3}(r - jx)\left(g + jb - \frac{jb_c}{6}\right)\right\};$$

$$E_0 = E' + \frac{r - jx}{2}I'';$$

$$E_0 = E\left\{1 + (r - jx)\left(g + jb - \frac{jb_c}{2}\right) - \frac{jb_c}{6}(r - jx)^2\left(g + jb - \frac{jb_c}{6}\right)\right\};$$

$$I_0 = I'' - \frac{jb_c}{6}E_0;$$

$$I_0 = E\left\{(g + jb - jb) - \frac{jb_c}{6}(r - jx)\left(3\,g + 3\,jb - \frac{5\,jb_c}{6}\right)\right.$$
$$\left. - \frac{b_c^2}{36}(r - jx)^2\left(g + jb - \frac{jb_c}{6}\right)\right\}.$$

As will be seen, the first terms in the expression of E_0 and of I_0 are the same in (A) and in (B).

149. *Distributed condensive reactance, inductive reactance, leakage, and resistance.*

In some cases, especially in very long circuits, as in lines conveying alternating-power currents at high potential over extremely long distances by overhead conductors or underground cables, or with very feeble currents at extremely high frequency, such as telephone currents, the consideration of the *line resistance* — which consumes e.m.fs. in phase with the current — and of the *line reactance* — which consumes e.m.fs. in quadrature with the current — is not sufficient for the explanation of the phenomena taking place in the line, but several other factors have to be taken into account.

In long lines, especially at high potentials, the *electrostatic capacity* of the line is sufficient to consume noticeable currents. The charging current of the line condenser is proportional to the difference of potential, and is one-fourth period ahead of the e.m.f. Hence, it will either increase or decrease the main current, according to the relative phase of the main current and the e.m.f.

As a consequence, the current changes in intensity as well as in phase, in the line from point to point; and the e.m.f. consumed by the resistance and inductive reactance therefore also changes in phase and intensity from point to point, being dependent upon the current.

Since no insulator has an infinite resistance, and as at high potentials not only leakage, but even direct *escape of electricity* into the air, takes place by "silent discharge," we have to recognize the existence of a current approximately proportional and in phase with the e.m.f. of the line. This current represents consumption of power, and is, therefore, analogous to the e.m.f. consumed by resistance, while the condenser current and the e.m.f. of self-induction are wattless or reactive.

Furthermore, the alternate current in the line produces in all neighboring conductors secondary currents, which react upon the primary current, and thereby introduce e.m.fs. of *mutual inductance* into the primary circuit. Mutual inductance is neither in phase nor in quadrature with the current, and can therefore be resolved into a *power component* of mutual inductance in phase with the current, which acts as an increase of

resistance, and into a *wattless component* in quadrature with the current, which decreases the self-inductance.

This mutual inductance is not always negligible, as, for instance, its disturbing influence in telephone circuits shows.

The alternating potential of the line induces, by *electrostatic influence*, electric charges in neighboring conductors outside of the circuit, which retain corresponding opposite charges on the line wires. This electrostatic influence requires a current proportional to the e.m.f. and consisting of a *power component*, in phase with the e.m.f., and a *wattless component*, in quadrature thereto.

The alternating electromagnetic field of force set up by the line current produces in some materials a loss of energy by magnetic hysteresis, or an expenditure of e.m.f. in phase with the current, which acts as an increase of resistance. This electro-magnetic hysteretic loss may take place in the conductor proper if iron wires are used, and will then be very serious at high frequencies, such as those of telephone currents.

The effect of *eddy currents* has already been referred to under "mutual inductive reactance," of which it is a power component.

The alternating electrostatic field of force expends energy in dielectrics by what is called *dielectric hysteresis*. In concentric cables, where the electrostatic gradient in the dielectric is comparatively large, the dielectric hysteresis may at high potentials consume appreciable amounts of energy. The dielectric hysteresis appears in the circuit as consumption of a current, whose component in phase with the e.m.f. is the *dielectric power current*, which may be considered as the power component of the capacity current.

Besides this, there is the increase of ohmic resistance due to *unequal distribution of current*, which, however, is usually not large enough to be noticeable.

Furthermore, the electric field of the conductor progresses with a finite velocity, the velocity of light, hence lags behind the flow of power in the conductor, and so also introduces power components, depending on current as well as on potential difference.

150. This gives, as the most general case, and per unit length of line:

e.m.fs. *consumed in phase with the current*, I, and $= rI$, representing consumption of power, and due to:

Resistance, and its increase by unequal current distribution; to the power component of *mutual inductive reactance* or to *induced currents;* to the power component of *self-inductive reactance* or to *electromagnetic hysteresis*, and to *radiation*.

e.m.fs. *consumed in quadrature with the current*, I, and $= xI$, wattless, and due to:

Self-inductance, and *mutual inductance*.

Currents consumed in phase with the e.m.f., E, and $= g E$, representing consumption of power, and due to:

Leakage through the insulating material, including silent discharge; power component of *electrostatic influence;* power component of *capacity* or *dielectric hysteresis*, and to *radiation*.

Currents consumed in quadrature to the e.m.f., E, and $= bE$, being wattless, and due to:

Capacity and *electrostatic influence*.

Hence we get four constants:

Effective resistance, r,
Effective reactance, x,
Effective conductance, g,
Effective susceptance, $- b$,

per unit length of line, which represents the coefficients, per unit length of line, of

e.m.f. consumed in phase with current;
e.m.f. consumed in quadrature with current;
current consumed in phase with e.m.f.;
current consumed in quadrature with e.m.f.;

or,

$$Z = r - jx,$$
$$Y = g - jb,$$

and, absolute,

$$z = \sqrt{r^2 + x^2},$$
$$y = \sqrt{g^2 + b^2}.$$

151. The complete investigation of such a circuit or line containing distributed capacity, inductive reactance, resistance, etc., leads to functions which are products of exponential and of trigonometric functions. That is, the current and potential difference along the line u are given by expressions of the form:

$$\epsilon^{\pm au}(A \cos \beta u + B \sin \beta u).$$

Such functions of the distance, u, or position on the line, while alternating in time, differ from the true alternating waves in that the intensities of successive half-waves progressively increase or decrease with the distance. Such functions are called oscillating waves, and, as such, are beyond the scope of this book, but are more fully treated in "Theory and Calculation of Transient Electric Phenomena and Oscillations," Section III. There also will be found the discussion of the phenomena of distributed capacity in high-potential transformer windings, the effect of the finite velocity of propagation of the electric field, etc.

CHAPTER XVII.

THE ALTERNATING-CURRENT TRANSFORMER.

152. The simplest alternating-current apparatus is the transformer. It consists of a magnetic circuit interlinked with two electric circuits, a primary and a secondary. The primary circuit is excited by an impressed e.m.f., while in the secondary circuit an e.m.f. is generated. Thus, in the primary circuit power is consumed, and in the secondary a corresponding amount of power is produced.

Since the same magnetic circuit is interlinked with both electric circuits, the e.m.f. generated per turn must be the same in the secondary as in the primary circuit; hence, the primary generated e.m.f. being approximately equal to the impressed e.m.f., the e.m.fs. at primary and at secondary terminals have approximately the ratio of their respective turns. Since the power produced in the secondary is approximately the same as that consumed in the primary, the primary and secondary currents are approximately in inverse ratio to the turns.

153. Besides the magnetic flux interlinked with both electric circuits — which flux, in a closed magnetic circuit transformer, has a circuit of low reluctance — a magnetic cross-flux passes between the primary and secondary coils, surrounding one coil only, without being interlinked with the other. This magnetic cross-flux is proportional to the current in the electric circuit, or rather, the ampere-turns or m.m.f., and so increases with the increasing load on the transformer, and constitutes what is called the self-inductive or leakage reactance of the transformer; while the flux surrounding both coils may be considered as mutual inductive reactance. This cross-flux of self-induction does not generate e.m.f. in the secondary circuit, and is thus, in general, objectionable, by causing a drop of voltage and a decrease of output. It is this cross-flux, how-

ever, or flux of self-inductive reactance, which is utilized in special transformers, to secure automatic regulation, for constant power, or for constant current, and in this case is exaggerated by separating primary and secondary coils. In the constant potential transformer, however, the primary and secondary coils are brought as near together as possible, or even interspersed, to reduce the cross-flux.

As will be seen, by the self-inductive reactance of a circuit, not the total flux produced by, and interlinked with, the circuit is understood, but only that (usually small) part of the flux which surrounds one circuit without interlinking with the other circuit.

154. The alternating magnetic flux of the magnetic circuit surrounding both electric circuits is produced by the combined magnetizing action of the primary and of the secondary current.

This magnetic flux is determined by the e.m.f. of the transformer, by the number of turns, and by the frequency.

If

Φ = maximum magnetic flux,

f = frequency,

n = number of turns of the coil,

the e.m.f. generated in this coil is

$$E = \sqrt{2}\,\pi f n \Phi\, 10^{-8} = 4.44\, f n \Phi\, 10^{-8} \text{ volts};$$

hence, if the e.m.f., frequency, and number of turns are determined, the maximum magnetic flux is

$$\Phi = \frac{E\, 10^8}{\sqrt{2}\,\pi f n}.$$

To produce the magnetism, Φ, of the transformer, a m.m.f. of $\mathcal{F}$ ampere-turns is required, which is determined by the shape and the magnetic characteristic of the iron, in the manner discussed in Chapter XIII.

For instance, in the closed magnetic circuit transformer, the maximum magnetic induction is $\mathcal{B}, = \frac{\Phi}{A}$, where A = the cross-section of magnetic circuit.

155. To induce a magnetic density, $\mathfrak{B}$, a magnetizing force of $\mathfrak{H}_m$ ampere-turns maximum is required, or, $\frac{\mathfrak{H}_m}{\sqrt{2}}$ ampere-turns effective, per unit length of the magnetic circuit; hence, for the total magnetic circuit, of length, l,

$$\mathfrak{F} = \frac{l\mathfrak{H}_m}{\sqrt{2}} \text{ ampere-turns;}$$

or

$$I = \frac{\mathfrak{F}}{n} = \frac{l\mathfrak{H}_m}{n\sqrt{2}} \text{ amp. eff.}$$

where n = number of turns.

At no-load, or open secondary circuit, this m.m.f., $\mathfrak{F}$, is furnished by the *exciting current*, I_{00}, improperly called the *leakage current*, of the transformer; that is, that small amount of primary current which passes through the transformer at open secondary circuit.

In a transformer with open magnetic circuit, such as the "hedgehog" transformer, the m.m.f., $\mathfrak{F}$, is the sum of the m.m.f. consumed in the iron and in the air part of the magnetic circuit (see Chapter XIII.).

The power component of the exciting current represents the power consumed by hysteresis and eddy currents and the small ohmic loss.

The exciting current is not a sine wave, but is, at least in the closed magnetic circuit transformer, greatly distorted by hysteresis, though less so in the open magnetic circuit transformer. It can, however, be represented by an equivalent sine wave, I_{00}, of equal intensity and equal power with the distorted wave, and a wattless higher harmonic, mainly of triple frequency.

Since the higher harmonic is small compared with the total exciting current, and the exciting current is only a small part of the total primary current, the higher harmonic can, for most practical cases, be neglected, and the exciting current represented by the equivalent sine wave.

This equivalent sine wave, I_{00}, leads the wave of magnetism, Φ, by an angle, α, the angle of hysteretic advance of phase, and consists of two components, — the hysteretic power current

in quadrature with the magnetic flux, and therefore in phase with the generated e.m.f. $= I_{00} \sin \alpha$; and the magnetizing current, in phase with the magnetic flux, and therefore in quadrature with the generated e.m.f., and so wattless, $= I_{00} \cos \alpha$.

The exciting current, I_{00}, is determined from the shape and magnetic characteristic of the iron, and the number of turns; the hysteretic power current is

$$I_{00} \sin \alpha = \frac{\text{Power consumed in the iron}}{\text{Generated e.m.f.}}.$$

156. Graphically, the polar diagram of m.m.fs., of a transformer is constructed thus:

Let, in Fig. 104, $\overline{O\Phi}$ = the magnetic flux in intensity and phase (for convenience, as intensities, the effective values are used throughout), assuming its phase as the vertical; that is, counting the time from the moment where the rising magnetism passes its zero value.

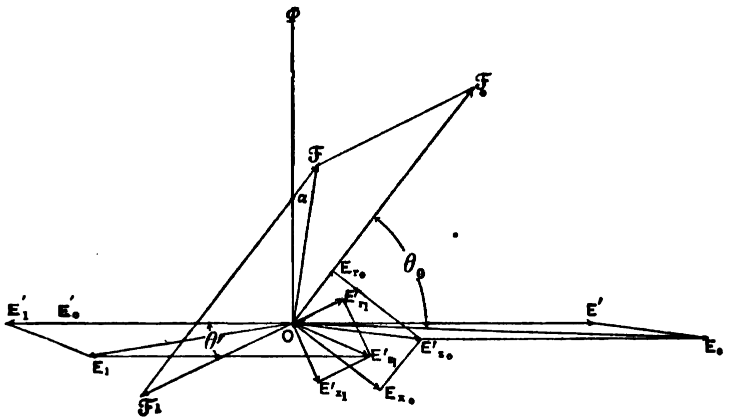


FIG. 104.

Then the resultant m.m.f. is represented by the vector, $\overline{O\mathfrak{F}}$, leading $\overline{O\Phi}$ by the angle, $\mathfrak{F}O\Phi = \alpha$.

The generated e.m.fs. have the phase 180°, that is, are plotted towards the left, and represented by the vectors, $\overline{OE_0'}$ and $\overline{OE_1'}$.

If, now, θ' = angle of time lag in the secondary circuit, due to the total (internal and external) secondary reactance,

the secondary current, I_1, and hence the secondary m.m.f., $\mathfrak{F}_1 = n_1 I_1$, lags behind E_1' by an angle θ', and have the phase, $180° + \theta'$, represented by the vector $\overline{O\mathfrak{F}_1}$. Constructing a parallelogram of m.m.fs., with $\overline{O\mathfrak{F}}$ as the diagonal and $\overline{O\mathfrak{F}_1}$ as one side, the other side or $\overline{O\mathfrak{F}_0}$ is the primary m.m.f., in intensity and phase, and hence, dividing by the number of primary turns, n_0, the primary current is $I_0 = \frac{\mathfrak{F}_0}{n_0}$.

To complete the diagram of e.m.fs., we have now,

In the primary circuit:

e.m.f. consumed by resistance is $I_0 r_0$, in time phase with I_0, and represented by the vector, $\overline{OE_{r_0}}$;

e.m.f. consumed by reactance is $I_0 x_0$, 90° ahead of I_0, and represented by the vector, $\overline{OE_{x_0}}$;

e.m.f. consumed by induced e.m.f. is E', equal and opposite to E_0', and represented by the vector, $\overline{OE'}$.

Hence, the total primary impressed e.m.f. by combination of $\overline{OE_{r_0}}$, $\overline{OE_{x_0}}$ and $\overline{OE'}$ by means of the parallelogram of e.m.fs. is

$$E_0 = \overline{OE_0},$$

and the difference of phase between the primary impressed e.m.f. and the primary current is

$$\theta_0 = E_0 O \mathfrak{F}_0.$$

In the secondary circuit:

Counter e.m.f. of resistance is $I_1 r_1$ in opposition with I_1, and represented by the vector, $\overline{OE'_{r_1}}$;

Counter e.m.f. of reactance is $I_1 x_1$, 90° behind I_1, and represented by the vector, $\overline{OE'_{x_1}}$.

Generated e.m.fs., E_1' represented by the vector, $\overline{OE_1'}$.

Hence, the secondary terminal voltage, by combination of $\overline{OE'_{r_1}}$, $\overline{OE'_{x_1}}$ and $\overline{OE_1'}$ by means of the parallelogram of e.m.fs. is

$$E_1 = \overline{OE_1},$$

and the difference of phase between the secondary terminal voltage and the secondary current is

$$\theta' = E_1 O \mathfrak{F}_1.$$

As seen, in the primary circuit the "components of impressed e.m.f. required to overcome the counter e.m.fs." were used for convenience, and in the secondary circuit the "counter e.m.fs."

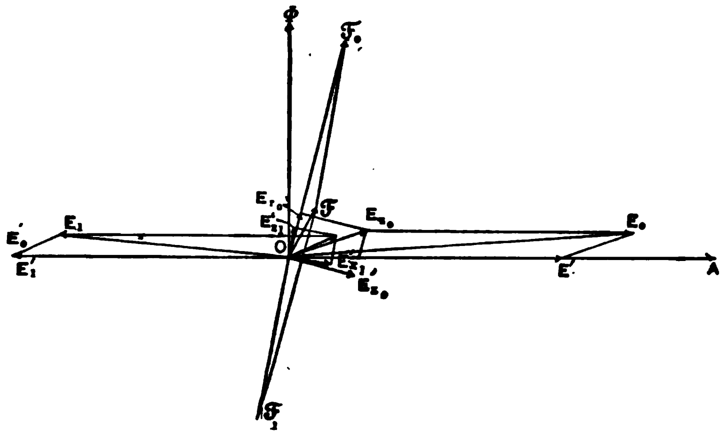


FIG. 105. — Transformer Diagram with 80° Lag in Secondary Circuit.

157. In the construction of the transformer diagram, it is usually preferable not to plot the secondary quantities, current

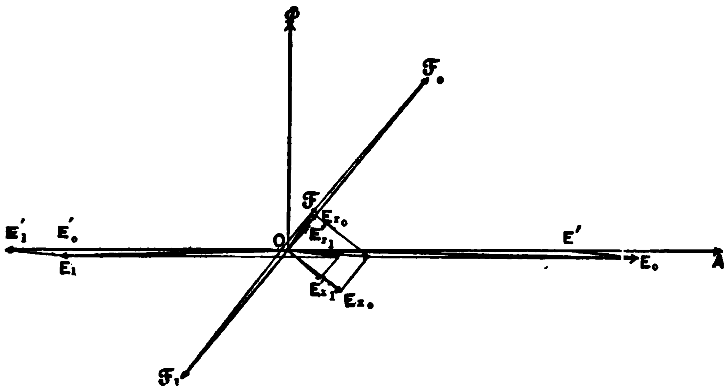


FIG. 106. — Transformer Diagram with 50° Lag in Secondary Circuit.

and e.m.f., direct, but to reduce them to correspondence with the primary circuit by multiplying by the ratio of turns, $a = \frac{n_0}{n_1}$, for the reason that frequently primary and secondary e.m.fs.,

etc., are of such different magnitude as not to be easily represented on the same scale; or the primary circuit may be

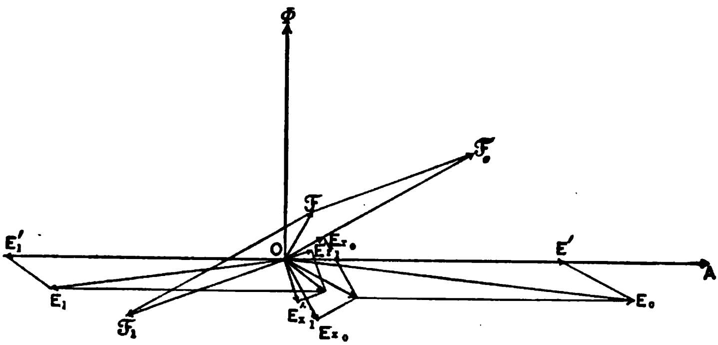


FIG. 107. — Transformer Diagram with 20° Lag in Secondary Circuit.

reduced to the secondary in the same way. In either case, the vectors representing the two generated e.m.fs. coincide, or $\overline{OE}_1' = \overline{OE}_0'$.

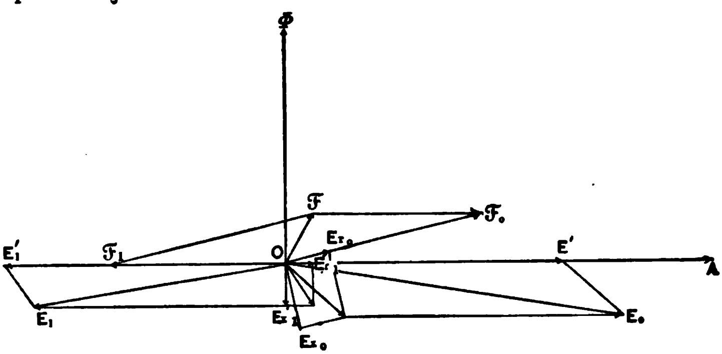


FIG. 108. — Transformer Diagram with Secondary Current in Phase with E.M.F.

Figs. 106 to 111 give the polar diagram of a transformer having the constants —

$r_0 = 0.2$ ohm, $b_0 = .0173$ mhos,
$x_0 = 0.33$ ohm, $E_1' = 100$ volts,
$r_1 = 0.00167$ ohm, $I_1 = 60$ amperes,
$x_1 = 0.0025$ ohm, $\alpha = 30$ degrees.
$g_0 = 0.0100$ mhos,

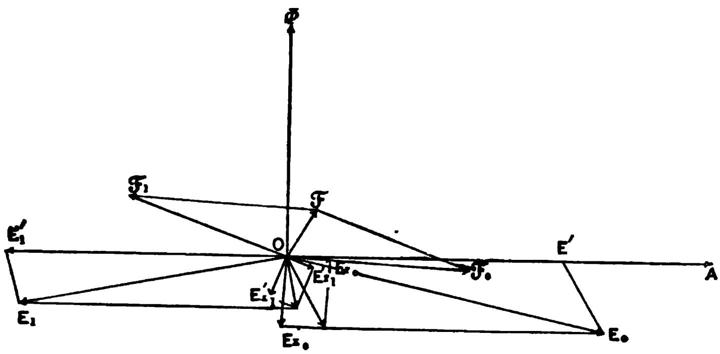


FIG. 109. — Transformer Diagram with 20° Lead in Secondary Current.

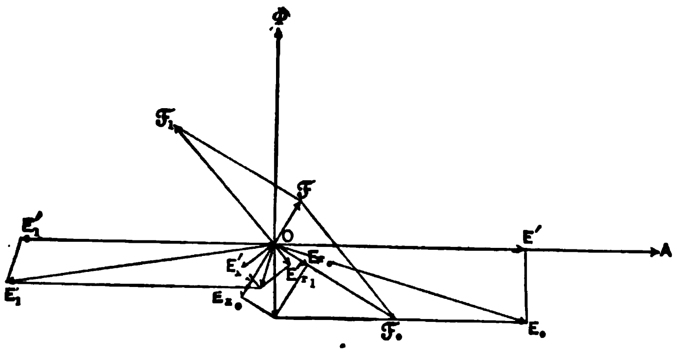


FIG. 110. — Transformer Diagram with 50° Lead in Secondary Current.

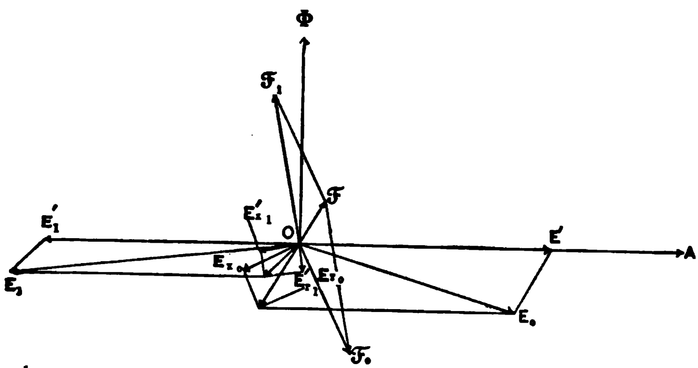


FIG. 111. — Transformer Diagram with 80° Lead in Secondary Current.

For the conditions of secondary circuit,

$\theta_1' = 80°$ lag	in Fig. 105	$\theta_1' = 20°$ lead	in Fig. 109
50° lag	" 106	50° lead	" 110
20° lag	" 107	80° lead	" 111
O, or in phase,	" 108		

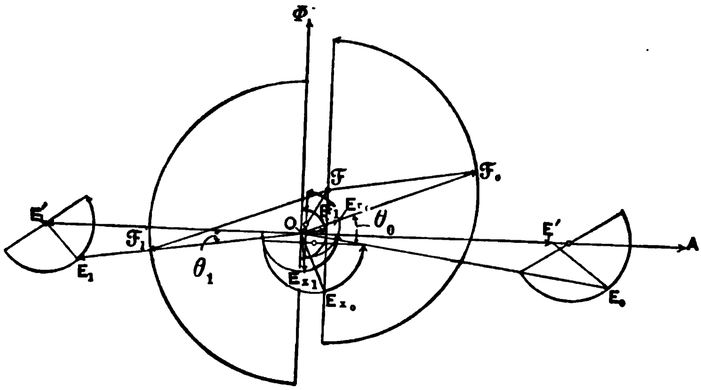


FIG. 112.

As shown, with a change of θ_1' the other quantities, E_0, I_1, I_0, etc., change in intensity and direction. The loci described by them are circles, and are shown in Fig. 112, with the point corresponding to non-inductive load marked. The part of the locus corresponding to a lagging secondary current is shown in thick full lines, and the part corresponding to leading current in thin full lines.

158. This diagram represents the condition of constant secondary generated e.m.f., E_1', that is, corresponding to a constant maximum magnetic flux.

By changing all the quantities proportionally from the diagram of Fig. 112, the diagrams for the constant primary impressed e.m.f. (Fig. 113), and for constant secondary terminal voltage (Fig. 114), are derived. In these cases, the locus gives curves of higher order.

Fig. 115 gives the locus of the various quantities when the load is changed from full load, $I_1 = 60$ amperes in a non-inductive secondary external circuit to no-load or open-circuit:

(a) By increase of secondary current; (b) by increase of secondary inductive resistance; (c) by increase of secondary condensive reactance.

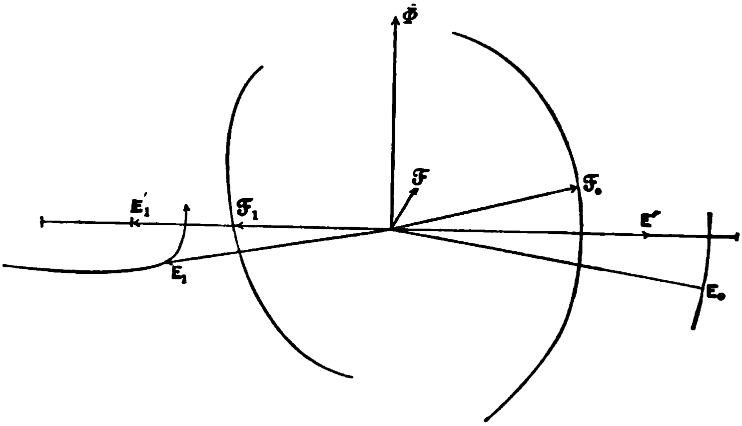


FIG. 113.

As shown in (a), the locus of the secondary terminal voltage, E_1, and thus of E_0, etc., are straight lines; and in (b) and (c), parts of one and the same circle (a) is shown in full lines, (b) in

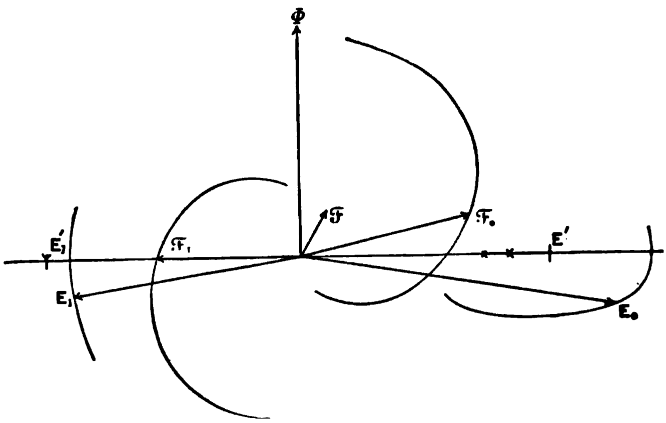


FIG. 114.

heavy full lines, and (c) in light full lines. This diagram corresponds to constant maximum magnetic flux; that is, to constant secondary generated e.m.f. The diagrams representing constant

primary impressed e.m.f. and constant secondary terminal voltage can be derived from the above by proportionality.

159. It must be understood, however, that for the purpose of making the diagrams plainer, by bringing the different values to somewhat nearer the same magnitude, the constants chosen for these diagrams represent, not the magnitudes found in actual transformers, but refer to greatly exaggerated internal losses.

In practice, about the following magnitudes would be found:

$r_0 = 0.01$ ohm;	$x_1 = 0.00025$ ohm;
$x_0 = 0.033$ ohm;	$g_0 = 0.001$ mho;
$r_1 = 0.00008$ ohm;	$b_0 = 0.00173$ mho;

that is, about one-tenth as large as assumed. Thus the changes of the values of E_0, E_1, etc., under the different conditions will be very much smaller.

Symbolic Method.

160. In symbolic representation by complex quantities the transformer problem appears as follows:

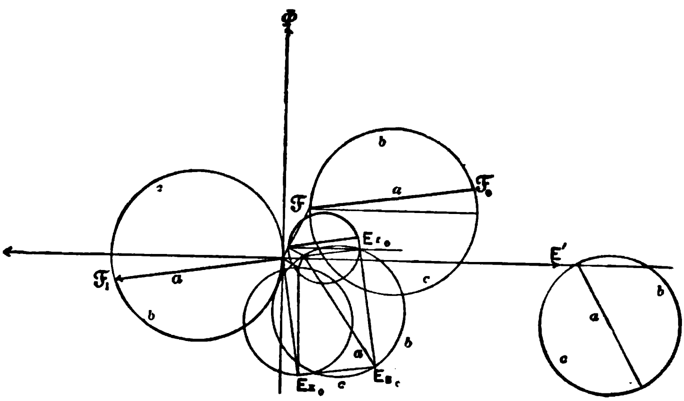


FIG. 115.

The exciting current, I_{00}, of the transformer depends upon the primary e.m.f., which dependance can be represented by an admittance, the "primary admittance," $Y_0 = g_0 + jb_0$, of the transformer.

The resistance and reactance of the primary and the secondary circuit are represented in the impedance by

$$Z_0 = r_0 - jx_0, \quad \text{and} \quad Z_1 = r_1 - jx_1.$$

Within the limited range of variation of the magnetic density in a constant-potential transformer, admittance and impedance can usually, and with sufficient exactness, be considered as constant.

Let

n_0 = number of primary turns in series;

n_1 = number of secondary turns in series;

$a = \frac{n_0}{n_1}$ = ratio of turns;

$Y_0 = g_0 + jb_0$ = primary admittance

$= \frac{\text{Exciting current}}{\text{Primary induced e.m.f.}}$;

$Z_0 = r_0 - jx_0$ = primary impedance

$= \frac{\text{E.M.F. consumed in primary coil by resistance and reactance}}{\text{Primary current}}$;

$Z_1 = r_1 - jx_1$ = secondary impedance

$= \frac{\text{E.M.F. consumed in secondary coil by resistance and reactance}}{\text{Secondary current}}$;

where the reactances, x_0 and x_1, refer to the true self-induction only, or to the cross-flux passing between primary and secondary coils; that is, interlinked with one coil only.

Let also

$Y = g + jb$ = total admittance of secondary circuit, including the internal impedance;

$\dot{E}_0$ = primary impressed e.m.f.;

$\dot{E}'$ = e.m.f. consumed by primary counter e.m.f.;

$\dot{E}_1$ = secondary terminal voltage;

$\dot{E}_1'$ = secondary generated e.m.f.;

$\dot{I}_0$ = primary current, total;

$\dot{I}_{00}$ = primary exciting current;

$\dot{I}_1$ = secondary current.

Since the primary counter e.m.f., $\dot{E}'$, and the secondary generated e.m.f., $\dot{E}_1'$, are proportional by the ratio of turns, a,

$$\dot{E}_0' = +a\dot{E}_1'. \qquad (1)$$

$$\dot{E}_0' = -\dot{E}'.$$

The secondary current is

$$\dot{I}_1 = Y\dot{E}_1'. \qquad (2)$$

consisting of a power component, $g\dot{E}_1'$, and a reactive component, $b\dot{E}_1'$.

To this secondary current corresponds the component of primary current,

$$\dot{I}_0' = \frac{-Y\dot{E}_1'}{a} = \frac{Y\dot{E}'}{a^2}. \qquad (3)$$

The primary exciting current is

$$\dot{I}_{00} = Y_0\dot{E}'. \qquad (4)$$

Hence, the total primary current is

$$\dot{I}_0 = \dot{I}_0' + \dot{I}_{00} \qquad (5)$$

$$= \frac{Y\dot{E}'}{a^2} + Y_0\dot{E}',$$

or,

$$\dot{I}_0 = \frac{\dot{E}'}{a^2}\{Y + a^2Y_0\} \qquad (6)$$

$$= -\frac{\dot{E}_1'}{a}\{Y + a^2Y_0\}.$$

The e.m.f. consumed in the secondary coil by the internal impedance is $Z_1\dot{I}_1$.

The e.m.f. generated in the secondary coil by the magnetic flux is $\dot{E}_1'$.

Therefore, the secondary terminal voltage is

$$\dot{E}_1 = \dot{E}_1' - Z_1\dot{I}_1;$$

or, substituting (2), we have

$$\dot{E}_1 = \dot{E}_1' \{1 - Z_1 Y\}. \tag{7}$$

The e.m.f. consumed in the primary coil by the internal impedance is $Z_0 \dot{I}_0$.

The e.m.f. consumed in the primary coil by the counter e.m.f. is $\dot{E}'$.

Therefore, the primary impressed e.m.f. is

$$\dot{E}_0 = \dot{E}' + Z_0 \dot{I}_0,$$

or, substituting (6),

$$\left.\begin{aligned} \dot{E}_0 &= \dot{E}' \left\{1 + Z_0 Y_0 + \frac{Z_0 Y}{a^2}\right\} \\ &= - a\dot{E}_1' \left\{1 + Z_0 Y_0 + \frac{Z_0 Y}{a^2}\right\}. \end{aligned}\right\} \tag{8}$$

161. We thus have,

primary e.m.f., $\dot{E}_0 = - a\dot{E}_1' \left\{1 + Z_0 Y_0 + \frac{Z_0 Y}{a^2}\right\}$, (8)

secondary e.m.f., $\dot{E}_1 = \dot{E}_1' \{1 - Z_1 Y\}$, (7)

primary current, $\dot{I}_0 = - \frac{\dot{E}_1'}{a} \{Y + a^2 Y_0\}$, (6)

secondary current, $\dot{I}_1 = Y \dot{E}_1^1$, (2)

as functions of the secondary generated e.m.f., $\dot{E}_1'$, as parameter.

From the above we derive

Ratio of transformation of e.m.fs.:

$$\frac{\dot{E}_0}{\dot{E}_1} = - a \frac{1 + Z_0 Y_0 + \frac{Z_0 Y}{a^2}}{1 - Z_1 Y}. \tag{9}$$

Ratio of transformations of currents:

$$\frac{\dot{I}_0}{\dot{I}_1} = - \frac{1}{a} \left\{1 + a^2 \frac{Y_0}{Y}\right\}. \tag{10}$$

From this we get, at constant primary impressed e.m.f.,

$$\dot{E}_0 = \text{constant};$$

secondary generated e.m.f.,

$$\dot{E}_1' = -\frac{\dot{E}_0}{a}\,\frac{1}{1 + Z_0 Y_0 + \dfrac{Z_0 Y}{a^2}};$$

e.m.f. generated per turn,

$$\delta \dot{E} = -\frac{\dot{E}_0}{n_0}\,\frac{1}{1 + Z_0 Y_0 + \dfrac{Z_0 Y}{a^2}};$$

secondary terminal voltage,

$$\dot{E}_1 = -\frac{\dot{E}_0}{a}\,\frac{1 - Z_1 Y}{1 + Z_0 Y_0 + \dfrac{Z_0 Y}{a^2}};$$

primary current,

$$\dot{I}_0 = \frac{\dot{E}_0}{a^2}\,\frac{Y + a^2 Y_0}{1 + Z_0 Y_0 + \dfrac{Z_0 Y}{a^2}} = \dot{E}_0\,\frac{\dfrac{Y}{a^2} + Y_0}{1 + Z_0 Y_0 + \dfrac{Z_0 Y}{a^2}};$$

secondary current,

$$\dot{I}_1 = -\frac{\dot{E}_0}{a}\,\frac{Y}{1 + Z_0 Y_0 + \dfrac{Z_0 Y}{a^2}}. \qquad (11)$$

At constant secondary terminal voltage,

$$\dot{E}_1 = \text{const.};$$

secondary generated e.m.f., $\dot{E}_1' = \dfrac{\dot{E}_1}{1 - Z_1 Y}$;

e.m.f. generated per turn, $\delta \dot{E} = \dfrac{\dot{E}_1}{n_1}\,\dfrac{1}{1 - Z_1 Y}$;

primary impressed e.m.f., $\dot{E}_0 = -a\dot{E}_1\,\dfrac{1 + Z_0 Y_0 + \dfrac{Z_0 Y}{a^2}}{1 - Z_1 Y}$;

primary current, $\dot{I}_0 = -\dfrac{\dot{E}_1}{a}\,\dfrac{Y + a^2 Y_0}{1 - Z_1 Y}$;

secondary current, $\dot{I}_1 = \dot{E}_1\,\dfrac{Y}{1 - Z_1 Y}$. (12)

162. Some interesting conclusions can be drawn from these equations.

The apparent impedance of the total transformer is

$$Z_t = \frac{\dot{E}_0}{\dot{I}_0} = a^2 \frac{1 + Z_0 Y_0 + \dfrac{Z_0 Y}{a^2}}{Y + a^2 Y_0} \tag{13}$$

$$= \frac{1 + Z_0\left(Y_0 + \dfrac{Y}{a^2}\right)}{Y_0 + \dfrac{Y}{a^2}};$$

$$Z_t = \frac{1}{Y_0 + \dfrac{Y}{a^2}} + Z_0. \tag{14}$$

Substituting now, $\dfrac{Y}{a^2} = Y'$, the total secondary admittance, reduced to the primary circuit by the ratio of turns, it is

$$Z_t = \frac{1}{Y_0 + Y'} + Z_0. \tag{15}$$

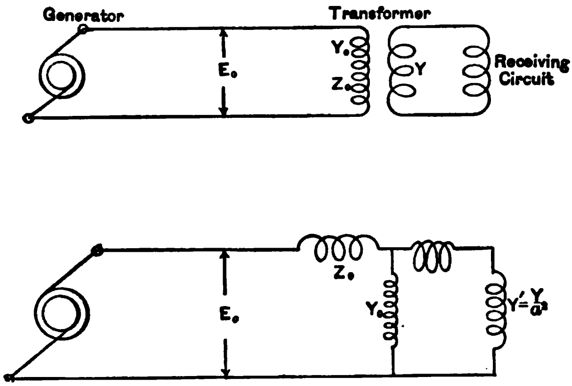


FIG. 116.

$Y_0 + Y'$ is the total admittance of a divided circuit with the exciting current of admittance, Y_0, and the secondary current of admittance, Y' (reduced to primary), as branches. Thus,

$$\frac{1}{Y_0 + Y'} = Z_0' \tag{16}$$

is the impedance of this divided circuit, and

$$Z_t = Z_0' + Z_0. \tag{17}$$

That is,

The alternate-current transformer, of primary admittance Y_0, total secondary admittance Y, and primary impedance Z_0, is equivalent to, and can be replaced by, a divided circuit with the branches of admittance Y_0, the exciting current, and admittance $Y' = \frac{Y}{a^2}$, the secondary current, fed over mains of the impedance Z_0, the internal primary impedance.

This is shown diagrammatically in Fig. 116.

163. Separating now the internal secondary impedance from the external secondary impedance, or the impedance of the consumer circuit, it is

$$\frac{1}{Y} = Z_1 + Z; \tag{18}$$

where $Z =$ external secondary impedance,

$$Z = \frac{\dot{E}_1}{\dot{I}_1}. \tag{19}$$

Reduced to primary circuit, it is

$$\frac{1}{Y'} = \frac{a^2}{Y} = a^2 Z_1 + a^2 Z$$
$$= Z_1' + Z'. \tag{20}$$

That is,

An alternate-current transformer, of primary admittance Y_0, primary impedance Z_0, secondary impedance Z_1, and ratio of turns a, can, when the secondary circuit is closed by an impedance, Z (the impedance of the receiver circuit), be replaced, and is equivalent to a circuit of impedance, $Z' = a^2 Z$, fed over mains of the impedance, $Z_0 + Z_1'$, where $Z_1' = a^2 Z_1$, shunted by a circuit of admittance, Y_0, which latter circuit branches off at the points, a, b, between the impedances, Z_0 and Z_1'.

This is represented diagrammatically in Fig. 117.

It is obvious, therefore, that if the transformer contains several independent secondary circuits, they are to be considered as branched off at the points a, b, in diagram, Fig. 117, as shown in diagram, Fig. 118.

It therefore follows:

An alternate-current transformer, of s *secondary coils, of the internal impedances,* Z_1^{I}, Z_1^{II}, . . . Z_1^{s}, *closed by external secondary circuits of the impedances,* Z^{I}, Z^{II}, . . . Z^{s}, *is equivalent to a divided circuit of* s + 1 *branches, one branch of admittance,* Y_0, *the exciting current, the other branches of the impedances,* $Z_1^{I} + Z^{I}$, $Z_1^{II} + Z^{II}$, . . . $Z_1^{s} + Z^{s}$, *the latter impedances being reduced to the primary circuit by the ratio of turns, and the whole divided circuit being fed by the primary impressed e.m.f.,* E_0, *over mains of the impedance,* Z_0.

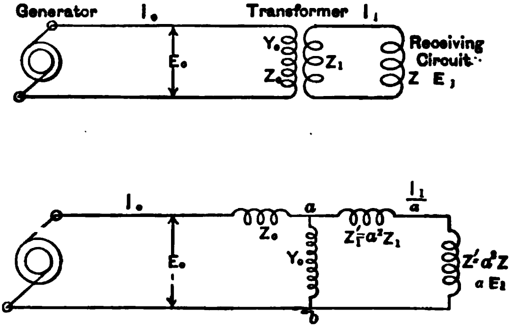


FIG. 117.

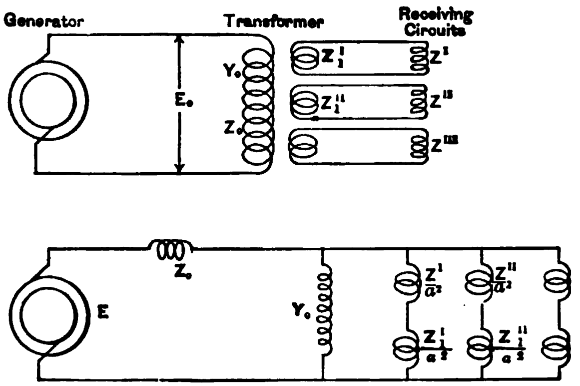


FIG. 118.

Consequently, transformation of a circuit merely changes all the quantities proportionally, introduces in the mains the impedance, $Z_0 + Z_1'$, and a branch circuit between Z_0 and Z_1', of admittance Y_0.

Thus, double transformation will be represented by diagram, Fig. 119.

With this the discussion of the alternate-current transformer ends, by becoming identical with that of a divided circuit containing resistances and reactances.

Such circuits have been discussed in detail in Chapter IX., and the results derived there are now directly applicable to the transformer, giving the variation and the control of secondary terminal voltage, resonance phenomena, etc.

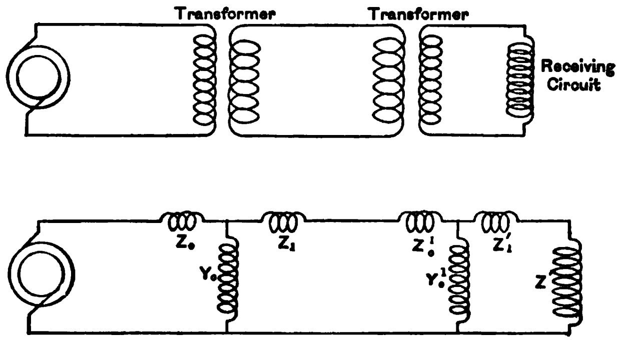


FIG. 119.

Thus, for instance, if $Z_1' = Z_0$, and the transformer contains an additional secondary coil, constantly closed by a condensive reactance of such size that this auxiliary circuit, together with the exciting circuit, gives the reactance, $-x_0$, with a non-inductive secondary circuit, $Z_1 = r_1$, we get the condition of transformation from constant primary potential to constant secondary current, and inversely, as previously discussed.

Non-inductive Secondary Circuit.

164. In a non-inductive secondary circuit, the external secondary impedance is

$$Z = R_1,$$

or, reduced to primary circuit,

$$a^2 Z = a^2 R_1 = R.$$

Assuming the secondary impedance, reduced to primary circuit, as equal to the primary impedance,

$$a^2 Z_1 = Z_0 = r_0 - jx_0,$$

it is

$$\frac{Y}{a^2} = \frac{1}{a^2(Z + Z_1)} = \frac{1}{R + r_0 - jx_0}.$$

Substituting these values in equations (9), (10), and (13), we have

Ratio of e.m.fs.:

$$\frac{\dot{E}_0}{\dot{E}_1} = -a \frac{1 + \dfrac{r_0 - jx_0}{R + r_0 - jx_0} + (r_0 - jx_0)(g_0 + jb_0)}{1 - \dfrac{(r_0 - jx_0)}{R + r_0 - jx_0}}$$

$$= -a \left\{ 1 + \frac{r_0 - jx_0}{R + r_0 - jx_0} + (r_0 - jx_0)(g_0 + jb_0) \right\}$$

$$\left\{ 1 + \frac{r_0 - jx_0}{R + r_0 - jx_0} + \left(\frac{r_0 - jx_0}{R + r_0 - jx_0} \right)^2 + \ldots \right\};$$

or, expanding, and neglecting terms of higher than third order,

$$\frac{\dot{E}_0}{\dot{E}_1} = -a \left\{ 1 + 2 \frac{r_0 - jx_0}{R + r_0 - jx_0} + 2 \left(\frac{r_0 - jx_0}{R + r_0 - jx_0} \right)^2 + (r_0 - jx_0)(g_0 + jb_0) \right\};$$

or, expanded,

$$\frac{\dot{E}_0}{\dot{E}_1} = -a \left\{ 1 + 2 \frac{r_0 - jx_0}{R} + (r_0 - jx_0)(g_0 + jb_0) \right\}.$$

Neglecting terms of tertiary order also,

$$\frac{\dot{E}_0}{\dot{E}_1} = -a \left\{ 1 + 2 \frac{r_0 - jx_0}{R} \right\}.$$

Ratio of currents:

$$\frac{\dot{I}_0}{\dot{I}_1} = -\frac{1}{a} \{1 + (g_0 + jb_0)(R + r_0 - jx_0)\};$$

or, expanded,

$$\frac{\dot{I}_0}{\dot{I}_1} = -\frac{1}{a} \{1 + R(g_0 + jb_0) + (r_0 - jx_0)(g_0 + jb_0)\}.$$

Neglecting terms of tertiary order also,

$$\frac{\dot{I}_0}{\dot{I}_1} = -\frac{1}{a}\{1 + R(g_0 + jb_0)\}.$$

Total apparent primary impedance:

$$Z_t = \frac{\dot{E}_0}{\dot{I}_0} = \frac{1 + \dfrac{r_0 - jx_0}{R + r_0 - jx_0} + (r_0 - jx_0)(g_0 + jb_0)}{\dfrac{1}{R + r_0 - jx_0} + g_0 + jb_0}$$

$$= \{R + (r_0 - jx_0) + R(r_0 - jx_0)(g_0 + jb_0)\}\{1 - (g_0 + jb_0)(R + r_0 - jb_0) + (g_0 + jb_0)^2 (R + r_0 - jb_0)^2 + \ldots\}$$

$$= \{R + 2(r_0 - jx_0) - R^2 (g_0 + jb_0) - 2R(r_0 - jx_0)(g_0 + jb_0) + R^3 (g_0 + jb_0)^2\};$$

or,

$$Z_t = R\left\{1 + 2\frac{r_0 - jx_0}{R} - R(g_0 + jb_0) - 2(r_0 - jx_0)(g_0 + jb_0) + R^2 (g_0 + jb_0)^2\right\}.$$

Neglecting terms of tertiary order also,

$$Z_t = R\left\{1 + 2\frac{r_0 - jx_0}{R} - R(g_0 + jb_0)\right\}.$$

Angle of lag in primary circuit:

$$\tan\theta_0 = \frac{x_t}{r_t}, \text{ hence,}$$

$$\tan\theta_0 = \frac{2\dfrac{x_0}{R} + Rb_0 + 2r_0b_0 - 2x_0g_0 - 2R^2g_0b_0}{1 + \dfrac{2r_0}{R} - Rg_0 - 2r_0g_0 - 2x_0b_0 - R^2g_0^2 + R_1^2b_0^2}.$$

Neglecting terms of tertiary order also,

$$\tan\theta_0 = \frac{2\dfrac{x_0}{R} + Rb_0}{1 + 2\dfrac{r_0}{R} - Rg_0}.$$

165. If, now, we represent the external resistance of the secondary circuit at full load (reduced to the primary circuit) by R_0, and denote,

$$\frac{2\,r_0}{R_0} = \mathsf{p} = \text{ratio } \frac{\text{Internal resistance of transformer}}{\text{External resistance of secondary circuit}} = \text{percentage internal resistance,}$$

$$\frac{2\,x_0}{R_0} = \mathsf{q} = \text{ratio } \frac{\text{Internal reactance of transformer}}{\text{External resistance of secondary circuit}} = \text{percentage internal reactance,}$$

$$R_0 g_0 = \mathsf{h} = \text{ratio } \frac{\text{Hysteretic power current}}{\text{Total secondary current}} = \text{percentage hysteresis,}$$

$$R_0 b_0 = \mathsf{g} = \text{ratio } \frac{\text{Magnetizing current}}{\text{Total secondary current}} = \text{percentage magnetizing current,}$$

and if d represents the load of the transformer, as fraction of full load, we have

$$R = \frac{R_0}{d},$$

and

$$\frac{2\,r_0}{R} = \mathsf{p}d,$$

$$\frac{2\,x_0}{R} = \mathsf{q}d,$$

$$Rg_0 = \frac{\mathsf{h}}{d},$$

$$Rb_0 = \frac{\mathsf{g}}{d}.$$

Substituting these values we get, as the equations of the transformer on non-inductive load,

Ratio of e.m.fs.:

$$\frac{\dot{E}_0}{\dot{E}_1} = -\,a\left\{1 + d\,(\mathsf{p} - j\mathsf{q}) + \frac{(\mathsf{p} - j\mathsf{q})\,(\mathsf{h} + j\mathsf{g})}{2}\right\}$$

$$\cong -\,a\,\{1 + d\,(\mathsf{p} - j\mathsf{q})\}:$$

or, eliminating imaginary quantities,

$$\frac{e_0}{e_1} = a\sqrt{\left(1 + d\mathsf{p} + \frac{\mathsf{ph} + \mathsf{qg}}{2}\right)^2 + \left(d\mathsf{q} - \frac{\mathsf{pg} - \mathsf{qh}}{2}\right)^2}$$

$$\cong a\sqrt{(1 + d\mathsf{p})^2 + d^2\mathsf{q}^2}$$

$$\cong a\left\{1 + d\mathsf{p} + \frac{\mathsf{ph} + \mathsf{qg} + d^2\mathsf{q}^2}{2}\right\}$$

$$\cong a\,\{1 + d\mathsf{p}\}.$$

Ratio of currents:

$$\frac{\dot{I}_0}{\dot{I}_1} = -\frac{1}{a}\left\{1 + \frac{(\mathsf{h} + j\mathsf{g})}{d} + \frac{(\mathsf{p} - j\mathsf{q})(\mathsf{h} + j\mathsf{g})}{2}\right\}$$

$$\cong -\frac{1}{a}\left\{1 + \left(\frac{\mathsf{h} + j\mathsf{g}}{d}\right)\right\};$$

or, eliminating imaginary quantities,

$$\frac{i_0}{i_1} = \frac{1}{a}\sqrt{\left(1 + \frac{\mathsf{h}}{d} + \frac{\mathsf{ph} + \mathsf{qg}}{2}\right)^2 + \left(\frac{\mathsf{g}}{d} + \frac{\mathsf{pg} - \mathsf{qh}}{2}\right)^2}$$

$$\cong \frac{1}{a}\sqrt{\left(1 + \frac{\mathsf{h}}{d}\right)^2 + \left(\frac{\mathsf{g}}{d}\right)^2}$$

$$\cong \frac{1}{a}\left\{1 + \frac{\mathsf{h}}{d} + \frac{\mathsf{ph} + \mathsf{qg} + \mathsf{g}^2}{2\,d^2}\right\}$$

$$\cong \frac{1}{a}\left\{1 + \frac{\mathsf{h}}{d}\right\}.$$

Total apparent primary impedance:

$$Z_t = \frac{R_0}{d}\left\{1 + d\,(\mathsf{p} - j\mathsf{g}) - \frac{\mathsf{h} + j\mathsf{g}}{d} - (\mathsf{p} - j\mathsf{g})(\mathsf{h} + j\mathsf{g}) + \left(\frac{\mathsf{h} + j\mathsf{g}}{d^2}\right)^2\right\}$$

$$\cong \frac{R_0}{d}\left\{1 + d\,(\mathsf{p} - j\mathsf{g}) - \frac{\mathsf{h} + jg}{d}\right\};$$

or, eliminating imaginary quantities,

$$z_t = \frac{R_0}{d}\sqrt{\left(1 + d\mathsf{p} - \frac{\mathsf{h}}{d} - \mathsf{ph} - \mathsf{qg} + \frac{\mathsf{h}^2 - \mathsf{g}^2}{d^2}\right)^2 + \left(d\mathsf{q} - \frac{\mathsf{g}}{d} - \mathsf{pg} + \mathsf{gh} + 2\frac{\mathsf{hg}}{d^2}\right)^2}$$

$$\cong \frac{R_0}{d}\sqrt{\left(1 + d\mathsf{p} - \frac{\mathsf{h}}{d}\right)^2 + \left(d\mathsf{q} - \frac{\mathsf{g}}{d}\right)^2}$$

$$\cong \frac{R_0}{d}\left\{1 + d\mathsf{p} - \frac{\mathsf{h}}{d} - \mathsf{ph} - 2\,\mathsf{qg} + \frac{\mathsf{h}^2 - \mathsf{g}^2}{d^2} + \frac{d^2\mathsf{q}^2}{2} + \frac{g^2}{2\,d^2}\right.$$

$$\left. + \frac{d^2\mathsf{p}^2}{2} + \frac{\mathsf{h}^2}{2\,d^2}\right\}$$

$$\cong \frac{R_0}{d}\left\{1 + d\mathsf{p} - \frac{\mathsf{h}}{d}\right\}.$$

Angle of time-lag in primary circuit:

$$\tan\theta_0 = \frac{dq + \frac{g}{d} + pg - gh - 2\frac{hg}{d^2}}{1 + dp - \frac{h}{d} - ph - qg + \frac{hg}{d}\frac{h^2 + g^2}{d^2}},$$

$$\cong \frac{dg + \frac{g}{d}}{1 + dp - \frac{h}{d}}.$$

That is,

An alternate-current transformer, feeding into a non-inductive secondary circuit, is represented by the constants:

R_0 = secondary external resistance at full load;
p = percentage resistance;
q = percentage reactance;
h = percentage hysteresis;
g = percentage magnetizing current;
d = percentage secondary load.

All these qualities being considered as reduced to the primary circuit by the square of the ratio of turns, a^2.

166. As an example, a transformer of the following constants may be given:

$e_0 = 1{,}000$;	$R_0 = 120$;	$q = 0.06$;
$a = 10$;	$p = 0.02$;	$h = 0.02$;
		$g = 0.04$.

Substituting these values, gives:

$$e_1 = \frac{100}{\sqrt{(1.0014 + 0.02\,d)^2 + (0.0002 + 0.06\,d)^2}};$$

$$i_1 = \frac{e_1 d a^2}{R_0} = \frac{e_1 d}{1.2};$$

$$i_0 = .1\,i_1\sqrt{\left(1.0014 + \frac{0.02}{d}\right)^2 + \left(\frac{0.04}{d} - 0.0002\right)^2};$$

$$\tan\theta_0 = \frac{0.06\,d + \frac{0.04}{d} - 0.0004 - \frac{0.0016}{d^2}}{1.9972 + 0.02\,d + \frac{0.002}{d^2} - \frac{0.02}{d}}.$$

In diagram, Fig. 120, are shown, for the values from $d = 0$ to $d = 1.5$, with the secondary current, i_1, as abscissas, the values: secondary terminal voltage, in volts; secondary drop of voltage, in per cent; primary current, in amperes; excess of primary current over proportionality with secondary, in per cent, and primary angle of lag.

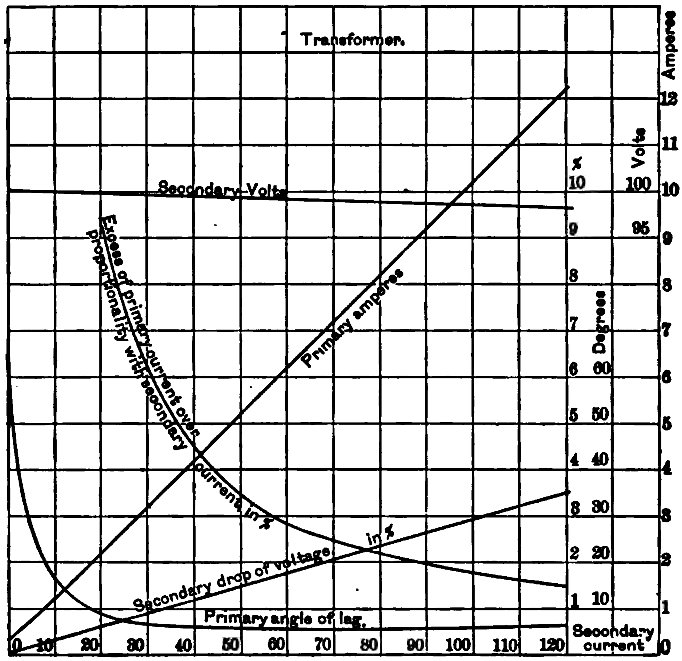


FIG. 120.—Load Diagram of Transformer.

The power-factor of the transformer, $\cos \theta_0$, is 0.45 at open secondary circuit, and is above 0.99 from 25 amperes, upwards, with a maximum of 0.995 at full load.

167. As seen, the alternating-current transformer is characterized by the constants:

Ratio of turns: $a = \frac{n_0}{n_1}$.

Exciting admittance: $Y_0 = g_0 + jb_0$.

Self-inductive impedances: $Z_0 = r_0 - jx_0$.

$Z_1 = r_1 - jx_1$.

Since the effect of the secondary impedance is essentially the same as that of the primary impedance (the only difference being, that no voltage is consumed by the exciting current in the secondary impedance, but voltage is consumed in the primary impedance, though very small in a constant-potential transformer), the individual values of the two impedances, Z_0 and Z_1, are of less importance than the resultant or total impedance of the transformer, that is, the sum of the primary impedance plus the secondary impedance reduced to the primary circuit:

$$Z' = Z_0 + a^2 Z_1,$$

and the transformer accordingly is characterized by the two constants:

Exciting admittance,	$Y_0 = g_0 + jb_0.$
Total self-inductive impedance,	$Z' = r' - jx'.$

Especially in constant-potential transformers with closed magnetic circuit — as usually built — the combination of both impedances into one, Z', is permissible as well within the errors of observation.

Experimentally, the exciting admittance, $Y_0 = g_0 + jb_0$, and the total self-inductive impedance, $Z' = r' - jx'$, are determined by operating the transformer at its normal frequency:

(1) With open secondary circuit, and measuring volts e_0, amperes i_0, and watts w_0, input.

(2) With the secondary short-circuited, and measuring volts e_1, amperes i_1, and watts p_1, input. (In this case, usually a far lower impressed voltage is required.)

It is then:

$$\left.\begin{aligned} y_0 &= \frac{i_0}{e_0}, \\ g_0 &= \frac{p_0}{e_0^2}, \\ b_0 &= \sqrt{y_0^2 - g_0^2}. \end{aligned}\right\} \qquad \left.\begin{aligned} z' &= \frac{e_1}{i_1}, \\ r' &= \frac{p_1}{i_1^2}, \\ x' &= \sqrt{z^{1^2} - r^{1^2}}. \end{aligned}\right\}$$

If a separation of the total impedance Z' into the primary impedance and the secondary impedance is desired, as a rule the

secondary reactance reduced to the primary can be assumed as equal to the primary reactance:

$$a^2 x_1 = x_0,$$

except if from the construction of the transformer it can be seen that one of the circuits has far more reactance than the other, and then judgment or approximate calculation must guide in the division of the total reactance between the two circuits.

If the total effective resistance, r', as derived by wattmeter, equals the sum of the ohmic resistances of primary and of secondary reduced to the primary:

$$r' = r_0 + a^2 r,$$

the ohmic resistances, r_0 and r_1, as measured by Wheatstone bridge or by direct current, are used.

If the effective resistance is greater than the resultant of the ohmic resistances:

$$r' > r_0 + a^2 r_1,$$

the difference:

$$r'' = r' - (r_0 + a^2 r_1)$$

may be divided between the two circuits in proportion to the ohmic resistances, that is, the effective resistance distributed between the two circuits in the proportion of their ohmic resistances, so giving the effective resistances of the two circuits, r_0' and r_1', by:

$$r_0' \div r_1' = r_0 \div r_1;$$

or, if from the construction of the transformer as the use of large solid conductors, it can be seen that the one circuit is entirely or mainly the seat of the power loss by hysteresis, eddies, etc., which is represented by the additional effective resistance, r'', this resistance, r'', is entirely or mainly assigned to this circuit.

In general, it therefore may be assumed:

$$\left.\begin{aligned} x_0 &= \frac{x'}{2}, \\ x_1 &= \frac{x'}{2a^2}, \end{aligned}\right\} \qquad \left.\begin{aligned} r_0' &= r_0 \frac{r'}{r_0 + a^2 r_1}, \\ r_1' &= r_1 \frac{r'}{r_0 + a^2 r_1}. \end{aligned}\right\}$$

In the calculation of the transformer:

The exciting admittance, Y_0, is derived by calculating the total exciting current from the ampere-turns excitation, the magnetic characteristic of the iron and the dimensions of the main magnetic circuit, that is the magnetic circuit interlinked with primary and secondary coils. The conductance, g_0, is derived from the hysteresis loss in the iron, as given by magnetic density, hysteresis coefficient and dimensions of magnetic circuit, allowance being made for eddy currents in the iron.

The ohmic resistances, r_0 and r_1, are found from the dimensions of the electric circuit, and, where required, allowance made for the additional effective resistance, r''.

The reactances, x_0 and x_1, are calculated by calculating the leakage flux, that is the magnetic flux produced by the total primary respectively secondary ampere-turns, and passing between primary and secondary coils, and within the primary respectively secondary coil, in a magnetic circuit consisting largely of air. In this case, the iron part of the magnetic leakage circuit can as a rule be neglected.

CHAPTER XVIII.

THE GENERAL ALTERNATING-CURRENT TRANSFORMER OR FREQUENCY CONVERTER.

168. The simplest alternating-current apparatus is the alternating-current transformer. It consists of a magnetic circuit, interlinked with two electric circuits or sets of electric circuits. The one, the primary circuit, is excited by an impressed e.m.f., while in the other, the secondary circuit, an e.m.f. is generated. Thus, in the primary circuit, power is consumed, in the secondary circuit a corresponding amount of power produced; or in other words, power is transferred through space, from primary to secondary circuit. This transfer of power finds its mechanical equivalent in a repulsive thrust acting between primary and secondary. Thus, if the secondary coil is not held rigidly as in the stationary transformer, it will be repelled and move away from the primary. This mechanical effect is made use of in the induction motor, which represents a transformer whose secondary is mounted movably with regard to the primary in such a way that, while set in rotation, it still remains in the primary field of force. The condition that the secondary circuit, while revolving with regard to the primary, does not leave the primary field of magnetic force, requires that this field is not undirectional, but that an active field exists in every direction. One way of producing such a magnetic field is by exciting different primary circuits angularly displaced in space with each other by currents of different phase. Another way is to excite the primary field in one direction only, and get the cross-magnetization, or the angularly displaced magnetic field, by the reaction of the secondary current.

We see, consequently, that the stationary transformer and the induction motor are merely different applications of the same apparatus, comprising a magnetic circuit interlinked with two electric circuits. Such an apparatus can properly be called a "*general alternating-current transformer.*" The equations of the stationary transformer and those of the induction motor are

merely specializations of the general alternating-current transformer equations.

Quantitatively the main differences between induction motor and stationary transformer are those produced by the air-gap between primary and secondary, which is required to give the secondary mechanical movability. This air-gap greatly increases the magnetizing current over that in the closed magnetic-circuit transformer, and requires an ironclad construction of primary and secondary to keep the magnetizing current within reasonable limits. An iron-clad construction again greatly increases the self-induction of primary and secondary circuit. Thus the induction motor is a transformer of large magnetizing current and large self-induction; that is, comparatively large primary exciting susceptance and large reactance.

The general alternating-current transformer transforms between electrical and mechanical power, and changes not only e.m.fs. and currents, but frequencies also, and may therefore be called a "frequency converter." Obviously, it also may change the number of phases.

169. Besides the magnetic flux interlinked with both primary and secondary electric circuit, a magnetic cross-flux passes in the transformer between primary and secondary, surrounding one coil only, without being interlinked with the other. This magnetic cross-flux is proportional to the current in the electric circuit, and constitutes what is called the self-induction of the transformer. As seen, as self-induction of a transformer circuit, not the total flux produced by and interlinked with this circuit is understood, but only that — usually small — part of the flux which surrounds the one circuit without interlinking with the other, and is thus produced by the m.m.f. of one circuit only.

170. The mutual magnetic flux of the transformer is produced by the resultant m.m.f. of both electric circuits. It is determined by the counter e.m.f., the number of turns, and the frequency of the electric circuit, by the equation:

$$E = \sqrt{2}\,\pi f n \Phi\, 10^{-8},$$

where E = effective e.m.f.,

f = frequency,

n = number of turns,

Φ = maximum magnetic flux.

The m.m.f. producing this flux, or the resultant m.m.f. of primary and secondary circuit, is determined by shape and magnetic characteristic of the material composing the magnetic circuit, and by the magnetic induction. At open secondary circuit, this m.m.f. is the m.m.f. of the primary current, which in this case is called the exciting current, and consists of a power component, the magnetic power current, and a reactive component, the magnetizing current.

171. In the general alternating-current transformer, where the secondary is movable with regard to the primary, the rate of cutting of the secondary electric circuit with the mutual magnetic flux is different from that of the primary. Thus, the frequencies of both circuits are different, and the generated e.m.fs. are not proportional to the number of turns as in the stationary transformer, but to the product of number of turns into frequency.

172. Let, in a general alternating-current transformer,

$$s = \text{ratio } \frac{\text{secondary}}{\text{primary}} \text{ frequency, or "slip";}$$

thus, if

$$f = \text{primary frequency, or frequency of impressed e.m.f.,}$$

$$sf = \text{secondary frequency;}$$

and the e.m.f. generated per secondary turn by the mutual flux has to the e.m.f. generated per primary turn the ratio, s,

$s = 0$ represents synchronous motion of the secondary;

$s < 0$ represents motion above synchronism — driven by external mechanical power, as will be seen;

$s = 1$ represents standstill;

$s > 1$ represents backward motion of the secondary,

that is, motion against the mechanical force acting between primary and secondary (thus representing driving by external mechanical power).

Let

n_0 = number of primary turns in series per circuit;

n_1 = number of secondary turns in series per circuit;

$a = \frac{n_0}{n_1}$ = ratio of turns;

$Y_0 = g_0 + jb_0$ = primary exciting admittance per circuit;

where

g_0 = effective conductance;

b_0 = susceptance;

$Z_0 = r_0 - jx_0$ = internal primary self-inductive impedance per circuit,

where

r_0 = effective resistance of primary circuit;

x_0 = self-inductive reactance of primary circuit;

$Z_{11} = r_1 - jx_1$ = internal secondary self-inductive impedance per circuit at standstill, or for $s = 1$,

where

r_1 = effective resistance of secondary coil;

x_1 = self-inductive reactance of secondary coil at standstill, or full frequency, $s = 1$.

Since the reactance is proportional to the frequency, at the slip, s, or the secondary frequency, sf, the secondary impedance is

$$Z_1 = r_1 - jsx_1.$$

Let the secondary circuit be closed by an external resistance, r, and an external reactance, and denote the latter by x at frequency f, then at frequency sf, or slip s, it will be $= sx$, and thus

$$Z = r - jsx = \text{external secondary impedance.*}$$

* This applies to the case where the secondary contains inductive reactance only; or, rather, that kind of reactance which is proportional to the frequency. In a condenser the reactance is inversely proportional to the frequency, in a synchronous motor under circumstances independent of the frequency. Thus, in general, we have to set, $x = x' + x'' + x'''$, where x' is that part of the reactance which is proportional to the frequency, x'' that part of the reactance independent of the frequency, and x''' that part of the reactance which is inversely proportional to the frequency; and have thus, at slip s, or frequency sf, the external secondary reactance $sx' + x'' + \frac{x'''}{s}$.

Let

$\dot{E}_0$ = primary impressed e.m.f. per circuit,
$\dot{E}'$ = e.m.f. consumed by primary counter e.m.f.,
$\dot{E}_1$ = secondary terminal e.m.f.,
$\dot{E}_1'$ = secondary generated e.m.f.,
e = e.m.f. generated per turn by the mutual magnetic flux, at full frequency f,
$\dot{I}_0$ = primary current,
$\dot{I}_{00}$ = primary exciting current,
$\dot{I}_1$ = secondary current.

It is then,

Secondary generated e.m.f.

$$\dot{E}_1' = sn_1e.$$

Total secondary impedance

$$Z_1 + Z = (r_1 + r) - js\,(x_1 + x);$$

hence, secondary current

$$\dot{I}_1 = \frac{\dot{E}_1'}{Z_1 + Z} = \frac{sn_1e}{(r_1 + r) - js\,(x_1 + x)}.$$

Secondary terminal voltage

$$\dot{E}_1 = \dot{E}_1' - \dot{I}_1Z_1 = \dot{I}_1Z$$

$$= sn_1e\left\{1 - \frac{r_1 - jsx_1}{(r_1 + r) - js\,(x_1 + x)}\right\} = \frac{sn_1e\,(r - jsx)}{(r_1 + r) - js\,(x_1 + x)}.$$

e.m.f. consumed by primary counter e.m.f.

$$\dot{E}' = -n_0e;$$

hence, primary exciting current:

$$\dot{I}_{00} = \dot{E}'Y_0 = -n_0e\,(g_0 + jb_0).$$

Component of primary current corresponding to secondary current $\dot{I}_1$:

$$\dot{I}_0' = -\frac{I_1}{a}$$

$$= -\frac{n_0se}{a^2\,\{(r_1 + r) - js\,(x_1 + x)\}};$$

hence, total primary current,

$$\dot{I}_0 = \dot{I}_{00} + \dot{I}_0{}'$$
$$= - sn_0 e \left\{ \frac{1}{a_2} \frac{1}{(r_1 + r) - js\,(x_1 + x)} + \frac{g_0 + jb_0}{s} \right\}.$$

Primary impressed e.m.f.,

$$\dot{E}_0 = E' + \dot{I}_0 Z_0$$
$$= - n_0 e \left\{ 1 + \frac{s}{a^2} \frac{r_0 - jx_0}{(r_1 + r) - js\,(x_1 + x)} + (r_0 - jx_0)\,(g_0 + jb_0) \right\}.$$

We get thus, as the

Equations of the General Alternating-Current Transformer, of ratio of turns, a; and ratio of frequencies, s; with the e.m.f. generated per turn at full frequency, e, as parameter, the values:

Primary impressed e.m.f.,

$$\dot{E}_0 = - n_0 e \left\{ 1 + \frac{s}{a^2} \frac{r_0 - jx_0}{(r_1 + r) - js\,(x_1 + x)} + (r_0 - jx_0)\,(g_0 + jb_0) \right\}.$$

Secondary terminal voltage,

$$\dot{E}_1 = sn_1 e \left\{ 1 - \frac{r_1 - jsx_1}{(r_1 + r) - js\,(x_1 + x)} \right\} = sn_1 \frac{r - jsx}{(r_1 + r) - js\,(x_1 + x)}.$$

Primary current,

$$I_0 = - sn_0 e \left\{ \frac{1}{a^2} \frac{1}{(r_1 + r) - js\,(x_1 + x)} + \frac{g_0 + jb_0}{s} \right\}.$$

Secondary current,

$$\dot{I}_1 = \frac{sn_1 e}{(r_1 + r) - js\,(x_1 + x)}.$$

Therefrom, we get:

Ratio of currents,

$$\frac{\dot{I}_0}{\dot{I}_1} = - \frac{1}{a} \left\{ 1 + \frac{a^2}{s} (g_0 + jb_0) \left[(r_1 + r) - js\,(x_1 + x) \right] \right\}.$$

Ratio of e.m.fs.,

$$\frac{\dot{E}_0}{\dot{E}_1} = - \frac{a}{s} \left\{ \frac{1 + \dfrac{s}{a^2} \dfrac{r_0 - jx_0}{(r_1 + r) - js\,(x_1 + x)} + (r_0 - jx_0)\,(g_0 + jb_0)}{1 - \dfrac{r_1 - jsx_1}{(r_1 + r) - js\,(x_1 + x)}} \right\}.$$

Total apparent primary impedance,

$$Z_t = \frac{\dot{E}_0}{\dot{I}_0} = \frac{a^2}{s} \{(r_1 + r) - js(x_1 + x)\}$$

$$\left\{\frac{1 + \dfrac{s}{a^2} \dfrac{r_0 - jx_0}{(r_1 + r) + js(x_1 + x)} + (r_0 - jx_0)(g_0 + jb_0)}{1 + \dfrac{a^2}{s}(g_0 + jb_0)[(r_1 + r) - js(x_1 + x)]}\right\},$$

where $$x = x' + \frac{x''}{s} + \frac{x'''}{s^2}$$

in the general secondary circuit as discussed in foot-note, page 265.

Substituting in these equations:

$$s = 1,$$

gives the

General Equations of the Stationary Alternating-Current Transformer:

$$\dot{E}_0 = -n_0 e \left\{1 + \frac{1}{a^2} \frac{Z_0}{Z_1 + Z} + Z_0 Y_0\right\}.$$

$$\dot{E}_1 = n_1 e \left\{1 - \frac{Z_1}{Z_1 + Z}\right\} = n_1 e \frac{Z}{Z_1 + Z}.$$

$$\dot{I}_0 = -n_0 e \left\{\frac{1}{a^2 (Z_1 + Z)} + Y_0\right\}.$$

$$\dot{I}_1 = \frac{n_1 e}{Z_1 + Z},$$

$$\frac{\dot{I}_0}{\dot{I}_1} = -\frac{1}{a}\{1 + a^2 Y_0 (Z_1 + Z)\},$$

$$\frac{\dot{E}_0}{\dot{E}_1} = -a \left\{\frac{1 + \dfrac{Z_0}{a^2 (Z_1 + Z)} + Z_0 Y_0}{1 - \dfrac{Z_1}{Z_1 + Z}}\right\},$$

$$Z_t = \frac{\dot{E}_0}{\dot{I}_0} = a^2 (Z_1 + Z) \left\{\frac{1 + \dfrac{Z_0}{a^2 (Z_1 + Z)} + Z_0 Y_0}{1 + a^2 Y_0 (Z_1 + Z)}\right\}.$$

Substituting in the equations of the general alternating-current transformer,

$$Z = 0,$$

gives the

General Equations of the Induction Motor:

$$\dot{E}_0 = -n_0 e \left\{ 1 + \frac{s}{a^2} \frac{r_0 - jx_0}{r_1 - jsx_1} + (r_0 - jx_0)(g_0 + jb_0) \right\},$$

$$\dot{E}_1 = 0.$$

$$\dot{I}_0 = -sn_0 e \left\{ \frac{1}{a^2 (r_1 - jsx_1)} + \frac{g_0 + jb_0}{s} \right\},$$

$$\dot{I}_1 = \frac{sn_1 e}{r_1 - jsx_1},$$

$$\frac{\dot{I}_0}{\dot{I}_1} = -\frac{1}{a} \left\{ 1 + \frac{a^2}{s} (g_0 + jb_0)(r_1 - jsx_1) \right\},$$

$$Z_t = \frac{a^2}{s} (r_1 - jsx_1) \left\{ \frac{1 + \frac{s}{a^2} \frac{r_0 - jx_0}{r_1 - jsx_1} + (r_0 - jx_0)(g_0 + jb_0)}{1 + \frac{a^2}{s} (r_1 - jsx_1)(g_0 + jb_0)} \right\}.$$

Returning now to the general alternating-current transformer, we have, by substituting

$$(r_1 + r)^2 + s^2 (x_1 + x)^2 = z_k^2,$$

and separating the real and imaginary quantities,

$$\begin{aligned} \dot{E}_0 = -n_0 e \Bigg\{ \Bigg[1 &+ \frac{s}{a^2 z_k^2} (r_0 (r_1 + r) + sx_0 (x_1 + x)) \\ &+ (r_0 g_0 + x_0 b_0) \Bigg] + j \Bigg[\frac{s}{a^2 z_k^2} (sr_0 (x_1 + x) - x (r_1 + r)) \\ &+ (r_0 b_0 - x_0 g_0) \Bigg] \Bigg\}. \end{aligned}$$

$$\dot{I}_0 = -sn_0 e \left\{ \left[\frac{r_1 + r}{a^2 z_k^2} + \frac{g_0}{s} \right] + j \left[\frac{s (x_1 + x)}{a^2 z_k^2} + \frac{b_0}{s} \right] \right\},$$

$$\dot{I}_1 = \frac{sn_1 e}{z_k^2} \left\{ (r_1 + r) + js (x_1 + x) \right\}.$$

Neglecting the exciting current, or rather considering it as a separate and independent shunt circuit outside of the trans-

former, as can approximately be done, and assuming the primary impedance reduced to the secondary circuit as equal to the secondary impedance,

$$Y_0 = 0, \qquad \frac{Z_0}{a^2} = Z_1.$$

Substituting this in the equations of the general transformer, we get,

$$\dot{E}_0 = -n_0 e \left\{ 1 + \frac{s}{z_k^2}[r_1(r_1 + r) + sx_1(x_1 + x)] + \frac{js}{z_k^2}[sr_1(x_1 + x) - x_1(r_1 + r)] \right\},$$

$$\dot{E}_1 = \frac{sn_1 e}{z_k^2}\{[r(r_1 + r) + s^2 x(x_1 + x)] + js[rx_1 - xr_1]\},$$

$$\dot{I}_0 = -\frac{sn_1 e}{az_k^2}\{(r_1 + r) + js(x_1 + x)\},$$

$$\dot{I}_1 = \frac{sn_1 e}{z_k^2}\{(r_1 + r) + js(x_1 + x)\}.$$

173. The true power is, in symbolic representation (see Chapter XV):

$$P = [\dot{E}\dot{I}]^1,$$

denoting,

$$\frac{sn_1^2 e^2}{z_k^2} = w$$

gives:

Secondary output of the transformer,

$$P_1 = [\dot{E}_1 \dot{I}_1]^1 = \left(\frac{sn_1 e}{z_k}\right)^2 r = srw;$$

Internal loss in secondary circuit,

$$P_1^1 = i_1^2 r_1 = \left(\frac{sn_1 e}{z_k}\right)^2 r_1 = sr_1 w;$$

Total secondary power,

$$P_1 + P_1^1 = \left(\frac{sn_1 e}{z_k}\right)^2 (r + r_1) = sw(r + r_1);$$

Internal loss in primary circuit,

$$P_0^1 = i_0^2 r_0 = i_0^2 r_1 a^2 = \left(\frac{sn_1 e}{z_k}\right)^2 r_1 = sr_1 w;$$

Total electrical output, plus loss,

$$P^1 = P_1 + P_1^1 + P_0^1 = \left(\frac{sn_1 e}{z_k}\right)^2 (r + 2r_1) = sw\,(r + 2r_1);$$

Total electrical input of primary,

$$P_0 = [E_0 I_0]^1 = s\left(\frac{n_1 e}{z_k}\right)^2 (r + r_1 + sr_1) = w\,(r + r_1 + sr_1);$$

Hence, mechanical output of transformer,

$$P = P_0 - P^1 = w\,(1 - s)\,(r + r_1);$$

Ratio,

$$\frac{\text{mechanical output}}{\text{total secondary power}} = \frac{P}{P_1 + P_1^1} = \frac{1 - s}{s} = \frac{\text{speed}}{\text{slip}}.$$

174. Thus,

In a general alternating transformer of ratio of turns, a, and ratio of frequencies, s, neglecting exciting current, it is:

Electrical input in primary,

$$P_0 = \frac{sn_1^2 e^2 (r + r_1 + r_1 s)}{(r_1 + r)^2 + s^2 (x_1 + x)^2};$$

Mechanical output,

$$P = \frac{s\,(1 - s) n_1^2 e^2 (r + r_1)}{(r_1 + r)^2 + s^2 (x_1 + x)^2};$$

Electrical output of secondary,

$$P_1 = \frac{s^2 n_1^2 e^2 r}{(r_1 + r)^2 + s^2 (x_1 + x)^2};$$

Losses in transformer,

$$P_0^1 + P_1^1 = P^1 = \frac{2\,s^2 n_1^2 e^2 r_1}{(r_1 + r)^2 + s^2 (x_1 + x)^2}.$$

Of these quantities, P^1 and P_1 are always positive; P_0 and P can be positive or negative, according to the value of s. Thus the apparatus can either produce mechanical power, acting as a motor, or consume mechanical power; and it can either consume electrical power or produce electrical power, as a generator.

175. At

$$s = 0, \text{ synchronism, } P_0 = 0, \quad P = 0, \quad P_1 = 0.$$

At $0 < s < 1$, between synchronism and standstill.

P_1, P and P_0 are positive; that is, the apparatus consumes electrical power P_0 in the primary, and produces mechanical power P and electrical power $P_1 + P_1^1$ in the secondary, which is partly, P_1^1, consumed by the internal secondary resistance, partly, P_1, available at the secondary terminals.

In this case

$$\frac{P_1 + P_1^1}{P} = \frac{s}{1 - s};$$

that is, of the electrical power consumed in the primary circuit, P_0, a part P_0^1 is consumed by the internal primary resistance, the remainder transmitted to the secondary, and divides between electrical power, $P_1 + P_1^1$, and mechanical power, P, in the proportion of the slip, or drop below synchronism, s, to the speed: $1 - s$.

In this range, the apparatus is a motor.

At $s > 1$; or, backwards driving, $P < 0$, or negative; that is, the apparatus requires mechanical power for driving.

Then: $$P_0 - P_0^1 - P_1^1 < P_1;$$

that is, the secondary electrical power is produced partly by the primary electrical power, partly by the mechanical power, and the apparatus acts simultaneously as transformer and as alternating-current generator, with the secondary as armature.

The ratio of mechanical input to electrical input is the ratio of speed to synchronism.

In this case, the secondary frequency is higher than the primary.

At $s < 0$, beyond synchronism,

$P < 0$; that is, the apparatus has to be driven by mechanical power.

$P_0 < 0$; that is, the primary circuit produces electrical power from the mechanical input.

At $r + r_1 + sr_1 = 0$, or, $s < -\frac{r + r_1}{r_1}$;

the electrical power produced in the primary becomes less than required to cover the losses of power, and P_0 becomes positive again.

We have thus:

$$s < -\frac{r + r_1}{r_1}$$

consumes mechanical and primary electric power; produces secondary electric power.

$$-\frac{r + r_1}{r_1} < s < 0$$

consumes mechanical, and produces electrical power in primary and in secondary circuit.

$$0 < s < 1$$

consumes primary electric power, and produces mechanical and secondary electrical power.

$$1 < s$$

consumes mechanical and primary electrical power; produces secondary electrical power.

176. As an example, in Fig. 121 are plotted, with the slip s as abscissas, the values of

Secondary electrical output	as Curve I.;
total internal loss	as Curve II.;
mechanical output	as Curve III.;
primary electrical input	as Curve IV.;

for the values:

$$n_1 e = 100.0; \qquad r = 0.4;$$
$$r_1 = 0.1; \qquad x = 0.3;$$
$$x_1 = 0.2;$$

hence,

$$P_1 = \frac{16{,}000\, s^2}{1 + s^2};$$

$$P_0^1 + P_1^1 = \frac{8{,}000\, s^2}{1 + s^2};$$

$$P_0 = \frac{4{,}000\, s\,(5 + s)}{1 + s^2};$$

$$P = \frac{20{,}000\, s\,(1 - s)}{1 + s^2}.$$

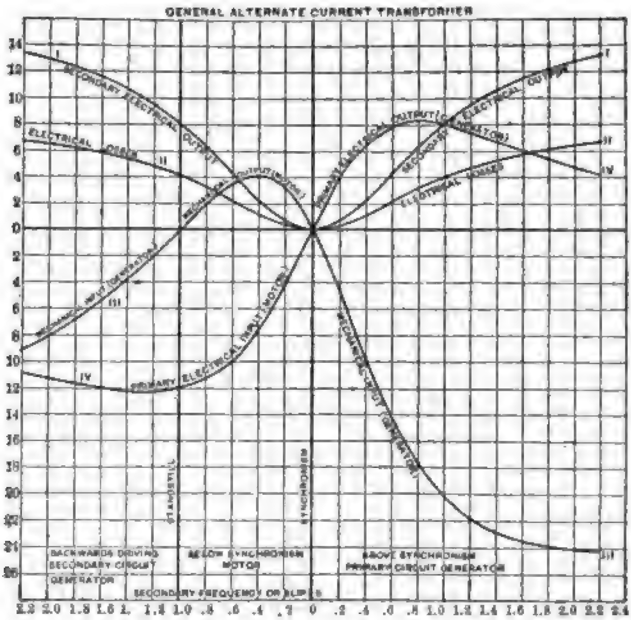


FIG. 121.

177. Since the most common practical application of the general alternating-current transformer is that of frequency converter, that is, to change from one frequency to another, either with or without change of the number of phases, the

following characteristic curves of this apparatus are of great interest.

1. The regulation curve; that is, the change of secondary terminal voltage as function of the load at constant impressed primary voltage.

2. The compounding curve; that is, the change of primary impressed voltage required to maintain constant secondary terminal voltage.

In this case the impressed frequency and the speed are constant, and consequently the secondary frequency is also constant. Generally the frequency converter is used to change from a low frequency, as 25 cycles, to a higher frequency, as 62.5 cycles, and is then driven backward, that is, against its torque, by mechanical power. Mostly a synchronous motor is employed, connected to the primary mains, which by over-excitation compensates also for the lagging current of the frequency converter.

Let,

$Y_0 = g_0 + jb_0$ = primary exciting admittance per circuit of the frequency converter.

$Z_1 = r_1 - jx_1$ = internal self-inductive impedance per secondary circuit, at the secondary frequency.

$Z_0 = r_0 - jx_0$ = internal self-inductive impedance per primary circuit at the primary frequency.

a = ratio of secondary to primary turns per circuit.

b = ratio of number of secondary to number of primary circuits.

c = ratio of secondary to primary frequencies.

Let,

e = generated e.m.f. per secondary circuit at secondary frequency.

$Z = r - jx$ = external impedance per secondary circuit at secondary frequency, that is load on secondary system, where $x = 0$ for non-inductive load.

We then have,

the total secondary impedance,

$$Z + Z_1 = (r + r_1) - j(x + x_1);$$

the secondary current,

$$\dot{I}_1 = \frac{e}{Z + Z_1} = e(a_1 + ja_2);$$

where,

$$a_1 = \frac{r + r_1}{(r + r_1)^2 + (x + x_1)^2} \quad \text{and} \quad a_2 = \frac{x + x_1}{(r + r_1)^2 + (x + x_1)^2};$$

and the secondary terminal voltage,

$$\dot{E}_1 = \dot{I}_1 \dot{Z} = e \frac{Z}{Z + Z_1};$$

$$= e(r - jx)(a_1 + ja_2) = e(b_1 + jb_2);$$

where,

$$b_1 = (ra_1 + xa_2) \quad \text{and} \quad b_2 = (ra_2 - xa_1);$$

primary generated e.m.f. per circuit,

$$\dot{E}^1 = \frac{e}{ac};$$

primary load current per circuit,

$$\dot{I}^1 = ab\dot{I}_1 = abe(a_1 + ja_2);$$

primary exciting current per circuit,

$$\dot{I}_{00} = \frac{Y_0 e}{ac} = (g_0 + jb_0)\frac{e}{ac}:$$

thus, total primary current,

$$\dot{I}_0 = \dot{I}^1 + \dot{I}_{00} = e(c_1 + jc_2);$$

where,

$$c_1 = aba_1 + \frac{g_0}{ac} \quad \text{and} \quad c_2 = aba_2 + \frac{b_0}{ac};$$

and the primary terminal voltage:

$$\dot{E}_0 = \dot{E}^1 + \dot{I}_0 Z_0$$
$$= e\,(d_1 + jd_2)$$

where,

$$d_1 = \frac{1}{ac} + r_0 c_1 + x_0 c_2 \quad \text{and} \quad d_2 = r_0 c_2 - x_0 c_1;$$

or the absolute value is

$$e_0 = e\sqrt{d_1^2 + d_2^2}, \quad e = \frac{e_0}{\sqrt{d_1^2 + d_2^2}}:$$

substituting this value of e in the preceding equations, gives, as function of the primary impressed e.m.f., e_0:

secondary current,

$$\dot{I}_1 = \frac{e_0(a_1 + ja_2)}{\sqrt{d_1^2 + d_2^2}} \text{ or, absolute, } I_1 = e_0\sqrt{\frac{a_1^2 + a_2^2}{d_1^2 + d_2^2}};$$

secondary terminal voltage,

$$\dot{E}_1 = \frac{e_0\,(b_1 + jb_2)}{\sqrt{d_1^2 + d_2^2}} \qquad E_1 = e_0\sqrt{\frac{b_1^2 + b_2^2}{d_1^2 + d_2^2}};$$

primary current,

$$\dot{I}_0 = \frac{e_0\,(c_1 + jc_2)}{\sqrt{d_1^2 + d_2^2}} \qquad I_0 = e_0\sqrt{\frac{c_1^2 + c_2^2}{d_1^2 + d_2^2}};$$

primary impressed e.m.f.

$$\dot{E}_0 = \frac{e_0\,(d_1 + jd_2)}{\sqrt{d_1^2 + d_2^2}};$$

secondary output,

$$P_1 = [\dot{E}_1\dot{I}_1]^1 = \frac{e_0^2\,(a_1b_1 + a_2b_2)}{d_1^2 + d_2^2};$$

primary electrical input,

$$P_0 = [\dot{E}_0\dot{I}_0]^1 = \frac{e_0^2\,(c_1d_1 + c_2d_2)}{d_1^2 + d_2^2};$$

primary apparent input, volt-amperes,

$$P_{a_0} = e_0 I_0.$$

Substituting thus different values for the secondary external impedance, Z, gives the regulation curve of the frequency converter.

Such a curve, taken from tests of a 200-kw. frequency converter

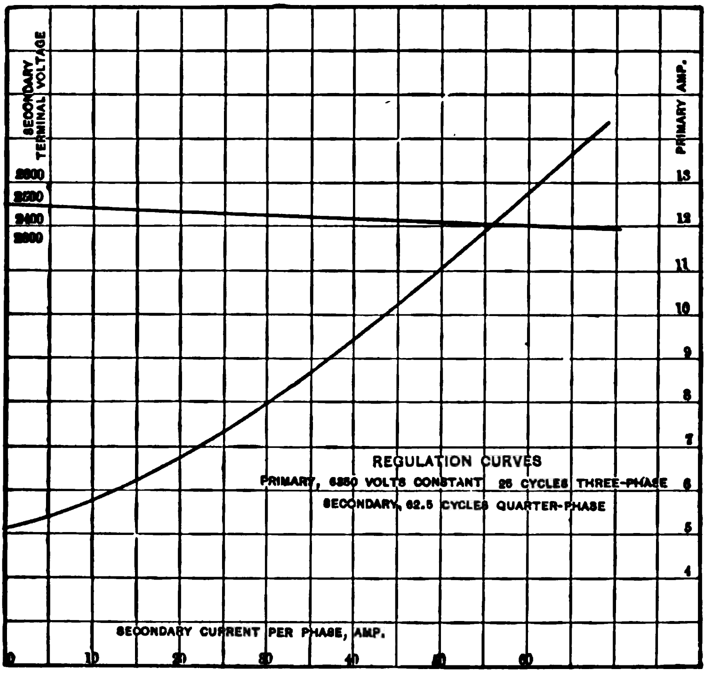


FIG. 122.

changing from 6300 volts, 25 cycles, three-phase, to 2500 volts, 62.5 cycles, quarter-phase, is given in Fig. 122.

From the secondary terminal voltage,

$$\dot{E}_1 = e\,(b_1 + jb_2),$$

it follows, absolute,

$$e_1 = e\sqrt{b_1^2 + b_2^2}, \qquad e = \frac{e_1}{\sqrt{b_1^2 + b_2^2}}.$$

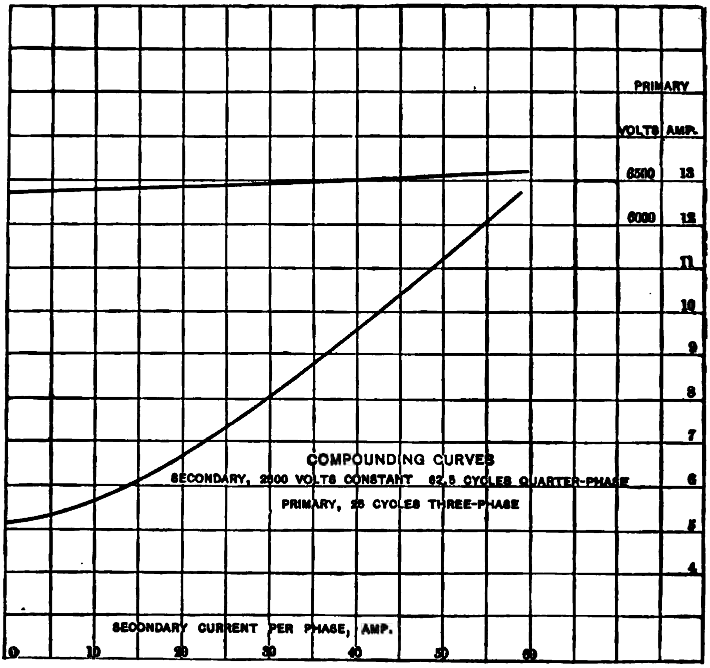


FIG. 123.

Substituting these values in the above equation gives the quantities as functions of the secondary terminal voltage, that is, at constant e_1, or the compounding curve.

The compounding curve of the frequency converter above mentioned is given in Fig. 123.

CHAPTER XIX.

INDUCTION MACHINES.

178. A specialization of the general alternating-current transformer is the induction motor. It differs from the stationary alternating-current transformer, which is also a specialization of the general transformer, in so far as in the stationary transformer only the transfer of electrical energy from primary to secondary is used, but not the mechanical force acting between the two, and therefore primary and secondary coils are held rigidly in position with regard to each other. In the induction motor, only the mechanical force between primary and secondary is used, but not the transfer of electrical energy, and thus the secondary circuits closed upon themselves. Transformer and induction motor thus are the two limiting cases of the general alternating-current transformer. Hence the induction motor consists of a magnetic circuit interlinked with two electric circuits or sets of circuits, the primary and the secondary circuit, which are movable with regard to each other. In general a number of primary and a number of secondary circuits are used, angularly displaced around the periphery of the motor, and containing e.m.fs. displaced in phase by the same angle. This multi-circuit arrangement has the object always to retain secondary circuits in inductive relation to primary circuits and *vice versa*, in spite of their relative motion.

The result of the relative motion between primary and secondary is, that the e.m.fs. generated in the secondary or the motor armature are not of the same frequency as the e.m.fs. impressed upon the primary, but of a frequency which is the difference between the impressed frequency and the frequency of rotation, or equal to the "slip," that is, the difference between synchronism and speed (in cycles).

Hence, if

f = frequency of main or primary e.m.f.,

s = percentage slip;

sf = frequency of armature or secondary e.m.f.,

and, $(1 - s) f$ = frequency of rotation of armature.

In its reaction upon the primary circuit, however, the armature current is of the same frequency as the primary current, since it is carried around mechanically, with a frequency equal to the difference between its own frequency and that of the primary. Or rather, since the reaction of the secondary on the primary must be of primary frequency — whatever the speed of rotation — the secondary frequency is always such as to give at the existing speed of rotation a reaction of primary frequency.

179. Let the primary system consist of p_0 equal circuits, displaced angularly in space by $\frac{1}{p_0}$ of a period, that is, $\frac{1}{p_0}$ of the width of two poles, and excited by p_0 e.m.fs. displaced in phase by $\frac{1}{p_0}$ of a period; that is, in other words, let the field circuits consist of a symmetrical p_0-phase system. Analogously, let the armature or secondary circuits consist of a symmetrical p_1-phase system.

Let

n_0 = number of primary turns per circuit or phase;

n_1 = number of secondary turns per circuit or phase;

$a = \frac{n_0 p_0}{n_1 p_1}$ = ratio of total primary turns to total secondary turns or ratio of transformation.

Since the number of secondary circuits and number of turns of the secondary circuits, in the induction motor — as in the stationary transformer — is entirely unessential, it is preferable

to reduce all secondary quantities to the primary system, by the ratio of transformation, a; thus

if $\dot{E}_1{}'$ = secondary e.m.f. per circuit,

$\dot{E}_1 = a\dot{E}_1{}'$ = secondary e.m.f. per circuit reduced to primary system;

if $\dot{I}_1{}'$ = secondary current per circuit,

$\dot{I}_1 = \frac{\dot{I}_1{}'}{a}$ = secondary current per circuit reduced to primary system;

if $r_1{}'$ = secondary resistance per circuit,

$r_1 = a^2 r'_1$ = secondary resistance per circuit reduced to primary system;

if $x_1{}'$ = secondary reactance per circuit,

$x_1 = a^2 x_1{}'$ = secondary reactance per circuit reduced to primary system;

if $z_1{}'$ = secondary impedance per circuit,

$z_1 = a^2 z_1{}'$ = secondary impedance per circuit reduced to primary system;

that is, the number of secondary circuits and of turns per secondary circuit is assumed the same as in the primary system.

In the following discussion, as secondary quantities, the values reduced to the primary system shall be exclusively used, so that, to derive the true secondary values, these quantities have to be reduced backwards again by the factor

$$a = \frac{n_0 p_0}{n_1 p_1}.$$

180. Let

Φ = total maximum flux of the magnetic field per motor pole.

We then have

$E = \sqrt{2}\, \pi n_0 f\ \Phi\ 10^{-8}$ = effective e.m.f. generated by the magnetic field per primary circuit.

Counting the time from the moment where the rising magnetic flux of mutual induction, Φ (flux interlinked with both

electric circuits, primary and secondary), passes through zero, in complex quantities, the magnetic flux is denoted by

$$\dot{\Phi} = j\Phi,$$

and the primary generated e.m.f.,

$$E = - e;$$

where

$e = \sqrt{2}\,\pi n f \Phi\, 10^{-8}$ may be considered as the "active e.m.f. of the motor," or "counter e.m.f."

Since the secondary frequency is sf, the secondary induced e.m.f. (reduced to primary system) is $\dot{E}_1 = - se$.

Let

$\dot{I}_0 =$ exciting current, or current through the motor, per primary circuit, when doing no work (at synchronism),

and

$$Y = g + jb = \text{primary admittance per circuit} = \frac{\dot{I}_0}{e}.$$

We thus have,

ge = magnetic power current, ge^2 = loss of power by hysteresis (and eddy currents) per primary coil.

Hence

$p_0 ge^2$ = total loss of power by hysteresis and eddies, as calculated according to Chapter XIII.

be = magnetizing current, and

$n_0 be$ = effective m.m.f. per primary circuit;

hence $\frac{p_0}{2} n_0 be$ = total effective m.m.f.,

and

$\frac{p_0}{\sqrt{2}} n_0 be$ = total maximum m.m.f., as resultant of the m.m.fs. of the p_0-phases, combined by the parallelogram of m.m.fs.*

* Complete discussion hereof, see Chapter XXXIII.

If $\mathcal{R}$ = reluctance of magnetic circuit per pole, as discussed in Chapter XIII, it is

$$\frac{p_0}{\sqrt{2}} n_0 be = \mathcal{R}\Phi.$$

Thus, from the hysteretic loss, and the reluctance, the constants, g and b and thus the admittance, Y, are derived.

Let r_0 = resistance per primary circuit;

x_0 = reactance per primary circuit;

thus,

$Z_0 = r_0 - jx_0$ = impedance per primary circuit;

r_1 = resistance per secondary circuit reduced to primary system;

x_1 = reactance per secondary circuit reduced to primary system, at full frequency f;

hence,

sx_1 = reactance per secondary circuit at slip s,

and

$Z_1 = r_1 - jsx_1$ = secondary internal impedance.

181. We now have,

Primary generated e.m.f.,

$$\dot{E} = - e.$$

Secondary generated e.m.f.,

$$\dot{E}_1 = - se.$$

Hence,

Secondary current,

$$\dot{I}_1 = \frac{\dot{E}_1}{Z_1} = - \frac{se}{r_1 - jsx_1}.$$

Component of primary current, corresponding thereto, or primary load current,

$$\dot{I}' = - \dot{I}_1 = \frac{se}{r_1 - jsx_1};$$

Primary exciting current,

$$\dot{I}_0 = eY = e(g + jb); \text{ hence,}$$

Total primary current,

$$\dot{I} = \dot{I}' + \dot{I}_0$$
$$= e\left\{\frac{s}{r_1 - jsx_1} + (g + jb)\right\};$$

e.m.f. consumed by primary impedance,

$$\dot{E}_z = Z_0\dot{I}$$
$$= e(r_0 - jx_0)\left\{\frac{s}{r_1 - jsx_1} + (g + jb)\right\};$$

e.m.f. required to overcome the primary generated e.m.f.,

$$-\dot{E} = e;$$

hence,

Primary terminal voltage,

$$E_0 = e + \dot{E}_z$$
$$= e\left\{1 + \frac{s(r_0 - jx_0)}{r_1 - jsx_1} + (r_0 - jx_0)(g + jb)\right\}.$$

We get thus, in an induction motor, at slip s and active e.m.f. e,

Primary terminal voltage,

$$\dot{E}_0 = e\left\{1 + \frac{s(r_0 - jx_0)}{r_1 - jsx_1} + (r_0 - jx_0)(g + jb)\right\};$$

Primary current,

$$\dot{I} = e\left\{\frac{s}{r_1 - jsx_1} + (g + jb)\right\};$$

or, in complex expression,

Primary terminal voltage,

$$\dot{E}_0 = e\left\{1 + s\frac{Z_0}{Z_1} + Z_0 Y\right\};$$

Primary current,

$$\dot{I} = e\left\{\frac{s}{Z_1} + Y\right\}.$$

To eliminate e, we divide, and get,

Primary current, at slip s, and impressed e.m.f., $\dot{E}_0$;

$$\dot{I} = \frac{s + Z_1 Y}{Z_1 + sZ_0 + Z_0 Z_1 Y} \dot{E}_0;$$

or,

$$\dot{I} = \frac{s + (r_1 - jsx_1)(g + jb)}{(r_1 - jsx_1) + s(r_0 - jx_0) + (r_0 - jx_0)(r_1 - jsx_1)(g + jb)} E_0.$$

Neglecting, in the denominator, the small quantity $Z_0 Z_1 Y$, it is

$$\dot{I} = \frac{s + Z_1 Y}{Z_1 + sZ_0} E_0$$

$$= \frac{s + (r_1 - jsx_1)(g + jb)}{(r_1 - j\dot{s}x_1) + s(r_0 - jx_0)} E_0$$

$$= \frac{(s + r_1 g + sx_1 b) + j(r_1 b - sx_1 g)}{(r_1 + sr_0) - js(x_1 + x_0)} E_0,$$

or, expanded,

$$I = \frac{[(sr_1 + s^2 r_0) + r_1^2 g + sr_1 (r_0 g - x_0 b) + s^2 x_1 (x_0 g + x_1 g + r_0 b)] + + j[s^2 (x_0 + x_1) + r_1^2 b + sr_1 (x_0 g + r_0 b) + s^2 x_1 (x_0 b + x_0 b - r_1 g)]}{(r_1 + sr_0)^2 + s^2 (x_1 + x_0)^2} E_0.$$

Hence, displacement of phase between current and e.m.f.,

$$\tan \theta_0 = \frac{s^2 (x_0 + x_1) + r_1^2 b + sr_1 (x_0 g + r_0 b) + s^2 x_1 (x_0 b + x_1 b - r_0 g)}{(sr_1 + s^2 r_0) + r_1^2 g + sr_1 (r_0 g - x_0 b) + s^2 x_1 (x_0 g + x_1 b_1 - r_0 b)}.$$

Neglecting the exciting current, I_0, altogether, that is, setting $Y = 0$,

We have

$$\dot{I} = s\dot{E}_0 \frac{(r_1 + sr_0) + js(x_0 + x_1)}{(r_1 + sr_0)^2 + s^2 (x_0 + x_1)^2}$$

$$= \frac{E\dot{s}_0}{(r_1 + sr_0) - js(x_0 + x_1)};$$

$$\tan \theta_0 = \frac{s(x_0 + x_1)}{r_1 + sr_0}.$$

182. In graphic representation, the induction motor diagram appears as follows: —

Denoting the magnetism by the vertical vector $\overline{O\Phi}$ in Fig. 124, the m.m.f. in ampere-turns per circuit is represented by vector $\overline{OF}$, leading the magnetism, $\overline{O\Phi}$, by the angle of hysteretic advance α. The e.m.f. generated in the secondary is proportional to the slip s, and represented by $\overline{OE_1}$ at the amplitude of 180°. Dividing $\overline{OE_1}$ by a in the proportion of $r_1 \div sx_1$, and connecting a with the middle, b, of the upper arc of the circle, $\overline{OE_1}$, this line intersects the lower arc of the circle at the point, $\overline{I_1r_1}$. Thus, $\overline{OI_1r_1}$ is the e.m.f. consumed by the secondary resistance, and $\overline{OI_1x_1}$ equal and parallel to $\overline{E_1I_1r_1}$ is the e.m.f. consumed by the secondary reactance. The angle, $E_1OI_1r_1 = \theta_1$, is the angle of secondary lag.

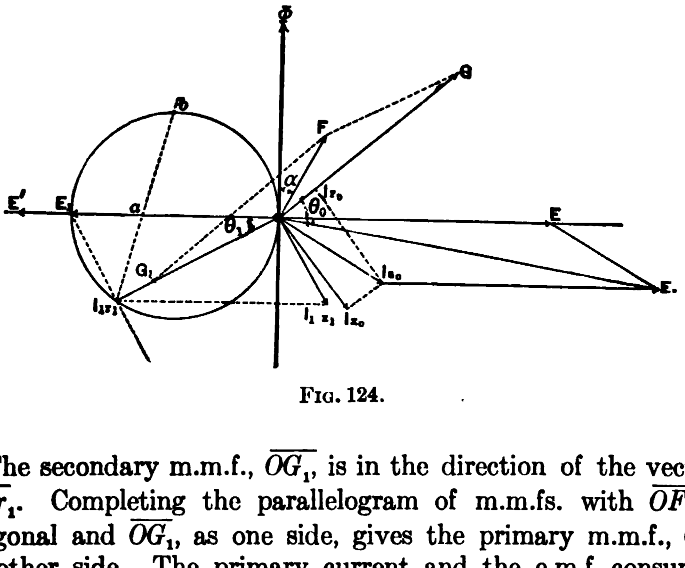


FIG. 124.

The secondary m.m.f., $\overline{OG_1}$, is in the direction of the vector, $\overline{OI_1r_1}$. Completing the parallelogram of m.m.fs. with $\overline{OF}$ as diagonal and $\overline{OG_1}$, as one side, gives the primary m.m.f., $\overline{OG}$, as other side. The primary current and the e.m.f. consumed by the primary resistance, represented by $\overline{OIr_0}$, is in line with $\overline{OG}$, the e.m.f. consumed by the primary reactance 90° ahead of $\overline{OG}$, and represented by $\overline{OIx_0}$, and their resultant, $\overline{OIz_0}$, is the e.m.f. consumed by the primary impedance. The e.m.f. generated in the primary circuit is $\overline{OE'}$, and the e.m.f. required to overcome this counter e.m.f. is $\overline{OE}$ equal and opposite to

$\overline{OE'}$. Combining $\overline{OE}$ with $\overline{OIz_0}$ gives the primary terminal voltage represented by vector $\overline{OE_0}$, and the angle of primary lag, $E_0OG = \theta_0$.

183. Thus far the diagram is essentially the same as the diagram of the stationary alternating-current transformer. Regarding dependence upon the slip of the motor, the locus of the different quantities for different values of the slip, s, is determined thus,

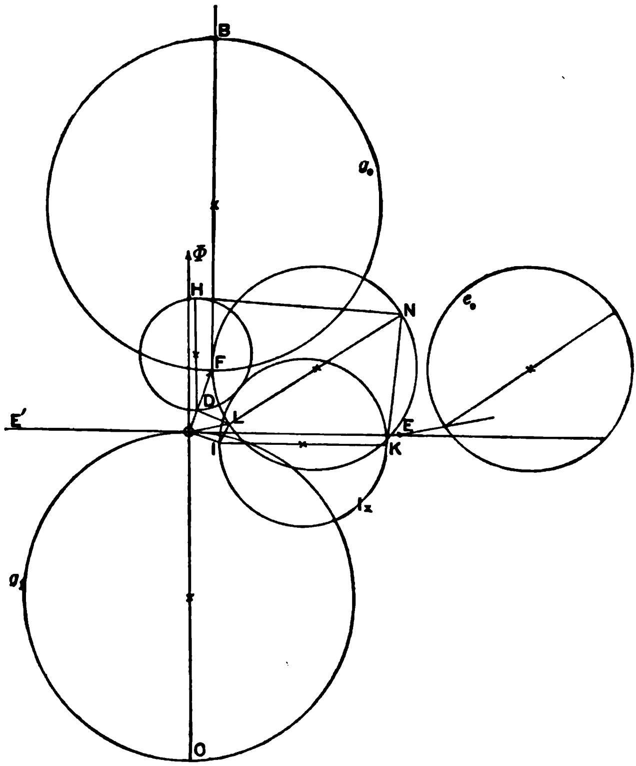


Fig. 125.

Let $$E_1 = sE'.$$

Assume in opposition to $\overline{O\Phi}$, a point, A, such that

$$OA \div I_1 r_1 = E_1 \div I_1 s x_1, \text{ then}$$

$$OA = \frac{I_1 r_1 \times E_1}{I_1 s x_1} = \frac{I_1 r_1 \times sE'}{I_1 s x_1} = \frac{r_1}{x_1} E' = \text{constant}.$$

That is, $I_1 r_1$ lies on a half-circle with $OA = \frac{r_1}{x_1} E'$ as diameter.

That means G_1 lies on a half-circle, g_1, in Fig. 125 with $\overline{OC}$ as diameter. In consequence hereof, G_0 lies on half-circle g_0 with $\overline{FB}$ equal and parallel to $\overline{OC}$ as diameter.

Thus Ir_0 lies on a half-circle with $\overline{DH}$ as diameter, which circle is perspective to the circle, $\overline{FB}$, and Ix_0 lies on a half-circle with $\overline{IK}$ as diameter, and Iz_0 on a half-circle with $\overline{LN}$ as diameter, which circle is derived by the combination of the circles, Ir_0 and Ix_0.

The primary terminal voltage, E_0, lies thus on a half-circle, e_0, equal to the half-circle, Iz_0, and having to point E the same relative position as the half-circle, Iz_0, has to point 0.

This diagram corresponds to constant intensity of the maximum magnetism, $\overline{O\Phi}$. If the primary impressed voltage, E_0, is kept constant, the circle, e_0, of the primary impressed voltage changes to an arc with O as eenter, and all the corresponding points of the other circles have to be reduced in accordance herewith, thus giving as locus of the other quantities curves of higher order which most conveniently are constructed point for point by reduction from the circle of the loci in Fig. 125.

Torque and Power.

184. The torque developed per pole by an electric motor equals the product of effective magnetism, $\frac{\Phi}{\sqrt{2}}$, times effective armature m.m.f., $\frac{\mathfrak{F}}{\sqrt{2}}$, times the sine of the angle between both,

$$D' = \frac{\Phi \mathfrak{F}}{2} \sin (\Phi \mathfrak{F}).$$

If n_1 = number of turns, I_1 = current, per circuit, with p_1 armature circuits, the total maximum current polarization, or m.m.f. of the armature, is

$$\mathfrak{F}_1 = \frac{p_1 n_1 I_1}{\sqrt{2}}.$$

Hence the torque per pole,

$$D' = \frac{p_1 n_1 \Phi I_1}{2\sqrt{2}} \sin (\Phi I_1).$$

If q = the number of poles of the motor, the total torque of the motor is,

$$D = \frac{q p_1 n_1 \Phi I_1}{2\sqrt{2}} \sin(\Phi I_1).$$

The secondary induced e.m.f., E_1, lags 90° behind the inducing magnetism, hence reaches a maximum displaced in space by 90° from the position of maximum magnetization. Thus, if the secondary current, I_1, lags behind its e.m.f., E_1, by angle, θ_1, the space displacement between armature current and field magnetism is

$$\measuredangle\,(I_1 \Phi) = 90° + \theta_1,$$

hence $$\sin(\Phi I_1) = \cos\theta_1.$$

We have, however,

$$\cos\theta_1 = \frac{r_1}{\sqrt{r_1^2 + s^2 x_1^2}},$$

$$I_1 = \frac{es\,10^{-1}}{\sqrt{r_1^2 + s^2 x_1^2}}$$

$$e = \sqrt{2}\,\pi n_1 \Phi f\,10^{-8},$$

thus, $$n_1 \Phi = \frac{e\,10^8}{\sqrt{2}\,\pi f};$$

substituting these values in the equation of the torque, it is

$$D = \frac{q p_1 s r_1 e^2\,10^7}{4\,\pi f\,(r_1^2 + s^2 x_1^2)};$$

or, in practical (c.g.s.) units,

$$D = \frac{q p_1 s r_1 e^2}{4\,\pi f\,(r_1^2 + s^2 x_1^2)},$$

is the torque of the induction motor.

At the slip, s, the frequency, f, and the number of poles, q, the linear speed at unit radius is

$$v = \frac{4\,\pi f}{q}(1 - s);$$

hence the output of the motor,

$$P = Dv,$$

or, substituted,

$$P = \frac{p_1 r_1 e^2 s\,(1 - s)}{r_1^2 + s^2 x_1^2},$$

is the power of the induction motor.

185. We can arrive at the same results in a different way:

By the counter e.m.f., e, of the primary circuit with current $I = I_0 + I_1$ the power is consumed, $eI = eI_0 + eI_1$. The power, eI_0, is that consumed by the primary hysteresis and eddys. The power, eI_1, disappears in the primary circuit by being transmitted to the secondary system.

Thus the total power impressed upon the secondary system, per circuit, is

$$P_1 = eI_1.$$

Of this power a part, $E_1 I_1$, is consumed in the secondary circuit by resistance. The remainder,

$$P' = I_1\,(e - E_1),$$

disappears as electrical power altogether; hence, by the law of conservation of energy, must reappear as some other form of energy, in this case as mechanical power, or as the output of the motor (including friction).

Thus the mechanical output per motor circuit is

$$P' = I_1\,(e - E_1).$$

Substituting,

$$E_1 = se;$$

$$I_1 = \frac{se}{r_1 - jsx_1};$$

it is

$$P' = \frac{e^2 s\,(1 - s)}{r_1 - jsx_1}$$

$$= \frac{e^2 s\,(1 - s)\,(r_1 + jsx_1)}{r_1^2 + s^2 x_1^2};$$

hence, since the imaginary part has no meaning as power,

$$P' = \frac{r_1 e^2 s (1 - s)}{r_1^2 + s^2 x_1^2};$$

and the total power of the motor,

$$P = \frac{p_1 r_1 e^2 s (1 - s)}{r_1^2 + s^2 x_1^2}.$$

At the linear speed,

$$v = \frac{4 \pi f}{q} (1 - s)$$

at unit radius the torque is

$$D = \frac{q p_1 r_1 e^2 s}{4 \pi f (r_1^2 + s^2 x_1^2)}.$$

In the foregoing, we found

$$E_0 = e \left\{ 1 + s \frac{Z_0}{Z_1} + Z_0 Y \right\},$$

or, approximately,

$$E_0 = e \left\{ 1 + s \frac{Z_0}{Z_1} \right\};$$

or,
$$e = \frac{E_0 Z_1}{s Z_0 + Z_1};$$

expanded,
$$e = E_0 \frac{r_1 - j s x_1}{(r_1 + s r_0) - j s (x_1 + x_0)};$$

or, eliminating imaginary quantities,

$$e = E_0 \sqrt{\frac{r_1^2 + s^2 x_1^2}{(r_1 + s r_0)^2 + s^2 (x_1 + x_0)^2}}.$$

Substituting this value in the equations of torque and of power, they become,

torque,
$$D = \frac{q p_1 r_1 E_0^2 s}{4 \pi f \{(r_1 + s r_0)^2 + s^2 (x_1 + x_0)^2\}};$$

power,
$$P = \frac{p_1 r_1 E_0^2 s (1 - s)}{(r_1 + s r_0)^2 + s^2 (x_1 + x_0)^2}.$$

Maximum Torque.

186. The torque of the induction motor is a maximum for that value of slip s, where

$$\frac{dD}{ds}=0,$$

or, since $$D=\frac{qp_1r_1E_0s}{4\pi f\{(r_1+sr_0)^2+s^2(x_1+x_0)^2\}},$$

for, $$\frac{d}{ds}\left\{\frac{(r_1+sr_0)^2+s^2(x_1+x_0)^2}{s}\right\}=0;$$

expanded, this gives,

$$-\frac{r_1^2}{s^2}+r_0^2+(x_1+x_0)^2=0,$$

or, $$s_t=\frac{r_1}{\sqrt{r_0^2+(x_1+x_0)^2}}.$$

Substituting this in the equation of torque, we get the value of maximum torque,

$$D_t=\frac{qp_1E_0^2}{8\pi f\{r_0+\sqrt{r_0^2+(x_1+x_0)^2}\}},$$

that is, independent of the secondary resistance, r_1.

The power corresponding hereto is, by substitution of s_t in P,

$$P_t=\frac{p_1E_0^2\{\sqrt{r_0^2+(x_1+x_0)^2}-r_1\}}{2\sqrt{r_0^2+(x_1+x_0)^2}\{\sqrt{r_0^2+(x_1+x_0)^2}+r_0\}}.$$

This power is not the maximum output of the motor, but is less than the maximum output. The maximum output is found at a lesser slip, or higher speed, while at the maximum torque point the output is already on the decrease, due to the decrease of speed.

With increasing slip, or decreasing speed, the torque of the induction motor increases; or inversely, with increasing load, the speed of the motor decreases, and thereby the torque increases, so as to carry the load down to the slip, s_t, corresponding to the maximum torque. At this point of load and slip

the torque begins to decrease again; that is, as soon as with increasing load, and thus increasing slip, the motor passes the maximum torque point, s_t, it "falls out of step," and comes to a standstill.

Inversely, the torque of the motor, when starting from rest, increases with increasing speed, until the maximum torque point is reached. From there towards synchronism the torque decreases again.

In consequence hereof, the part of the torque-speed curve below the maximum torque point is in general unstable, and can be observed only by loading the motor with an apparatus whose counter-torque increases with the speed faster than the torque of the induction motor.

In general, the maximum torque point, s_t, is between synchronism and standstill, rather nearer to synchronism. Only in motors of very large armature resistance, that is, low efficiency, $s_t > 1$, that is, the maximum torque, occurs below standstill, and the torque constantly increases from synchronism down to standstill.

It is evident that the position of the maximum torque point, s_t, can be varied by varying the resistance of the secondary circuit, or the motor armature. Since the slip of the maximum torque point, s_t, is directly proportional to the armature resistance, r_1, it follows that very constant speed and high efficiency brings the maximum torque point near synchronism, and gives small starting torque, while good starting torque means a maximum torque point at low speed; that is, a motor with poor speed regulation and low efficiency.

Thus, to combine high efficiency and close speed regulation with large starting torque, the armature resistance has to be varied during the operation of the motor, and the motor started with high armature resistance, and with increasing speed this armature resistance cut out as far as possible.

187. If
$$s_t = 1,$$
$$r_1 = \sqrt{r_0^2 + (x_1 + x_0)^2}.$$

In this case the motor starts with maximum torque, and when overloaded does not drop out of step, but gradually slows down more and more, until it comes to rest.

If $$s_t > 1,$$

then $$r_1 > \sqrt{r_0^2 + (x_1 + x_0)^2}.$$

In this case, the maximum torque point is reached only by driving the motor backwards, as counter torque.

As seen above, the maximum torque, D_t, is entirely independent of the armature resistance, and likewise is the current corresponding thereto, independent of the armature resistance. Only the speed of the motor depends upon the armature resistance.

Hence the insertion of resistance into the motor armature does not change the maximum torque, and the current corresponding thereto, but merely lowers the speed at which the maximum torque is reached.

The effect of resistance inserted into the induction motor is merely to consume the e.m.f., which otherwise would find its mechanical equivalent in an increased speed, analogous to resistance in the armature circuit of a continuous-current shunt motor.

Further discussion on the effect of armature resistance is found under "Starting Torque."

Maximum Power

188. The power of an induction motor is a maximum for that slip, s_p, where

$$\frac{dP}{ds} = 0;$$

or, since $$P = \frac{p_1 r_1 E_0^2 s (1 - s)}{(r_1 + s r_0)^2 + s^2 (x_1 + x_0)^2},$$

$$\frac{d}{ds}\left\{\frac{(r_1 + s r_0)^2 + s^2 (x_1 + x_0)^2}{s (1 - s)}\right\} = 0;$$

expanded, this gives

$$s_p = \frac{r_1}{r_1 + \sqrt{(r_1 + r_0)^2 + (x_1 + x_0)^2}};$$

substituted in P, we get the maximum power,

$$P_p = \frac{p_1 E_0^2}{2\left\{(r_1 + r_0) + \sqrt{(r_1 + r_0)^2 + (x_1 + x_0)^2}\right\}}.$$

This result has a simple physical meaning: $(r_1 + r_0) = r$ is the total resistance of the motor, primary plus secondary (the latter reduced to the primary). $(x_1 + x_0)$ is the total reactance, and thus $\sqrt{(r_1 + r_0)^2 + (x_1 + x_0)^2} = z$ is the total impedance of the motor. Hence

$$P_p = \frac{p_1 E_0^2}{2 \{r + z\}},$$

is the maximum output of the induction motor, at the slip,

$$s_p = \frac{r_1}{r_1 + z}.$$

The same value has been derived in Chapter XI., as the maximum power which can be transmitted into a non-inductive receiver circuit over a line of resistance, r, and impedance, z, or as the maximum output of a generator, or of a stationary transformer. Hence:

The maximum output of an induction motor is expressed by the same formula as the maximum output of a generator, or of a stationary transformer, or the maximum output which can be transmitted over an inductive line into a non-inductive receiver circuit.

The torque corresponding to the maximum output, P_p, is

$$D_p = \frac{q p_1 E_0^2 (r_1 + z)}{8 \pi f z (r + z)}.$$

This is not the maximum torque; but the maximum torque, D_t, takes place at a lower speed, that is, greater slip,

$$s_t = \frac{r_1}{\sqrt{r_0^2 + (x_1 + x_0)^2}},$$

since
$$\frac{r_1}{\sqrt{r_0^2 + (x_1 + x_0)^2}} > \frac{r_1}{r_1 + \sqrt{(r_1 + r_0)^2 + (x_1 + x_0)^2}};$$

that is,
$$s_t > s_p.$$

It is obvious from these equations, that, to reach as large an output as possible, r and z should be as small as possible; that is, the resistances, $r_1 + r_0$, and the impedances, z, and thus

the reactances, $x_1 + x_0$, should be small. Since $r_1 + r_0$ is usually small compared with $x_1 + x_0$ it follows, that the problem of induction motor design consists in constructing the motor so as to give the minimum possible reactances, $x_1 + x_0$.

Starting Torque.

189. In the moment of starting an induction motor, the slip is

$$s = 1;$$

hence, starting current,

$$\dot{I} = \frac{1 + (r_1 - jx_1)(g + jb)}{(r_1 - jx_1) + (r_0 - jx_0) + (r_1 - jx_1)(r_0 - jx_0)(g + jb)} \dot{E}_0;$$

or, expanded, with the rejection of the last term in the denominator, as insignificant,

$$\dot{I} = \frac{[(r_1 + r_0) + g(r_1[r_1 + r_0] + x_1[x_1 + x_0]) + b(r_0x_1 - x_0r_1)] - j[(x_1 + x_0) + b(r_1[r_1 + r_0] + x_1[x_1 + x_0]) - g(r_0x_1 - x_0r_1)]}{(r_1 + r_0)^2 + (x_1 + x_0)^2} \dot{E}_0;$$

and, displacement of phase, or angle of lag,

$$\tan\theta_0 = \frac{(x_1 + x_0) + b(r_1[r_1 + r_0] + x_1[x_1 + x_0]) - g(r_0x_1 - x_0r_1)}{(r_1 + r_0) + g(r_1[r_1 + r_0] + x_1[x_1 + x_0]) + b(r_0x_1 - x_0r_1)}.$$

Neglecting the exciting current, $g = 0 = b$, these equations assume the form,

$$\dot{I} = \frac{(r_1 + r_0) + j(x_1 + x_0)}{(r_1 + r_0)^2 + (x_1 + x_0)^2} \dot{E}_0 = \frac{\dot{E}_0}{(r_1 + r_0) - j(x_1 + x_0)};$$

or, eliminating imaginary quantities,

$$I = \frac{E_0}{\sqrt{(r_1 + r_0)^2 + (x_1 + x_0)^2}} = \frac{E_0}{z};$$

and $$\tan\theta_0 = \frac{x_1 + x_0}{r_1 + r_0}.$$

That means, that in starting the induction motor without additional resistance in the armature circuit, — in which case $x_1 + x_0$ is large compared with $r_1 + r_0$, and the total impedance, z, small, — the motor takes excessive and greatly lagging currents.

The starting torque is

$$D_0 = \frac{q p_1 r_1 E_0^2}{4 \pi f \{(r_1 + r_0)^2 + (x_1 + x_0)^2\}}$$

$$= \frac{q p_1 E_0^2}{4 \pi f} \frac{r_1}{z^2}.$$

That is, the starting torque is proportional to the armature resistance, and inversely proportional to the square of the total impedance of the motor.

It is obvious thus, that, to secure large starting torque, the impedance should be as small, and the armature resistance as large, as possible. The former condition is the condition of large maximum output and good efficiency and speed regulation; the latter condition, however, means inefficiency and poor regulation, and thus cannot properly be fulfilled by the internal resistance of the motor, but only by an additional resistance which is short-circuited while the motor is in operation.

Since, necessarily,

$$r_1 < z,$$

we have,

$$D_0 < \frac{q p_1 E_0^2}{4 \pi f z};$$

and since the starting current is, approximately,

$$I = \frac{E_0}{z},$$

we have,

$$D_0 < \frac{q p_1}{4 \pi f} E_0 I.$$

$$D_{00} = \frac{q p_1}{4 \pi f} E_0 I$$

would be the theoretical torque developed at 100 per cent efficiency and power-factor, by e.m.f. E_0, and current I, at synchronous speed.

Thus, $D_0 < D_{00}$,

and the ratio between the starting torque, D_0, and the theoretical maximum torque, D_{00}, gives a means to judge the perfection of a motor regarding its starting torque.

This ratio, $\frac{D_0}{D_{00}}$, exceeds 0.9 in the best motors.

Substituting $I = \frac{E_0}{z}$ in the equation of starting torque, it assumes the form,

$$D_0 = \frac{qp_1}{4\pi f} I^2 r_1.$$

Since $\frac{4\pi f}{q}$ = synchronous speed, it is:

The starting torque of the induction motor is equal to the resistance loss in the motor armature, divided by the synchronous speed.

The armature resistance which gives maximum starting torque is

$$\frac{dD_0}{dr_1} = 0$$

or since, $$D_0 = \frac{qp_1 E_0^2}{4\pi f} \frac{r_1}{(r_1 + r_0)^2 + (x_1 + x_0)^2},$$

$$\frac{d}{dr_1}\left\{\frac{(r_1 + r_0)^2 + (x_1 + x_0)^2}{r_1}\right\} = 0;$$

expanded, this gives,

$$r_1 = \sqrt{r_0^2 + (x_1 + x_0)^2},$$

the same value as derived in the paragraph on "maximum torque."

Thus, adding to the internal armature resistance, r_1', in starting the additional resistance,

$$r_1'' = \sqrt{r_0^2 + (x_1 + x_0)^2} - r_1',$$

makes the motor start with maximum torque, while with increasing speed the torque constantly decreases, and reaches zero at synchronism. Under these conditions, the induction motor behaves similarly to the continuous-current series motor,

varying in speed with the load, the difference being, however, that the induction motor approaches a definite speed at no-load, while with the series motor the speed indefinitely increases with decreasing load.

The additional armature resistance, r_1'', required to give a certain starting torque, if found from the equation of starting torque:

Denoting the internal armature resistance by r_1', the total armature resistance is $r_1 = r_1' + r_1''$,

and thus, $$D_0 = \frac{q p_1 E_0^2}{4 \pi f} \frac{r_1' + r_1''}{(r_1' + r_1'' + r_0)^2 + (x_1 + x_0)^2};$$

hence,

$$r_1'' = -r_1' - r_0 + \frac{q p_1 E_0^2}{8 \pi f D_0} \pm \sqrt{\left(\frac{q p_1 E_0^2}{8 \pi f D_0}\right)^2 - \frac{q p_1 E_0^2 r_0}{4 \pi f D_0} - (x_1 + x_0)^2 f}.$$

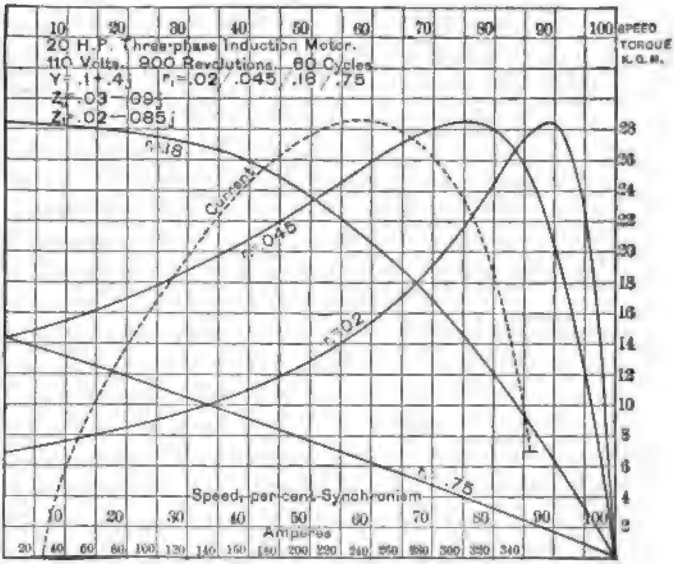


FIG. 126. — Speed Characteristics of Induction Motor.

This gives two values, one above, the other below, the maximum torque point.

Choosing the positive sign of the root, we get a larger armature resistance, a small current in starting, but the torque constantly decreases with the speed.

Choosing the negative sign, we get a smaller resistance, a large starting current, and with increasing speed the torque first increases, reaches a maximum, and then decreases again towards synchronism.

These two points correspond to the two points of the speed-torque curve of the induction motor, in Fig. 126, giving the desired torque, D_0.

The smaller value of r_1'' gives fairly good speed regulation, and thus in small motors, where the comparatively large starting current is no objection, the permanent armature resistance may be chosen to represent this value.

The larger value of r_1'' allows to start with minimum current, but requires cutting out of the resistance after the start, to secure speed regulation and efficiency.

190. Approximately, the *torque* of the induction motor at any slip, s:

$$D = \frac{qp_1r_1E_0^2s}{4\pi f\left\{(r_1 + sr_0)^2 + s^2(x_1 + x_0)^2\right\}},$$

can be expressed in a simple and so convenient form as function of the *maximum torque:*

$$D_t = \frac{qp_1E_0^2}{8\pi f\left\{r_0 + \sqrt{r_0^2 + (x_1 + x)^2}\right\}},$$

or of the *starting torque*: $s = 1$:

$$D_0 = \frac{qp_1r_1E_0^2}{4\pi f\left\{(r_1 + r_0)^2 + (x_1 + x_0)^2\right\}}.$$

Dividing D by D_t we have

$$D = \frac{2r_1s\left\{r_0 + \sqrt{r_0^2 + (x_1 + x_0)^2}\right\}}{(r_1 + sr_0)^2 + s^2(x_1 + x_0)^2}D_t.$$

Since r_0, the primary resistance, is small compared with

$$x = x_1 + x_0,$$

the total self-inductive reactance of the motor, it can be neglected under the square root, and the equation so gives:

$$D = \frac{2r_1s(r_0 + x)}{(r_1 + sr_0)^2 + s^2x^2}D_t,$$

or, still more approximately:

$$D = \frac{2\,sr_1x}{r_1^2 + s^2x^2}\,D_t,$$

and the starting torque, for : $s = 1$:

$$D_0 = \frac{2\,r_1x}{r_1^2 + x^2}\,D_t,$$

hence, dividing,

$$D = \frac{s\,(r_1^2 + x^2)}{r_1^2 + s^2x^2}\,D_0,$$

or, if r_1 is small compared with x, that is, in a motor of low-resistance armature:

$$D = \frac{sx^2}{r_1^2 + s^2x^2}\,D_0,$$

From the equation:

$$D = \frac{2\,sr_1x}{r_1^2 + s^2x^2}\,D_t.$$

it follows that for small values of s, or near synchronism:

$$D = \frac{2\,sx}{r_1}\,D_t.$$

By neglecting s^2x^2 compared with r_1^2:

For low values of speed, or high values of s, it follows, by neglecting r_1^2 compared with s^2x^2:

$$D = \frac{2\,r_1}{sx}\,D_t.$$

that is, approximately, near synchronism, the torque is directly proportional to the slip, and inversely proportional to the armature resistance, that is, proportional to the ratio $\frac{\text{slip}}{\text{armature resistance}}$; near standstill, the torque is inversely proportional to the slip, but directly proportional to the armature resistance, and so is increased by increasing the armature resistance in a motor of low-armature resistance.

Synchronism.

191. At synchronism, $s = 0$, we have,

$$I_s = E_0 (g + jb);$$

or,

$$I_s = E_0 \sqrt{g^2 + b^2};$$
$$P = 0, \ D = 0;$$

that is, power and torque are zero. Hence, the induction motor can never reach complete synchronism, but must slip sufficiently to give the torque consumed by friction.

Running near Synchronism.

192. When running near synchronism, at a slip, s, above the maximum output point, where s is small, from 0.01 to 0.05 at full load, the equations can be simplified by neglecting terms with s, as of higher order.

We then have, current,

$$\dot{I} = \frac{s + r_1 (g + jb)}{r_1} \dot{E}_0;$$

or, eliminating imaginary quantities,

$$I = \sqrt{\left(\frac{s}{r_1} + g\right)^2 + b^2}\, E_0;$$

angle of lag,

$$\tan \theta_0 = \frac{s^2 (x_1 + x_0) + r_1^2 b}{s r_1 + r_1^2 g} = \frac{s^2 \dfrac{x_1 + x_0}{r_1} + r_1 b}{s + r_1 g};$$

$$P = \frac{p_1 E_0^2 s}{r_1};$$

$$D = \frac{q p_1 E_0^2 s}{4 \pi f r_1};$$

or, inversely,

$$s = \frac{r_1 P}{p_1 E_0^2};$$

$$s = \frac{4\,\pi f r_1 D}{q p_1 E_0^2},$$

that is,

Near synchronism, the slip, s, of an induction motor, or its drop in speed, is proportional to the armature resistance, r_1, and to the power, P, or torque, D.

Example.

193. As an example are shown, in Fig. 126, characteristic curves of a 20-horsepower three-phase induction motor, of 900 revolutions synchronous speed, 8 poles, frequency of 60 cycles.

The impressed e.m.f. is 110 volts between lines, and the motor star connected, hence the e.m.f. impressed per circuit:

$$\frac{110}{\sqrt{3}} = 63.5; \text{ or } E_0 = 63.5.$$

The constants of the motor are:

Primary admittance,	$Y = 0.1 + 0.4\,j.$	
Primary impedance,	$Z = 0.03 - 0.09\,j.$	
Secondary impedance,	$Z_1 = 0.02 - 0.085\,j.$	

In Fig. 126 is shown, with the speed in per cent of synchronism, as abscissas, the torque in kilogram-meters as ordinates in drawn lines, for the values of armature resistance:

$r_1 = 0.02$: short-circuit of armature, full speed.

$r_1 = 0.045$: 0.025 ohms additional resistance.

$r_1 = 0.18$: 0.16 ohms additional, maximum starting torque.

$r_1 = 0.75$: 0.73 ohms additional, same starting torque as $r_1 = 0.045$.

On the same figure is shown the current per line, in dotted lines, with the verticals or torque as abscissas, and the hori-

zontals or amperes as ordinates. To the same current always corresponds the same torque, no matter what the speed may be.

On Fig. 127 is shown, with the current input per line as abscissas, the torque in kilogram-meters and the output in horse-

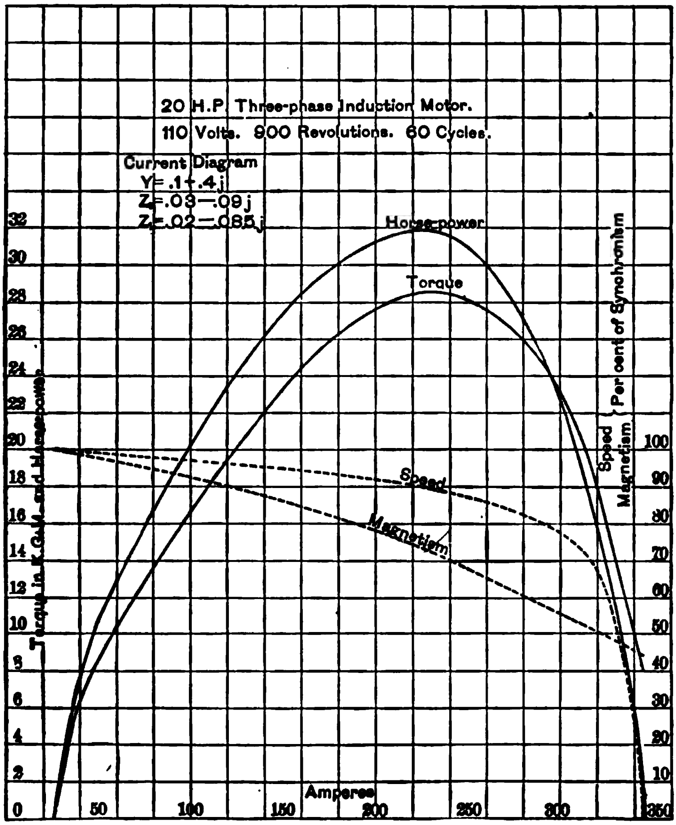


Fig. 127. — Current Characteristics of Induction Motor.

power as ordinates in drawn lines, and the speed and the magnetism, in per cent of their synchronous values, as ordinates in dotted lines, for the armature resistance, $r_1 = 0.02$, or short-circuit.

In Fig. 128 is shown, with the speed, in per cent of synchronism, as abscissas, the torque in drawn line, and the output in dotted line, for the value of armature resistance $r_1 = 0.045$,

for the whole range of speed from 120 per cent backwards speed to 220 per cent beyond synchronism, showing the two

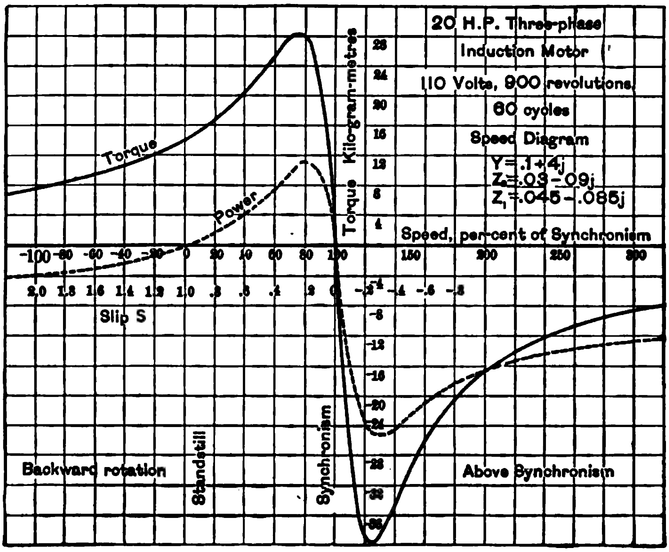


FIG. 128. — Speed Characteristics of Induction Motor.

maxima, the motor maximum at $s = 0.25$, and the generator maximum at $s = -0.25$.

194. As seen in the preceding, the induction motor is characterized by the three complex imaginary constants,

$Y_0 = g_0 + jb_0$, the primary exciting admittance,
$Z_0 = r_0 - jx_0$, the primary self-inductive impedance, and
$Z_1 = r_1 - jx_1$, the secondary self-inductive impedance,

reduced to the primary by the ratio of secondary to primary turns.

From these constants and the impressed e.m.f., e_0, the motor can be calculated as follows:

Let,

e = counter e.m.f. of motor, that is, e.m.f. generated in the primary by the mutual magnetic flux.

At the slip s the e.m.f. generated in the secondary circuit is,

$$se.$$

Thus the secondary current,

$$I_1 = \frac{se}{r_1 - jsx_1} = e\,(a_1 + ja_2),$$

where,

$$a_1 = \frac{sr_1}{r_1^2 + s^2x_1^2} \quad \text{and} \quad a_2 = \frac{s^2x_1}{r_1^2 + s^2x_1^2}.$$

The primary exciting current is,

$$I_{00} = eY_0 = e\,(g_0 + jb_0);$$

thus, the total primary current,

$$I_0 = I_1 + I_{00} = e\,(b_1 + jb_2),$$

where,

$$b_1 = a_1 + g_0, \quad \text{and} \quad b_2 = a_2 + b_0.$$

The e.m.f. consumed by the primary impedance is,

$$E^1 = I_0Z_0 = e\,(r_0 - jx_0)\,(b_1 + jb_2);$$

the primary counter e.m.f. is e, thus the primary impressed e.m.f.,

$$E_0 = e + E^1 = e\,(c_1 + jc_2),$$

where,

$$c_1 = 1 + r_0b_1 + x_0b_2 \quad \text{and} \quad c_2 = r_0b_2 - x_0b_1,$$

or, the absolute value is,

$$e_0 = e\sqrt{c_1^2 + c_2^2},$$

hence,

$$e = \frac{e_0}{\sqrt{c_1^2 + c_2^2}}.$$

Substituting this value gives,

Secondary current,

$$\dot{I}_1 = e_0 \frac{a_1 + ja_2}{\sqrt{c_1^2 + c_2^2}}, \quad I_1 = e_0 \sqrt{\frac{a_1^2 + a_2^2}{c_1^2 + c_2^2}};$$

Primary current,

$$\dot{I}_0 = e_0 \frac{b_1 + jb_2}{\sqrt{c_1^2 + c_2^2}}, \quad I_0 = e_0 \sqrt{\frac{b_1^2 + b_2^2}{c_1^2 + c_2^2}};$$

Impressed e.m.f.,

$$\dot{E}_0 = e_0 \frac{c_1 + jc_2}{\sqrt{c_1^2 + c_2^2}}$$

Thus torque, in synchronous watts (that is, the watts output the torque would produce at synchronous speed),

$$D = [e\dot{I}_1]^1$$

$$= \frac{e_0^2 a_1}{c_1^2 + c_2^2},$$

hence, the torque in absolute units,

$$D_0 = \frac{D}{f} = \frac{e_0^2 a_1}{(c_1^2 + c_2^2) f},$$

where f = frequency.

The power output is torque times speed, thus:

$$P_1 = D(1 - s) = \frac{e_0^2 a_1 (1 - s)}{c_1^2 + c_2^2}.$$

The power input is,

$$\dot{P}_0 = [\dot{E}_0 \dot{I}_0] = [\dot{E}_0 \dot{I}_0]^1 + j[\dot{E}_0 \dot{I}_0]^j = P_0^1 + jP_0^j$$

$$= \frac{e_0^2 (b_1 c_1 + b_2 c_2)}{c_1^2 + c_2^2} + j\frac{e_0^2 (b_2 c_1 - b_1 c_2)}{c_1^2 + c_2^2}.$$

The volt-ampere input,

$$P_{a_0} = e_0 I_0 = \frac{e_0^2 \sqrt{b_1^2 + b_2^2}}{\sqrt{c_1^2 + c_2^2}}$$

hence, the efficiency is,

$$\frac{P_1}{P_0^1} = \frac{a_1 (1 - s)}{b_1 c_1 + b_2 c_2};$$

the power-factor,

$$\frac{P_0^1}{P_{a_0}} = \frac{b_1 c_1 + b_2 c_2}{\sqrt{(b_1^2 + b_2^2)(c_1^2 + c_2^2)}};$$

the apparent efficiency,

$$\frac{P_1}{P_{a_0}} = \frac{a_1 (1 - s)}{\sqrt{(b_1^2 + b_2^2)(c_1^2 + c_2^2)}};$$

the torque efficiency,*

$$\frac{D}{P_0^1} = \frac{a_1}{b_1 c_1 + b_2 c_2},$$

and the apparent torque efficiency,†

$$\frac{D}{P_{a_0}} = \frac{a_1}{\sqrt{(b_1^2 + b_2^2)(c_1^2 + c_2^2)}}.$$

195. Most instructive in showing the behavior of an induction motor are the load curves and the speed curves.

The load curves are curves giving, with the power output as abscissas, the current input, speed, torque, power-factor, efficiency, and apparent efficiency, as ordinates.

The speed curves give, with the speed as abscissas, the torque, current input, power-factor, torque efficiency, and apparent torque efficiency, as ordinates.

* That is the ratio of actual torque to torque which would be produced, if there were no losses of energy in the motor, at the same power input.

† That is the ratio of actual torque to torque which would be produced if there were neither losses of energy nor phase displacement in the motor, at the same volt-ampere input.

The load curves characterize the motor especially at its normal running speeds near synchronism, while the speed curves characterize it over the whole range of speed.

In Fig. 129 are shown the load curves, and in Fig. 130 the speed curves of a motor having the constants: $Y_0 = 0.01 + 0.1\,j$; $Z_0 = 0.1 - 0.3\,j$; and $Z_1 = 0.1 - 0.3\,j$.

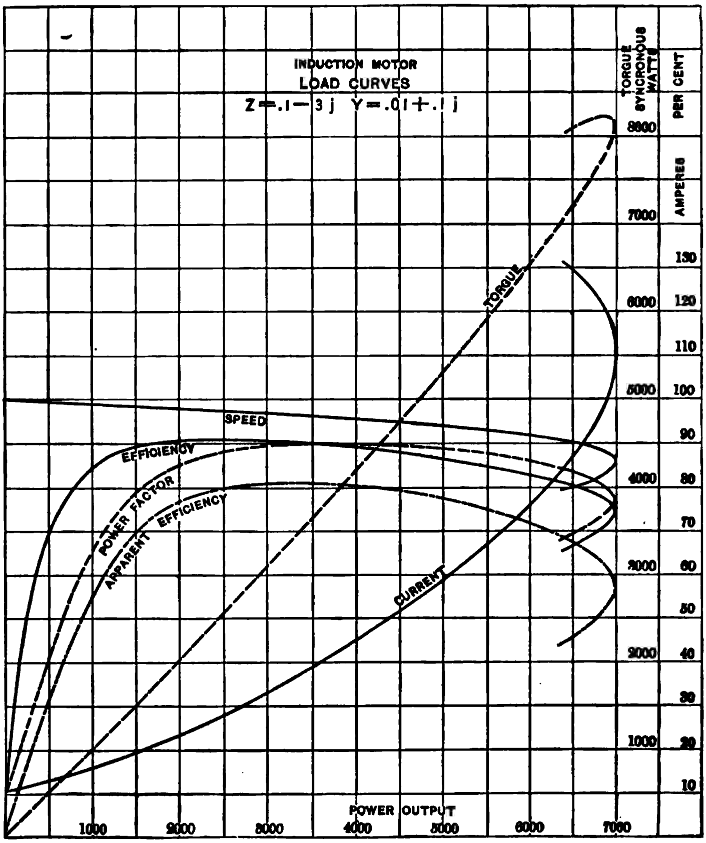


Fig. 129.

INDUCTION GENERATOR.

196. In the foregoing, the range of speed from $s = 1$, standstill, to $s = 0$, synchronism, has been discussed. In this range the motor does mechanical work.

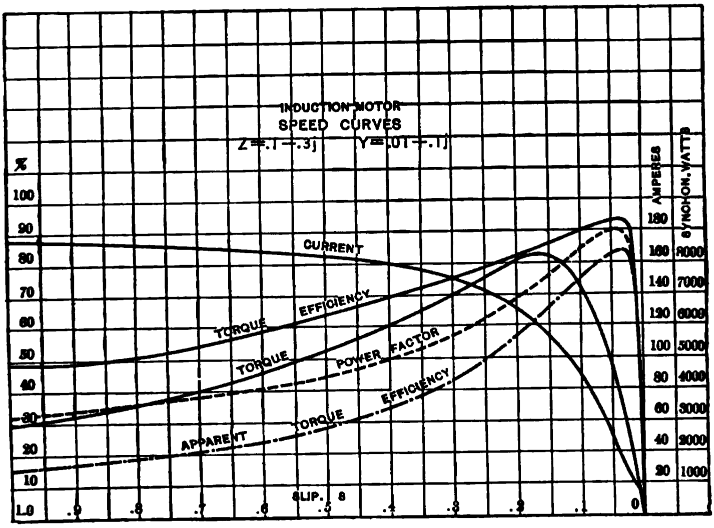


FIG. 130.

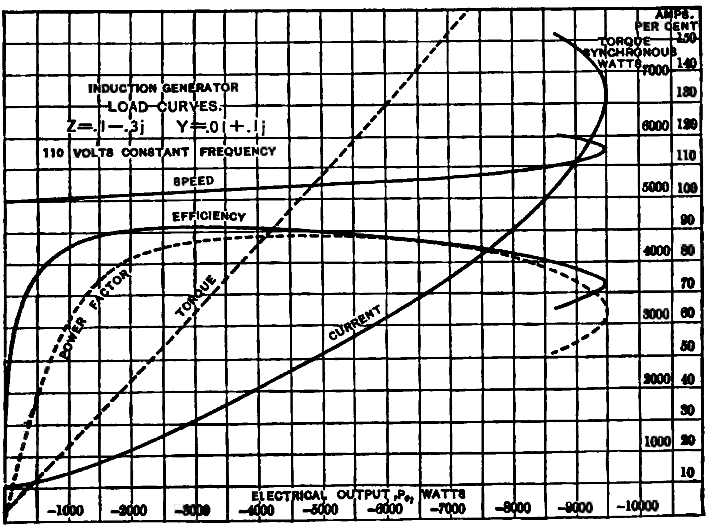


FIG. 131.

It consumes mechanical power, that is, acts as generator or as brake outside of this range.

For $s > 1$, backwards driving, P_1 becomes negative, representing consumption of power, while D remains positive; hence, since the direction of rotation has changed, represents consumption of power also. All this power is consumed in the motor, which thus acts as brake.

For $s < 0$, or negative, P_1 and D become negative, and the machine becomes an electric generator, converting mechanical into electric energy.

The calculation of the induction generator at constant frequency, that is, at a speed increasing with the load by the negative slip, s_1, is the same as that of the induction motor except that s_1 has negative values, and the load curves for the machine shown as motor in Fig. 129 are shown in Fig. 131 for negative slip s_1 as induction generator.

Again, a maximum torque point and a maximum output point are found, and the torque and power increase from zero at synchronism up to a maximum point, and then decrease again, while the current constantly increases.

197. The induction generator differs essentially from the ordinary synchronous alternator in so far as the induction generator has a definite power-factor, while the synchronous alternator has not. That is, in the synchronous alternator the phase relation between current and terminal voltage entirely depends upon the condition of the external circuit. The induction generator, however, can operate only if the phase relation of current and e.m.f., that is, the power-factor required by the external circuit, exactly coincides with the internal power-factor of the induction generator. This requires that the power-factor either of the external circuit or of the induction generator varies with the voltage, so as to permit the generator and the external circuit to adjust themselves to equality of power-factor.

Beyond magnetic saturation the power-factor decreases; that is, the lead of current increases in the induction machine. Thus, when connected to an external circuit of constant power-factor the induction generator will either not generate at all, if its power-factor is lower than that of the external circuit, or,

if its power-factor is higher than that of the external circuit, the voltage will rise until by magnetic saturation in the induction generator its power-factor has fallen to equality with that of the external circuit. This, however, requires magnetic saturation in the induction generator, in some part of the magnetic circuit, as for instance in the armature teeth.

To operate below saturation, — that is, at constant internal power-factor, — the induction generator requires an external circuit with leading current, whose power-factor varies with the voltage, as a circuit containing synchronous motors or synchronous converters. In such a circuit, the voltage of the induction generator remains just as much below the counter e.m.f. of the synchronous motor as is necessary to give the required leading exciting current of the induction generator, and the synchronous motor can thus to a certain extent be called the exciter of the induction generator.

When operating self-exciting, that is, shunt-wound, converters from the induction generator, below saturation of both the converter and the induction generator, the conditions are unstable also, and the voltage of one of the two machines must rise beyond saturation of its magnetic field.

When operating in parallel with synchronous alternating current generators, the induction generator obviously takes its leading exciting current from the synchronous alternator, which thus carries a lagging wattless current.

198. To generate constant frequency, the speed of the induction generator must increase with the load. Inversely, when driven at constant speed, with increasing load on the induction generator, the frequency of the current generated thereby decreases. Thus, when calculating the characteristic curves of the constant-speed induction generator, due regard has to be taken of the decrease of frequency with increase of load, or what may be called the slip of frequency, s.

Let, in an induction generator,

$$Y_0 = g_0 + jb_0 = \text{primary exciting admittance,}$$
$$Z_0 = r_0 - jx_0 = \text{primary self-inductive impedance,}$$
$$Z_1 = r_1 - jx_1 = \text{secondary self-inductive impedance,}$$

reduced to primary, all these quantities being reduced to the frequency of synchronism with the speed of the machine, f.

Let e = generated e.m.f., reduced to full frequency.

s = slip of frequency, thus: $(1-s)\,f$ = frequency generated by machine.

We then have

the secondary generated e.m.f.,

$$se:$$

thus, the secondary current,

$$\dot{I}_1 = \frac{se}{r_1 - jsx_1} = e\,(a_1 + ja_2),$$

where,

$$a_1 = \frac{sr_1}{r_1^2 + s^2x_1^2} \quad \text{and} \quad a_2 = \frac{s^2x_1}{r_1^2 + s^2x_1^2};$$

the primary exciting current,

$$\dot{I}_{00} = EY_0 = e\,(g_0 + jb_0),$$

thus, the total primary current,

$$\dot{I}_0 = \dot{I}_1 + \dot{I}_{00} = e\,(b_1 + jb_2),$$

where,

$$b_1 = a_1 + g_0 \quad \text{and} \quad b_2 = a_2 + b_0;$$

the primary impedance voltage,

$$\dot{E}^1 = \dot{I}_0\,(r_0 - j\,[1-s]x_0);$$

the primary generated e.m.f. is,

$$e\,(1-s).$$

Thus, primary terminal voltage,

$$\dot{E}_0 = e\,(1-s) - \dot{I}_0\,(r_0 - j\,[1-s]x_0) = e\,(c_1 + jc_2),$$

where,

$$c_1 = 1 - s - r_0b_1 - (1-s)\,x_0b_2 \quad \text{and} \quad c_2 = (1-s)\,x_0b_1 - r_0b_2,$$

hence, the absolute value is,

$$e_0 = e\sqrt{c_1^2 + c_2^2},$$

and,

$$e = \frac{e_0}{\sqrt{c_1^2 + c_2^2}}.$$

Thus,

the secondary current,

$$\dot{I}_1 = \frac{e_0\,(a_1 + ja_2)}{\sqrt{c_1^2 + c_2^2}}, \qquad I_1 = e_0\sqrt{\frac{a_1^2 + a_2^2}{c_1^2 + c_2^2}};$$

the primary current,

$$\dot{I}_0 = \frac{e_0\,(b_1 + jb_2)}{\sqrt{c_1^2 + c_2^2}}, \qquad I_0 = e_0\sqrt{\frac{b_1^2 + b_2^2}{c_1^2 + c_2^2}};$$

the primary terminal voltage,

$$\dot{E}_0 = \frac{e_0\,(c_1 + jc_2)}{\sqrt{c_1^2 + c_2^2}};$$

the torque and mechanical power input,

$$D = P_1 = [e\dot{I}_1]^1 = \frac{e_0^2 a_1}{c_1^2 + c_2^2};$$

the electrical output,

$$\dot{P}_0 = P_0^1 + jP_0^j = [E_0 I_0] = [E_0 I_0]^1 + j\,[E_0 I_0]^j$$
$$= \frac{e_0^2}{c_1^2 + c_2^2}\left\{(b_1 c_1 + b_2 c_2) + j\,(b_2 c_1 - b_1 c_2)\right\};$$

the volt-ampere output,

$$P_{a_0} = e_0 I_0$$
$$= e_0^2\,\frac{\sqrt{b_1^2 + b_2^2}}{c_1^2 + c_2^2};$$

the efficiency,

$$\frac{P_0^1}{P_1} = \frac{b_1 c_1 + b_2 c_2}{a_1};$$

the power-factor,

$$\cos\theta = \frac{P_0^1}{P_{a_0}} = \frac{b_1 c_1 + b_2 c_2}{\sqrt{(b_1^2 + b_2^2)\,(c_1^2 + c_2^2)}},$$

or,

$$\tan\theta = \frac{P_0^j}{P_0^1} = \frac{b_2 c_1 - b_1 c_2}{b_1 c_1 + b_2 c_2}.$$

In Fig. 132 is plotted the load characteristic of a constant-speed induction generator, at constant terminal voltage $e_0 = 110$,

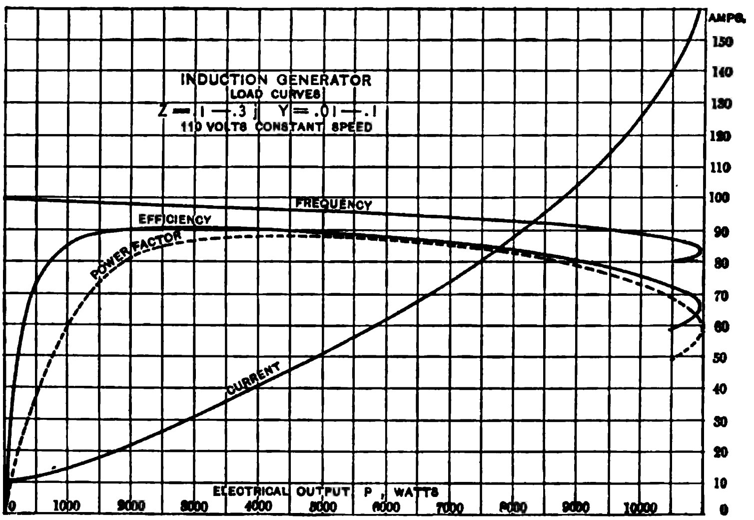


FIG. 132.

and the constants: $Y_0 = 0.01 + 0.1\,j$; $Z_0 = 0.1 - 0.3\,j$, and $Z_1 = 0.1 - 0.3\,j$.

199. As an example may be considered a power transmission from an induction generator of constants Y_0, Z_0, Z_1, over a line of impedance, $Z = r - jx$, into a synchronous motor of synchronous impedance, $Z_2 = r_2 - jx_2$, operating at constant-field excitation.

Let e_0 = counter e.m.f. or nominal generated e.m.f. of synchronous motor at full frequency; that is, frequency of synchronism with the speed of the induction generator. By the preceding paragraph the primary current of the induction generator was,

$$\dot{I}_0 = e\,(b_1 + jb_2);$$

the primary terminal voltage,

$$\dot{E}_0 = e\,(c_1 + jc_2);$$

thus, terminal voltage at synchronous motor terminals,

$$\dot{E}_0' = \dot{E}_0 - \dot{I}_0 (r - j[1 - s] x)$$
$$= e (d_1 + jd_2),$$

where,

$$d_1 = c_1 - rb_1 - (1 - s) xb_2 \text{ and } d_2 = (1 - s) xb_1 - rb_2;$$

the counter e.m.f. of the synchronous motor,

$$\dot{E}_2 = \dot{E}_0' - \dot{I}_0 (r_2 - j[1 - s] x_2)$$
$$= e (k_1 + jk_2);$$

where,

$$k_1 = d_1 - r_2b_1 - (1 - s) x_2b_2 \text{ and } k_2 = (1 - s) x_2b_1 - r_2b_2,$$

or the absolute value

$$E_2 = e\sqrt{k_1^2 + k_2^2},$$

since, however,

$$E_2 = e_0 (1 - s),$$

we have,

$$e = \frac{e_0 (1 - s)}{\sqrt{k_1^2 + k_2^2}}.$$

Thus, the current,

$$\dot{I}_0 = \frac{e_0 (1 - s) (b_1 + jb_2)}{\sqrt{k_1^2 + k_2^2}};$$

the terminal voltage at induction generator,

$$\dot{E}_0 = \frac{e_0 (1 - s) (c_1 + jc_2)}{\sqrt{k_1^2 + k_2^2}},$$

and the terminal voltage at the synchronous motor,

$$\dot{E}_0' = \frac{e_0 (1 - s) (d_1 + jd_2)}{\sqrt{k_1^2 + k_2^2}};$$

herefrom in the usual way the efficiencies, power-factor, etc., are derived.

When operated from an induction generator, a synchronous motor gives a load characteristic very similar to that of an

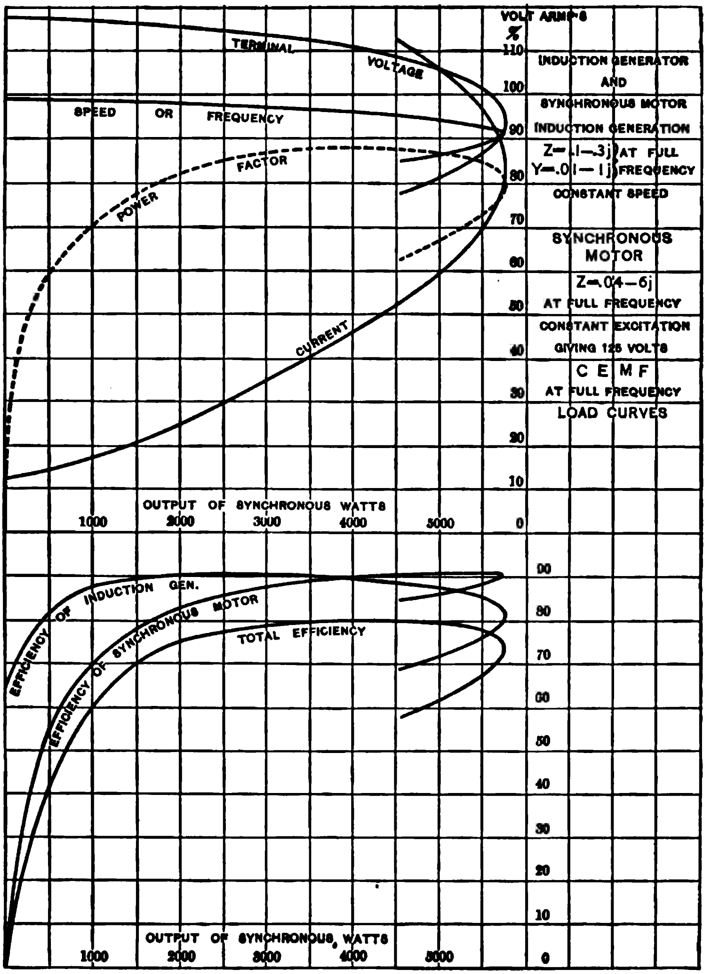


FIG. 133.

induction motor operated from a synchronous generator, but in the former case the current is leading, in the latter lagging.

In either case, the speed gradually falls off with increasing load (in the synchronous motor, due to the falling off of the

frequency of the induction generator), up to a maximum output point, where the motor drops out of step and comes to standstill.

Such a load characteristic of the induction generator in Fig. 132, feeding a synchronous motor of counter e.m.f. $e_0 = 125$ volts (at full frequency) and synchronous impedance $Z_2 = 0.04 - 6j$, over a line of negligible impedance is shown in Fig. 133.

CONCATENATION, OR TANDEM CONTROL OF INDUCTION MOTORS.

200. If of two induction motors the secondary of the first motor is connected to the primary of the second motor, the second machine operates as motor with the e.m.f. and frequency impressed upon it by the secondary of the first machine, which acts as general alternating-current transformer, converting a part of the primary impressed power into secondary electrical power for the supply of the second machine, and a part into mechanical work.

The frequency of the secondary e.m.f. of the first motor, and thus the frequency impressed upon the second motor, is the frequency of slip below complete synchronism, s. The frequency of the secondary generated e.m.f. of the second motor is the difference between its impressed frequency, s, and its speed; thus, if both motors are connected together mechanically to turn at the same speed, $1 - s$, the secondary frequency of the second motor, is $2s - 1$, hence equal to zero at $s = 0.5$. That is, the second motor reaches its synchronism at half speed. At this speed its torque becomes equal to zero, the power component of the current in it, and consequently the power component of the secondary current of the first motor, and thus the torque of the first motor becomes equal to zero also, when neglecting the hysteresis power current of the second motor. That is, a system of concatenated motors with short-circuited secondary of the second motor approaches half synchronism, in the same manner as the ordinary induction motor approaches synchronism. With increasing load, its slip below half synchronism increases.

More generally, any pair of induction motors connected in concatenation divide the speed so that the sum of their two respective speeds approaches synchronism at no-load; or,

still more generally, any number of concatenated motors run at such speeds that the sum of the speeds approaches synchronism at no-load.

With mechanical connection between the two motors, concatenation thus offers a means to operate a pair of induction motors at full efficiency at half speed in tandem, as well as at full speed in parallel, and thus gives the same advantage as the series-parallel control of the continuous-current motor.

In starting, a concatenated system is controlled by resistance in the armature of the second motor.

Since, with increasing speed, the frequency impressed upon the second motor decreases proportionally to the decrease of voltage, when neglecting internal losses in the first motor, the magnetic density of the second motor remains practically constant, and thus its torque the same as when operated at full voltage and full frequency under the same conditions.

At half synchronism the torque of the concatenated couple becomes zero, and above half-synchronism the second motor runs beyond its impressed frequency; that is, becomes a generator. In this case, due to the reversal of current in the secondary of the first motor, its torque becomes negative also, that is, the concatenated couple becomes an induction generator above half synchronism. At about two thirds synchronism, with low-resistance armature, the torque of the couple becomes zero again, and once more positive between about two thirds synchronism and full synchronism, and negative once more beyond full synchronism. With high resistance in the secondary of the second motor, the second range of positive torque, below full synchronism, disappears, more or less.

201. The calculation of a concatenated couple of induction motors is as follows,

Let

f = frequency of main circuit,
s = slip of the first motor from synchronism.

The frequency in the secondary of the first motor and thus impressed upon the primary of the second motor is sf.

The speed of the first motor is $(1 - s)f$; thus the slip of the second motor, or the frequency in its secondary, is

$$sf - (1 - s)f = (2s - 1)f.$$

Let

e = counter e.m.f. generated in the secondary of the second motor, reduced to full frequency.

$Z_0 = r_0 - jx_0$ = primary self-inductive impedance.
$Z_1 = r_1 - jx_1$ = secondary self-inductive impedance.
$Y = g + jb$ = primary exciting admittance of each motor, all reduced to full frequency and to the primary by the ratio of turns.

We then have for the second motor:

the secondary generated e.m.f.,

$$e\,(2\,s - 1);$$

the secondary current,

$$\dot{I}_1 = \frac{e\,(2\,s-1)}{r_1 - j\,(2\,s-1)x_1} = e\,(a_1 + ja_2),$$

where,

$$a_1 = \frac{(2\,s-1)r_1}{r_1^2 + (2\,s-1)^2x_1^2} \text{ and } a_2 = \frac{(2\,s-1)^2x_1}{r_1^2 + (2\,s-1)^2x_1^2};$$

the primary exciting current,

$$\dot{I}_0 = e\,(g + jb);$$

thus, the total primary current,

$$\dot{I}_2 = \dot{I}_1 + \dot{I}_0 = e\,(b_1 + jb_2),$$

where,

$$b_1 = a_1 + g \text{ and } b_2 = a_2 + b;$$

the primary generated e.m.f. is

$$se,$$

the primary impedance voltage is

$$\dot{I}_2\,(r_0 - jsx_0):$$

thus, the primary impressed e.m.f.,

$$\dot{E}_2 = se + \dot{I}_2\,(r_0 - jsx_0) = e\,(c_1 + jc_2),$$

where,

$$c_1 = s + r_0b_1 + sx_0b_2 \text{ and } c_2 = r_0b_2 - sx_0b_1.$$

For the first motor, the secondary current,

$$\dot{I}_2 = e\,(b_1 + jb_2);$$

the secondary generated e.m.f.,

$$\dot{E}_3 = \dot{E}_2 + \dot{I}_2\,(r_1 - jsx_1) = e\,(d_1 + jd_2),$$

where,

$$d_1 = c_1 + r_1 b_1 + sx_1 b_2 \text{ and } d_2 = c_2 + r_1 b_2 - sx_1 b_1;$$

the primary generated e.m.f.,

$$\dot{E}_4 = \frac{E_3}{s} = e\,(k_1 + jk_2),$$

where,

$$k_1 = \frac{d_1}{s} \text{ and } k_2 = \frac{d_2}{s}{}^*;$$

the primary exciting current,

$$\dot{I}_4 = \dot{E}_4\,(g + jb),$$

the total primary current,

$$\dot{I} = \dot{I}_2 + \dot{I}_4 = e\,(g_1 + jg_2),$$

where,

$$g_1 = b_1 + gk_1 - bk_2 \text{ and } g_2 = b_2 + gk_2 + bk_1;$$

the primary impedance voltage is,

$$\dot{I}\,(r_0 - jx_0);$$

thus, the primary impressed e.m.f.,

$$\dot{E}_0 = \dot{E}_4 + \dot{I}\,(r_0 - jx_0) = e\,(h_1 + jh_2),$$

where,

$$h_1 = k_1 + r_0 g_1 + x_0 g_2 \text{ and } h_2 = k_2 + r_0 g_2 - x_0 g_1;$$

or, the absolute value,

$$e_0 = e\sqrt{h_1^2 + h_2^2},$$

and,

$$e = \frac{e_0}{\sqrt{h_1^2 + h_2^2}}.$$

* At $s = 0$ these terms, k_1 and k_2, become indefinite, and thus at and very near synchronism have to be derived by substituting the complete expressions for k_1 and k_2.

Substituting now this value of e in the preceding gives the values of the currents and e.m.fs. in the different circuits of the motor series.

In the second motor, the torque

$$D_2 = [e\dot{I}_1]^1 = e^2 a_1,$$

hence, its power output,

$$P_3 = (1 - s)\, D_2 = (1 - s)\, e^2 a_1.$$

The power input is,

$$\begin{aligned} \dot{P}_2 &= [\dot{E}_2 \dot{I}_2] = [\dot{E}_2 \dot{I}_2]^1 + j\, [\dot{E}_2 \dot{I}_2]^j \\ &= e^2\, [(c_1 + jc_2)\,(b_1 + jb_2)], \end{aligned}$$

hence, the efficiency is,

$$\frac{P_3}{P_2^1} = \frac{(1 - s)\, e^2 a_1}{[\dot{E}_2 \dot{I}_2]^1} = \frac{(1 - s)\, a_1}{c_1 b_1 + c_2 b_2};$$

the power-factor is,

$$\frac{P_2^1}{P_{a_2}} = \frac{[\dot{E}_2 \dot{I}_2]^1}{E_2 I_2} = \frac{c_1 b_1 + c_2 b_2}{\sqrt{(c_1^2 + c_2^2)\,(b_1^2 + b_2^2)}},$$

etc.

In the first motor,

the torque,

$$\begin{aligned} D_1 &= [\dot{E}_4 \dot{I}_2]^1 = e^2\, [(k_1 + jk_2)\,(b_1 + jb_2)]^1 \\ &= e^2\, (k_1 b_1 + k_2 b_2); \end{aligned}$$

the power output,

$$\begin{aligned} P_4 &= D_1\, (1 - s) \\ &= e^2\, (1 - s)\, (k_1 b_1 + k_2 b_2); \end{aligned}$$

the power input,

$$\begin{aligned} \dot{P}_1 &= [\dot{E}_0 \dot{I}] = e^2\, [(h_1 + jh_2)\,(g_1 + jg_2)] \\ &= [\dot{E}_0 \dot{I}]^1 + j\, [\dot{E}_0 \dot{I}]^j. \end{aligned}$$

Thus, the efficiency is,

$$\frac{P_4}{[E_0 I]^1 - [E_2 I_2]^1} = \frac{(1-s)(k_1 b_1 + k_2 b_2)}{(h_1 g_1 + h_2 g_2) - (c_1 b_1 + c_2 b_2)};$$

the power-factor of the whole system is,

$$\frac{P_1}{E_0 I} = \frac{h_1 g_1 + h_2 g_2}{\sqrt{(h_1^2 + h_2^2)(g_1^2 + g_2^2)}};$$

the power-factor of the first motor is,

$$\frac{P_1 - P_2}{E_0 I - E_2 I_2} = \frac{(h_1 g_1 + h_2 g_2) - (c_1 b_1 + c_2 b_2)}{\sqrt{(h_1^2 + h_2^2)(g_1^2 + g_2^2)} - \sqrt{(c_1^2 + c_2^2)(b_1^2 + b_2^2)}};$$

the total efficiency of the system,

$$\frac{P_4 + P_3}{[E_0 I]^1} = \frac{(1-s)(k_1 b_1 + k_2 b_2 + a_1)}{h_2 g_1 + h_2 g_2},$$

etc.

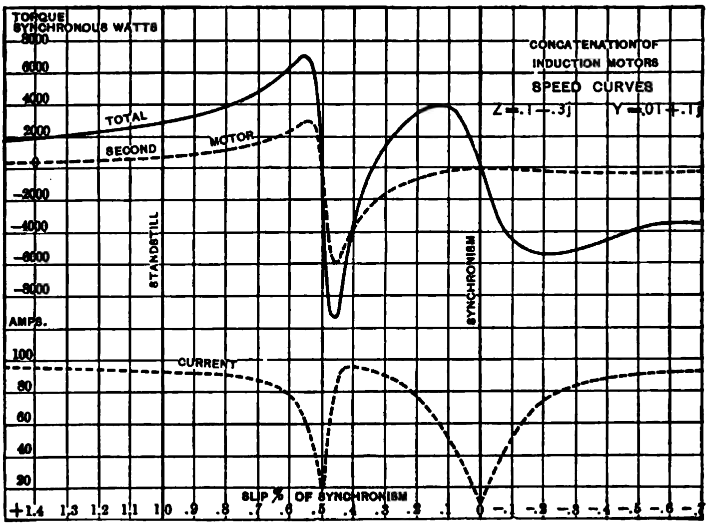


FIG. 134. — Concatenation of Induction Motors, Speed Curves.

202. As examples are given in Fig. 134, the curves of total torque, of torque of the second motor, and of current, for the range of slip from $s = +1.5$ to $s = -0.7$ for a pair of induction

motors in concatenation, having the constants: $Z_0 = Z_1 = 0.1 - 0.3\,j$, $Y = 0.01 + 0.1\,j$.

As seen, there are two ranges of positive torque for the whole system; one below half synchronism, and one from about two thirds to full synchronism, and two ranges of negative torque, or generator action of the motor, from half to two thirds synchronism, and above full synchronism.

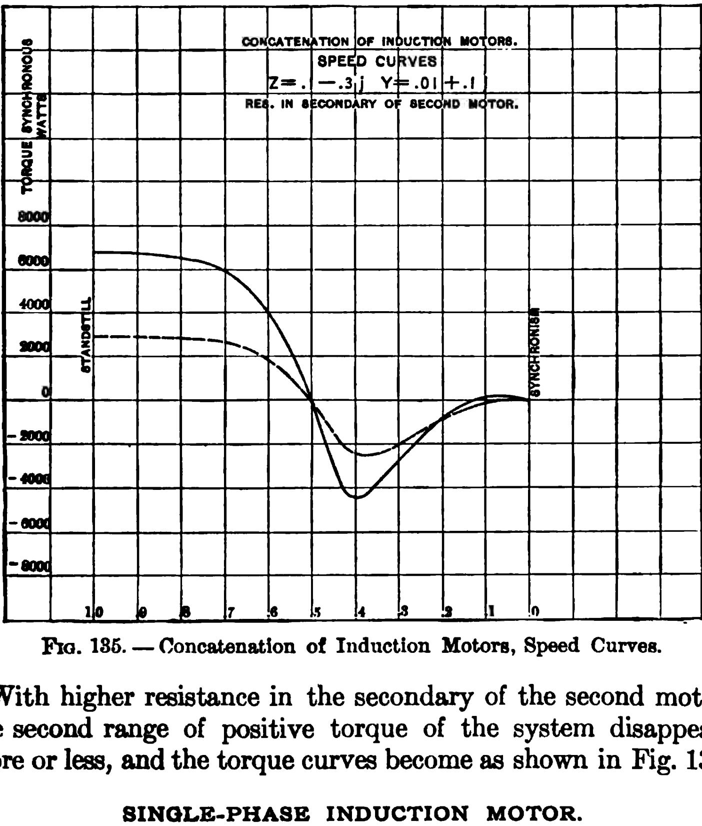


FIG. 135. — Concatenation of Induction Motors, Speed Curves.

With higher resistance in the secondary of the second motor, the second range of positive torque of the system disappears more or less, and the torque curves become as shown in Fig. 135.

SINGLE-PHASE INDUCTION MOTOR.

203. The magnetic circuit of the induction motor at or near synchronism consists of two magnetic fluxes superimposed upon each other in quadrature, in time, and in position. In the polyphase motor these fluxes are produced by e.m.fs. displaced in phase. In the monocyclic motor one of the fluxes is due to the primary power circuit, the other to the primary exciting

circuit. In the single-phase motor the one flux is produced by the primary circuit, the other by the currents produced in the secondary or armature, which are carried into quadrature position by the rotation of the armature. In consequence thereof, while in all these motors the magnetic distribution is the same at or near synchronism, and can be represented by a rotating field of uniform intensity and uniform velocity, it remains such in polyphase and monocyclic motors; but in the single-phase motor, with increasing slip, — that is, decreasing speed, — the quadrature field decreases, since the secondary armature currents are not carried to complete quadrature position; and thus only a component available for producing the quadrature flux. Hence, approximately, the quadrature flux of a single-phase motor can be considered as proportional to its speed; that is, it is zero at standstill.

Since the torque of the motor is proportional to the product of secondary current times magnetic flux in quadrature, it follows that the torque of the single-phase motor is equal to that of the same motor under the same condition of operation on a polyphase circuit, multiplied with the speed; hence equal to zero at standstill.

Thus, while single-phase induction motors are quite satisfactory at or near synchronism, their torque decreases proportionally with the speed, and becomes zero at standstill. That is, they are not self-starting, but some starting device has to be used.

Such a starting device may either be mechanical or electrical. All the electrical starting devices essentially consist in impressing upon the motor at standstill a magnetic quadrature flux. This may be produced either by some outside e.m.f., as in the monocyclic starting device, or by displacing the circuits of two or more primary coils from each other, either by mutual induction between the coils, — that is, by using one as secondary to the other, — or by impedances of different inductance factors connected with the different primary coils.

204. The starting devices of the single-phase induction motor by producing a quadrature magnetic flux can be subdivided into three classes:

1. Phase-Splitting Devices. Two or more primary circuits are used, displaced in position from each other, and either in

series or in shunt with each other, or in any other way related, as by transformation. The impedances of these circuits are made different from each other as much as possible to produce a phase displacement between them. This can be done either by inserting external impedances in the circuits, as a condenser and a reactive coil, or by making the internal impedances of the motor circuits different, as by making one coil of high and the other of low resistance.

2. Inductive Devices. The different primary circuits of the motor are inductively related to each other in such a way as to produce a phase displacement between them. The inductive relation can be outside of the motor or inside, by having the one coil submitted to the inductive action of the other; and in this latter case the current in the secondary coil may be made leading, accelerating coil, or lagging, shading coil.

3. Monocyclic Devices. External to the motor an essentially wattless e.m.f. is produced in quadrature with the main e.m.f. and impressed upon the motor, either directly or after combination with the single-phase main e.m.f. Such wattless quadrature e.m.f. can be produced by the common connection of two impedances of different power-factor, as an inductive reactance and a resistance, or an inductive and a condensive reactance connected in series across the mains.

The investigation of these starting-devices offers a very instructive application of the symbolic method of investigation of alternating-current phenomena, and a study thereof is thus recommended to the reader.*

205. Frequently, no special motors are built for single-phase operation, but polyphase motors used in single-phase circuits, since for starting the polyphase primary winding is required, the single primary-coil motor obviously not allowing the application of phase-displacing devices for producing the starting quadrature flux.

Since at or near synchronism, at the same impressed e.m.f. — that is, the same magnetic density — the total volt-amperes excitation of the single-phase induction motor must be the same

* See paper on the Single-phase Induction Motor, A. I. E. E. Transactions, 1898.

as of the same motor on polyphase circuit, it follows that by operating a quarter-phase motor from single-phase circuit on one primary coil, its primary exciting admittance is doubled. Operating a three-phase motor single-phase on one circuit its primary exciting admittance is trebled. The self-inductive primary impedance is the same single-phase as polyphase, but the secondary impedance reduced to the primary is lowered, since in single-phase operation all secondary circuits correspond to the one primary circuit used. Thus the secondary impedance in a quarter-phase motor running single-phase is reduced to one-half, in a three-phase motor running single-phase reduced to one-third. In consequence thereof the slip of speed in a single-phase induction motor is usually less than in a polyphase motor; but the exciting current is considerably greater, and thus the power-factor and the efficiency are lower.

The preceding considerations obviously apply only when running so near synchronism that the magnetic field of the single-phase motor can be assumed as uniform, that is, the cross magnetizing flux produced by the armature as equal to the main magnetic flux.

When investigating the action of the single-phase motor at lower speeds and at standstill, the falling off of the magnetic quadrature flux produced by the armature current, the change of secondary impedance, and where a starting device is used the effect of the magnetic field produced by the starting device, have to be considered.

The exciting current of the single-phase motor consists of the primary exciting current or current producing the main magnetic flux, and represented by a constant admittance, Y_0^1, the primary exciting admittance of the motor, and the secondary exciting current, that is, that component of primary current corresponding to the secondary current which gives the excitation for the quadrature magnetic flux. This latter magnetic flux is equal to the main magnetic flux, Φ_0, at synchronism, and falls off with decreasing speed to zero at standstill, if no starting device is used, or to $\Phi_1 = \Phi_0 t$ at standstill if by a starting device a quadrature magnetic flux is impressed upon the motor, and at standstill t = ratio of quadrature or starting magnetic flux to main magnetic flux.

Thus the secondary exciting current can be represented by an

admittance, Y_1^1, which changes from equality with the primary exciting admittance, Y_0^1, at synchronism to $Y_1^1 = 0$, respectively to $Y_1^1 = tY_0^1$ at standstill. Assuming thus that the starting device is such that its action is not impaired by the change of speed, at slip s the secondary exciting admittance can be represented by:

$$Y_1^1 = [1 - (1 - t)\, s]\, Y_0^1.$$

The secondary impedance of the motor at synchronism is the joint impedance of all the secondary circuits, since all secondary circuits correspond to the same primary circuit, hence $= \frac{Z_1}{3}$ with a three-phase secondary, and $= \frac{Z_1}{2}$ with a two-phase secondary with impedance Z_1 per circuit.

At standstill, however, the secondary circuits correspond to the primary circuit only with their projection in the direction of the primary flux, and thus as resultant only one-half of the secondary circuits are effective, so that the secondary impedance at standstill is equal to $\frac{2\,Z_1}{3}$ with a three-phase, and equal to Z_1 with a two-phase, secondary. Thus the effective secondary impedance of the single-phase motor changes with the speed and can at the slip, s, be represented by $Z_1^1 = \frac{(1 + s)\, Z_1}{3}$ in a three-phase secondary, and $Z_1^1 = \frac{(1 + s)\, Z_1}{2}$ in a two-phase secondary, with the impedance, Z_1, per secondary circuit.

In the single-phase motor without starting device, due to the falling off of the quadrature flux, the torque at slip s is:

$$D = a_1 e^2 (1 - s).$$

In a single-phase motor with a starting device which at standstill produces a ratio of magnetic fluxes t, the torque at standstill is

$$D_0 = tD_1,$$

where D_1 = total torque of the same motor on polyphase circuit.

Thus denoting the value $\frac{D_0}{a_1 e^2} = v$,

the single-phase motor torque at standstill is:

$$D_0 = a_1 e^2 v,$$

and the single-phase motor torque at slip s is:

$$D = a_1 e^2 [1 - (1 - v) s].$$

206. In the single-phase motor considerably more advantage is gained by compensating for the wattless magnetizing component of current by capacity than in the polyphase motor, where this wattless component of the current is relatively small. The use of shunted capacity, however, has the disadvantage of requiring a wave of impressed e.m.f. very close to sine shape; since even with a moderate variation from sine shape the wattless charging current of the condenser of higher frequency may lower the power-factor more than the compensation for the wattless component of the fundamental wave raises it, as will be seen in the chapter on General Alternating-Current Waves.

Thus the most satisfactory application of the condenser in the single-phase motor is not in shunt to the primary circuit, but in a tertiary circuit; that is, in a circuit stationary with regard to the primary impressed circuit but submitted to inductive action by the revolving secondary circuit.

In this case the condenser is supplied with an e.m.f. transformed twice, from primary to secondary, and from secondary to tertiary, through multitooth structures in a uniformly revolving field, and thus a very close approximation to sine wave produced at the condenser, irrespective of the wave-shape of primary impressed e.m.f.

With the condenser connected into a tertiary circuit of a single-phase induction motor, the wattless magnetizing current of the motor is supplied by the condenser in a separate circuit, and the primary coil carries the power current only, and thus the efficiency of the motor is essentially increased.

The tertiary circuit may be at right angles to the primary, or under any other angle. Usually it is applied on an angle of 60°, so as to secure a mutual induction between tertiary and

primary for starting, which produces in starting in the condenser a leading current, and gives the quadrature magnetic flux required.

207. The most convenient way to secure this arrangement is the use of a three-phase motor which with two of its terminals, 1–2, is connected to the single-phase mains, and with terminals 1 and 3 to a condenser.

Let $Y_0 = g_0 + jb_0$ = primary exciting admittance of the motor per delta circuit.

$Z_0 = r_0 - jx_0$ = primary self-inductive impedance per delta circuit.

$Z_1 = r_1 - jx_1$ = secondary self-inductive impedance per delta circuit reduced to primary.

Let

$Y_3 = g_3 - jb_3$ = admittance of the condenser connected between terminals 1 and 3.

If then, as single-phase motor,

t = ratio of auxiliary quadrature flux to main flux in starting,

h = ratio of e.m.f. generated in condenser circuit to e.m.f. generated in main circuit in starting,

$$v = \frac{\text{starting torque}}{a_1 e^2 \text{ in starting}} .$$

Operating single-phase

$Y_0^1 = 1.5\, Y_0 = 1.5\, (g_0 + jb_0)$ = primary exciting admittance,

$Y_1^1 = 1.5\, Y_0 [1 - (1 - t)\, s]$

$= 1.5\ (g_0 + jb_0)\ [1 - (1 - t)\ s]$ = secondary exciting admittance at slip s;

$Z_0^1 = \dfrac{2\, Z_0}{3} = \dfrac{2\, (r_0 - jx_0)}{3}$ = primary self-inductive impedance;

$Z_1^1 = \dfrac{(1 + s)}{3}\, Z_1 = \dfrac{(1 + s)}{3}\ (r_1 - jsx_1)$ = secondary self-inductive impedance;

$Z_2^1 = \dfrac{2\, Z_0}{3} = \dfrac{2\, (r_0 - jx_0)}{3}$ = tertiary self-inductive impedance of motor.

Thus,

$$Y_4 = \frac{1}{Z_2^1 + \dfrac{1}{Y_3}} = \text{total admittance of tertiary circuit.}$$

Since the e.m.f. generated in the tertiary circuit decreases from e at synchronism to he at standstill, the effective tertiary admittance or admittance reduced to a generated e.m.f., e, is, at slip, s

$$Y_4^1 = [1 - (1 - h)\, s]\, Y_4.$$

Let then,

e = counter e.m.f. of primary circuit,

s = slip.

We have,

the secondary load-current,

$$\dot{I}_1 = \frac{se}{Z_1^1} = \frac{3\, se}{(1 + s)\,(r_1 - jsx_1)} = e\,(a_1 + ja_2);$$

the secondary exciting current,

$$\dot{I}_1^1 = eY_1^1 = 1.5\, eY_0 [1 - (1 - t)\, s];$$

the secondary condenser current;

$$\dot{I}_4 = eY_4^1 = eY_4 [1 - (1 - h)\, s];$$

thus, the total secondary current,

$$\dot{I}^1 = \dot{I}_1 + \dot{I}_1^1 + \dot{I}_4;$$

the primary exciting current,

$$\dot{I}_0^1 = eY_0^1 = 1.5\, eY_0,$$

thus, the total primary current,

$$\dot{I}_0 = \dot{I}^1 + \dot{I}_0^1 = \dot{I}_1 + \dot{I}_4 + I_1^1 + I_0^1 = e\,(b_1 + jb_2);$$

the primary impressed e.m.f.,

$$\dot{E}_0 = e + Z_0^1 I_0 = e\,(c_1 + jc_2);$$

thus, the main counter e.m.f.,

$$e = \frac{\dot{E}_0}{c_1 + jc_2},$$

or,

$$\underset{\cdot}{E} = \frac{e_0}{c_1 + jc_2},$$

and the absolute value,

$$e = \frac{e_0}{\sqrt{c_1^2 + c_2^2}},$$

hence, the primary current,

$$\underset{\cdot}{I}_0 = \frac{e_0(b_1 + jb_2)}{c_1 + jc_2}, \text{ or, } I_0 = e_0\sqrt{\frac{b_1^2 + b_2^2}{c_1^2 + c_2^2}}.$$

The volt-ampere input,

$$P_{a_0} = e_0 I_0;$$

the power input,

$$P_0 = [\underset{\cdot}{I}_0 e_0]^1 = e_0^2 \frac{b_1 c_1 + b_2 c_2}{c_1^2 + c_2^2};$$

the torque at slip s,

$$D = D^1 [1 - (1 - v)\, s] = \frac{e_0^2 a_1}{c_1^2 + c_2^2} [1 - (1 - v)\, s],$$

and the power output,

$$\begin{aligned} P &= D\,(1 - s) \\ &= \frac{e_0^2 a_1}{c_1^2 + c_2^2} (1 - s)\, [1 - (1 - v)\, s], \end{aligned}$$

and herefrom in the usual manner may be derived the efficiency, apparent efficiency, torque efficiency, apparent torque efficiency, and power-factor.

The derivation of the constants, t, h, v, which have to be determined before calculating the motor, is as follows:

Let e_0 = single-phase impressed em.f.,

Y = total stationary admittance of motor per delta circuit,

$\underset{\cdot}{E}_s$ = e.m.f. at condenser terminals in starting.

In the circuit between the single-phase mains from terminal 1 over terminal 3 to 2, the admittances, $Y + Y_s$, and Y, are connected in series, and have the respective e.m.fs., $\dot{E}_s$ and $e_0 - \dot{E}_s$. It is thus,

$$Y + Y_s \div Y = e_0 - \dot{E}_s \div \dot{E}_s,$$

since with the same current in both circuits, the impressed e.m.fs. are inversely proportional to the respective admittances.

Thus,

$$\dot{E}_s = \frac{e_0 Y}{2\,Y + Y_s} = e_0\,(h_1 + jh_2),$$

and the quadrature e.m.f. is

$$e_0 h_2,$$

hence

$$E_s = e_0 \sqrt{h_1^2 + h_2^2},$$

and

$$h = \sqrt{h_1^2 + h_2^2}.$$

Since in the three-phase e.m.f. triangle, the altitude corresponding to the quadrature magnetic flux $= \frac{e_0}{2\sqrt{3}}$, and the quadrature and main fluxes are equal, in the single-phase motor the ratio of quadrature to main flux is

$$t = \frac{2\,h_2}{\sqrt{3}} = 1.155\,h_2.$$

From t, v is derived as shown in the preceding.

For further discussion on the theory and calculation of the single-phase induction motor, see American Institute Electrical Engineers Transactions, January, 1900.

SYNCHRONOUS INDUCTION MOTOR.

208. The induction motor discussed in the foregoing consists of one or a number of primary circuits acting upon a movable armature which comprises a number of closed secondary circuits displaced from each other in space so as to offer a resultant circuit in any direction. In consequence thereof the motor can be considered as a transformer, having to each primary circuit

a corresponding secondary circuit, — a secondary coil, moving out of the field of the primary coil, being replaced by another secondary coil moving into the field.

In such a motor the torque is zero at synchronism, positive below, and negative above, synchronism.

If, however, the movable armature contains one closed circuit only, it offers a closed secondary circuit only in the direction of the axis of the armature coil, but no secondary circuit at right angles therewith. That is, with the rotation of the armature the secondary circuit, corresponding to a primary circuit, varies from short-circuit at coincidence of the axis of the armature coil with the axis of the primary coil, to open-circuit in quadrature therewith, with the periodicity of the armature speed. That is, the apparent admittance of the primary circuit varies periodically from open-circuit admittance to the short-circuited transformer admittance.

At synchronism such a motor represents an electric circuit of an admittance varying with twice the periodicity of the primary frequency, since twice per period the axis of the armature coil and that of the primary coil coincide. A varying admittance is obviously identical in effect with a varying reluctance, which will be discussed in the chapter on reaction machines. That is, the induction motor with one closed armature circuit is, at synchronism, nothing but a reaction machine, and consequently gives zero torque at synchronism if the maxima and minima of the periodically varying admittance coincide with the maximum and zero values of the primary circuit, but gives a definite torque if they are displaced therefrom. This torque may be positive or negative according to the phase displacement between admittance and primary circuit; that is, the lag or lead of the maximum admittance with regard to the primary maximum. Hence an induction motor with single-armature circuit at synchronism acts either as motor or as alternating-current generator according to the relative position of the armature circuit with respect to the primary circuit. Thus it can be called a synchronous induction motor or synchronous induction generator, since it is an induction machine giving torque at synchronism.

Power-factor and apparent efficiency of the synchronous induction motor as reaction machine are very low. Hence it is

of practical application only in cases where a small amount of power is required at synchronous rotation, and continuous current for field excitation is not available.

The current produced in the armature of the synchronous induction motor is of double the frequency impressed upon the primary.

Below and above synchronism the ordinary induction motor, or induction generator, torque is superimposed upon the synchronous-induction machine torque. Since with the frequency of slip the relative position of primary and of secondary coil changes, the synchronous-induction machine torque alternates periodically with the frequency of slip. That is, upon the constant positive or negative torque below or above synchronism an alternating torque of the frequency of slip is superimposed, and thus the resultant torque pulsating with a positive mean value below, a negative mean value above, synchronism.

When started from rest, a synchronous induction motor will accelerate like an ordinary single-phase induction motor, but not only approach synchronism, as the latter does, but run up to complete synchronism under load. When approaching synchronism it makes definite beats with the frequency of slip, which disappear when synchronism is reached.

THE HYSTERESIS MOTOR.

209. In a revolving magnetic field, a circular iron disk, or iron cylinder of uniform magnetic reluctance in the direction of the revolving field, is set in rotation, even if subdivided so as to preclude the production of eddy currents. This rotation is due to the effect of hysteresis of the revolving disks or cylinder, and such a motor may thus be called a hysteresis motor.

Let I be the iron disk exposed to a rotating magnetic field or resultant m.m.f. The axis of resultant magnetization in the disk, I, does not coincide with the axis of the rotating field, but lags behind the latter, thus producing a couple. That is, the component of magnetism in a direction of the rotating disk, I, ahead of the axis of rotating m.m.f., is rising, thus below, and in a direction behind the axis of rotating m.m.f. decreasing; that is, above proportionality with the m.m.f., in consequence of the lag of magnetism in the hysteresis loop, and thus the axis of

resultant magnetism in the iron disk, $\dot{I}$, does not coincide with the axis of rotating m.m.f., but is shifted backwards by an angle, α, which is the angle of hysteretic lead in Chapter XIII., § 114.

The induced magnetism gives with the resultant m.m.f. a mechanical couple, —

$$D = m\mathfrak{F}\Phi \sin \alpha,$$

where

$$\mathfrak{F} = \text{resultant m.m.f.},$$

$$\Phi = \text{resultant magnetism},$$

$$\alpha = \text{angle of hysteretic advance of phase},$$

$$m = \text{a constant}.$$

The apparent or volt-ampere input of the motor is, —

$$P = m\mathfrak{F}\Phi.$$

Thus the apparent torque efficiency, —

$$\frac{D}{P_a} = \sin \alpha,$$

and the power of the motor is, —

$$P = (1 - s)\, D = (1 - s)\, m\mathfrak{F}\Phi \sin \alpha,$$

where

$$s = \text{slip as fraction of synchronism}.$$

The apparent efficiency is, —

$$\frac{P}{P_a} = (1 - s) \sin \alpha.$$

Since in a magnetic circuit containing an air-gap the angle, α, is extremely small, a few degrees only, it follows that the apparent efficiency of the hysteresis motor is extremely low, the motor consequently unsuitable for producing large amounts of mechanical power.

From the equation of torque it follows, however, that at constant impressed e.m.f., or current, — that is, constant $\mathfrak{F}$, — the torque is constant and independent of the speed; and there-

fore such a motor arrangement is suitable, and occasionally used as alternating-current meter.

For $$s \leqq 0, \text{ we have } \alpha < 0,$$

and the apparatus is an hysteresis generator.

210. The same result can be reached from a different point of view. In such a magnetic system, comprising a movable iron disk, I, of uniform magnetic reluctance in a revolving field, the magnetic reluctance — and thus the distribution of magnetism — is obviously independent of the speed, and consequently the current and energy expenditure of the impressed m.m.f. independent of the speed also. If, now, —

V = volume of iron of the movable part,

$\mathfrak{B}$ = magnetic density,

and η = coefficient of hysteresis,

the energy expended by hysteresis in the movable disk, I, is per cycle, —

$$W_0 = V\eta\mathfrak{B}^{1.6},$$

hence, if f = frequency, the power supplied by the m.m.f. to the rotating iron disk in the hysteretic loop of the m.m.f. is, —

$$P_0 = fV\eta\mathfrak{B}^{1.6}.$$

At the slip, sf, that is, the speed $(1 - s)f$, the power expended by hysteresis in the rotating disk is, however, —

$$P_1 = sfV\eta\mathfrak{B}^{1.6}.$$

Hence, in the transfer from the stationary to the revolving member the magnetic power,—

$$P = P_0 - P_1 = (1 - s)\, fV\eta\mathfrak{B}^{1.6},$$

has disappeared, and thus reappears as mechanical work, and the torque is, —

$$D = \frac{P}{(1 - s)\, f} = V\eta\mathfrak{B}^{1.6},$$

that is, independent of the speed.

Since, as seen in Chapter XIII, sin α is the ratio of the energy of the hysteretic loop to the total apparent energy of the magnetic cycle, it follows that the apparent efficiency of such a motor can never exceed the value $(1 - s) \sin \alpha$, or a fraction of the primary hysteretic energy.

The primary hysteretic energy of an induction motor, as represented by its conductance, g, being a part of the loss in the motor, and thus a very small part of its output only, it follows that the output of a hysteresis motor is a very small fraction only of the output which the same magnetic structure could give with secondary short-circuited winding, as regular induction motor.

As secondary effect, however, the rotary effort of the magnetic structure as hysteresis motor appears more or less in all induction motors, although usually it is so small as to be neglected.

If in the hysteresis motor the rotary iron structure has not uniform reluctance in all directions — but is, for instance, bar-shaped or shuttle-shaped — on the hysteresis-motor effect is superimposed the effect of varying magnetic reluctance, which tends to accelerate the motor to synchronism, and maintain it therein, as shall be more fully investigated under "Reaction Machine" in Chapter XXVII.

211. In the hysteresis motor, consisting of an iron disk of uniform magnetic reluctance, which revolves in a uniformly rotating magnetic field, below synchronism, the magnetic flux rotates in the armature with the frequency of slip, and the resultant line of magnetic induction in the disk thus lags, in space, behind the synchronously rotating line of resultant m.m.f. of the exciting coils, by the angle of hysteretic lead, α, which is constant, and so gives, at constant magnetic flux, that is, constant impressed e.m.f., a constant torque and a power proportional to the speed.

Above synchronism, the iron disk revolves faster than the rotating field, and the line of resulting magnetization in the disk being behind the line of m.m.f. with regard to the direction of rotation of the magnetism in the disk, therefore is ahead of it in space, that is, the torque and therefore the power reverses at synchronism, and above synchronism the apparatus is an

hysteresis generator, that is, changes at synchronism from motor to generator. At synchronism such a disk thus can give mechanical power as motor, with the line of induction lagging, or give electric power as generator, with the line of induction leading the line of rotation m.m.f.

Electrically, the power transferred between the electric circuit and the rotating disk is represented by the hysteresis loop. Below synchronism the hysteresis loop of the electric circuit has the normal shape, and of its constant power a part, proportional to the slip, is consumed in the iron, the other part, proportional to the speed, appears as mechanical power. At synchronism the hysteresis loop collapses and reverses, and above synchronism the electric supply current so traverses the normal hysteresis loop in reverse direction, representing generation of electric power. The mechanical power consumed by the hysteresis generator then is proportional to the speed, and of this power a part, proportional to the slip above synchronism, is consumed in the iron, the other part is constant and appears as electric power generated by the apparatus in the inverted hysteresis loop.

This apparatus is of interest especially as illustrating the difference between hysteresis and molecular magnetic friction: the hysteresis is the power represented by the loop between magnetic induction and m.m.f. or the electric power in the circuit, and so may be positive or negative, or change from the one to the other, as in the above instance, while molecular magnetic friction is the power consumed in the magnetic circuit by the reversals of magnetism. Hysteresis therefore is an electrical phenomenon, and is a measure of the molecular magnetic friction only if there is no other source or consumption of power in the magnetic circuit.

CHAPTER XX.

SYNCHRONOUS INDUCTION GENERATOR.

212. If an induction machine is driven above synchronism, the power component of the primary current reverses, that is, energy flows outward, and the machine becomes an induction generator. The component of current required for magnetization remains, however, the same; that is, the induction generator requires the supply of a reactive current for excitation, just as the induction motor, and so must be connected to some apparatus which gives a lagging, or, what is the same, consumes a leading current.

The frequency of the e.m.f. generated by the induction generator, f, is lower than the frequency of rotation or speed, f_0, by the frequency, f_1, of the secondary currents. Or, inversely, the frequency, f_1, of the secondary circuit is the frequency of slip, — that is, the frequency with which the speed of mechanical rotation slips behind the speed of the rotating field, in the induction motor, or the speed of the rotating field slips behind the speed of mechanical rotation, in the induction generator.

As in every transformer, so in the induction machine, the secondary current must have the same ampere-turns as the primary current less the exciting current, that is, the secondary current is approximately proportional to the primary current, or to the load of the induction generator.

In an induction generator with short-circuited secondary, the secondary currents are proportional, approximately, to the e.m.f. generated in the secondary circuit, and this e.m.f. is proportional to the frequency of the secondary circuit, that is, the slip of frequency behind speed. It so follows that the slip of frequency in the induction generator with short-circuited secondary is approximately proportional to the load, that is, such an induction generator does not produce constant synchronous frequency, but a frequency which decreases slightly

with increasing load, just as the speed of the induction motor decreases slightly with increase of load.

Induction generator and induction motor so have also been called asynchronous generator and asynchronous motor, but these names are wrong, since the induction machine is not independent of the frequency, but depends upon it just as much as a synchronous machine, — the difference being, that the synchronous machine runs exactly in synchronism, while the induction machine approaches synchronism. The real asynchronous machine is the commutating machine.

213. Since the slip of frequency with increasing load on the induction generator with short-circuited secondary is due to the increase of secondary frequency required to produce the secondary e.m.f. and therewith the secondary currents, it follows: if these secondary currents are produced by impressing an e.m.f. of constant frequency, f_1, upon the secondary circuit, the primary frequency, f, does not change with the load, but remains constant and equal to $f = f_0 - f_1$. The machine then is a *synchronous-induction machine* — that is, a machine in which the speed and frequency are rigid with regard to each other, just as in the synchronous machine, except that in the synchronous-induction machine, speed and frequency have a constant difference, while in the synchronous machine this difference is zero, that is, the speed equals the frequency.

By thus connecting the secondary of the induction machine with a source of constant low-frequency, f_1, as a synchronous machine, or a commutating machine with low-frequency field excitation, the primary of the induction machine at constant speed, f_0, generates electric power at constant frequency, f_1, independent of the load. If the secondary $f_1 = 0$, that is, a continuous current is supplied to the secondary circuit, the primary frequency is the frequency of rotation and the machine an ordinary synchronous machine. The synchronous machine so appears as a special case of the synchronous induction machine and corresponds to $f_1 = 0$.

In the synchronous induction generator, or induction machine with an e.m.f. of constant low frequency, f_1, impressed upon the secondary circuit, by a synchronous machine, etc., with increasing load, the primary and so the secondary currents change, and

the synchronous machine so receives more power as synchronous motor, if the rotating field produced in the secondary circuit revolves in the same direction as the mechanical rotation — that is, if the machine is driven above synchronism of the e.m.f. impressed upon the secondary circuit — or the synchronous machine generates more power as alternator, if the direction of rotation of the secondary revolving field is in opposition to the speed. In the former case, the primary frequency equals speed minus secondary impressed frequency: $f = f_0 - f_1$; in the latter case, the primary frequency equals the sum of speed and secondary impressed frequency: $f = f_0 + f_1$, and the machine is a frequency converter or general alternating-current transformer, with the frequency, f_1, as primary, and the frequency, f, as secondary, transforming up in frequency to a frequency, f, which is very high compared with the impressed frequency, so that the mechanical power-input into the frequency converter is very large compared with the electrical power-input.

The synchronous induction generator, that is, induction generator in which the secondary frequency or frequency of slip is fixed by an impressed frequency, so can also be considered as a frequency converter or general alternating-current transformer.

214. To transform from a frequency, f_1, to a frequency, f_2, the frequency, f_1, is impressed upon the primary of an induction machine, and the secondary driven at such a speed, or frequency of rotation, f_0, that the difference between primary impressed frequency, f_1, and frequency of rotation, f_2, that is, the frequency of slip, is the desired secondary frequency, f_2.

There are two speeds, f_0, which fulfill this condition: one below synchronism: $f_0 = f_1 - f_2$, and one above synchronism: $f_0 = f_1 + f_2$. That is, the secondary frequency becomes f_2, if the secondary runs slower than the primary revolving field of frequency, f_1, or if the secondary runs faster than the primary field, by the slip, f_2.

In the former case, the speed is below synchronism, that is, the machine generates electric power at the frequency, f_2, in the secondary, and consumes electric power at the frequency, f_1, in the primary. If $f_2 < f_1$, the speed $f_0 = f_1 - f_2$ is between standstill and synchronism, and the machine, in addition to electric power, generates mechanical power, as induction motor,

and as has been seen in the chapter on the General Alternating-Current Transformer, it is, approximately:

Electric power-input $\div$ electric power-output $\div$ mechanical power-output $= f_1 \div f_2 \div f_0$.

If $f_2 > f_1$, that is, the frequency converter increases the frequency, the rotation must be in backward direction, against the rotating field, so as to give a slip, f_2, greater than the impressed frequency, f_1, and the speed is $f_0 = f_2 - f_1$. In this case, the machine consumes mechanical power, since it is driven against the torque given by it as induction motor, and we have:

Electric power-input $\div$ mechanical power-input $\div$ electric power-output $= f_1 \div f_0 \div f_2$.

That is, the three powers, primary electric, secondary electric, and mechanical, are proportional to their respective frequencies.

As stated, the secondary frequency, f_2, is also produced by driving the machine above synchronism, f_1 — that is, with a negative slip, f_2, or at a speed, $f_0 = f_1 + f_2$. In this case, the machine is induction generator, that is, the primary circuit generates electric power at frequency f_1, the secondary circuit generates power at frequency f_2, and the machine consumes mechanical power, and the three powers again are proportional to their respective frequencies:

Primary electric output $\div$ secondary electric output $\div$ mechanical input $= f_1 \div f_2 \div f_0$.

Since in this case of over synchronous rotation, both electric circuits of the machine generate, it cannot be called a frequency converter, but is an electric generator, converting mechanical power into electric power at two different frequencies, f_1 and f_2, and so is called a synchronous induction machine, since the sum of the two frequencies generated by it equals the frequency of rotation or speed — that is, the machine revolves in synchronism with the sum of the two frequencies generated by it.

It is obvious that like all induction machines, this synchronous induction generator requires a reactive lagging current for excitation, which has to be supplied to it by some outside source, as a synchronous machine, etc.

That is, an induction machine driven at speed f_0, when supplied with reactive exciting current of the proper frequency, generates electric power in the stator as well as in the rotor, at the two respective frequencies, f_1 and f_2, which are such that their sum is in synchronism with the speed, that is:

$$f_1 + f_2 = f_0;$$

otherwise the frequencies, f_1 and f_2, are entirely independent. That is, connecting the stator to a circuit of frequency, f_1, the rotor generates frequency $f_2 = f_0 - f_1$, or connecting the rotor to a circuit of frequency f_2, the stator generates a frequency $f_1 = f_0 - f_2$.

215. The power generated in the stator, P_1, and the power generated in the rotor, P_2, are proportional to their respective frequencies:

$$P_1 : P_2 : P_0 = f_1 : f_2 : f_0,$$

where P_0 is the mechanical input (approximately, that is, neglecting losses).

As seen here the difference between the two circuits, stator and rotor, disappears — that is, either can be primary or secondary, that is, the reactive lagging current required for excitation can be supplied to the stator circuit at frequency f_1, or to the rotor circuit at frequency f_2, or a part to the stator and a part to the rotor circuit. Since this exciting current is reactive or wattless, it can be derived from a synchronous motor or converter, as well as from a synchronous generator, or an alternating commutating machine.

As the voltage required by the exciting current is proportional to the frequency, it also follows that the reactive power-input or the volt-amperes excitation, is proportional to the frequency of the exciting circuit. Hence, using the low-frequency circuit for excitation, the exciting volt-amperes are small.

Such a synchronous induction generator therefore is a two frequency generator, producing electric power simultaneously at two frequencies, and in amounts proportional to these frequencies. For instance, driven at 85 cycles, it can connect with the stator to a 25-cycle system, and with the rotor to a 60-cycle system, and feed into both systems power in the proportion of

25 ÷ 60, as is obvious from the equations of the general alternating-current transformer in the preceding chapter.

216. Since the amounts of electric power at the two frequencies are always proportional to each other, such a machine is hardly of much value for feeding into two different systems, but of importance are only the cases where the two frequencies generated by the machine can be reduced to one.

This is the case:

(1) If the two frequencies are the same: $f_1 = f_2 = \frac{f_0}{2}$. In this case, stator and rotor can be connected together, in parallel or in series, and the induction machine then generates electric power at half the frequency of its speed, that is, runs at double synchronism of its generated frequency. Such a "double synchronous alternator" so consists of an induction machine, in which the stator and the rotor are connected with each other in parallel or in series, supplied with the reactive exciting current by a synchronous machine — for instance, by using synchronous converters with overexcited field as load, — and driven at a speed equal to twice the frequency required. This type of machine may be useful for prime movers of very high speeds.

(2) If of the two frequencies, one is chosen so low that the amount of power generated at this frequency is very small, and can be taken up by a synchronous machine or other low-frequency machine, the latter then may also be called an exciter. For instance, connecting the rotor of an induction machine to a synchronous motor of $f_2 = 4$ cycles, and driving it at a speed of $f_0 = 64$ cycles, generates in the stator an e.m.f. at $f_1 = 60$ cycles, and the amount of power generated at 60 cycles is $\frac{60}{4} =$ 15 times the power generated by 4 cycles. The machine then is an induction generator driven at 15 times its synchronous speed. Where the power at frequency f_2 is very small, it would be no serious objection if this power were not generated, but consumed. That is, by impressing $f_2 = 4$ cycles upon the rotor, and driving it at $f_0 = 56$ cycles, in opposite direction to the rotating field produced in it by the impressed frequency of 4 cycles, the stator also generates an e.m.f. at $f_1 = 60$ cycles. In

this case, electric power has to be put into the machine by a generator at $f_2 = 4$ cycles, and mechanical power at a speed of $f_0 = 56$ cycles, and electric power is produced as output at $f_1 =$ 60 cycles. The machine thus operated is an ordinary frequency converter, which transforms from a very low frequency, $f_2 =$ 4 cycles, to frequency $f_1 = 60$ cycles or 15 times the impressed frequency, and the electric power-input so is only one-fifteenth of the electric power-output, the other fourteen-fifteenths are given by the mechanical power-input, and the generator supplying the impressed frequency, $f_2 = 4$ cycles, accordingly is so small that it can be considered as an exciter.

217. (3) If the rotor of frequency, f_2, driven at speed f_0, is connected to the external circuit through a commutator, the effective frequency supplied by the commutator brushes to the external circuit is $f_0 - f_2$; hence equals f_1, or the stator frequency. Stator and rotor so give the same effective frequency, f_1, and irrespective of the frequency, f_2 generated in the rotor, and the frequencies, f_1 and f_2, accordingly become indefinite, that is, f_1 may be any frequency. f_2 then becomes $f_0 - f_1$, but by the commutator is transformed to the same frequency, f_1. If the stator and rotor were used on entirely independent electric circuits, the frequency would remain indeterminate. As soon, however, as stator and rotor are connected together, a relation appears due to the transformer law, that the secondary ampere-turns must equal the primary ampere-turns (when neglecting the exciting ampere-turns). This makes the frequency dependent upon the number of turns of stator and rotor circuit.

Assuming the rotor circuit is connected in multiple with the stator circuit — as it always can be, since by the commutator brushes it has been brought to the same frequency. The rotor e.m.f. then must be equal to the stator e.m.f. The e.m.f., however, is proportional to the frequency times number of turns, and it is therefore:

$$n_2 f_2 = n_1 f_1,$$

where n_1 = number of effective stator turns,

n_2 = number of effective rotor turns, and f_1 and f_2 are the respective frequencies.

Herefrom follows:

$$f_1 \div f_2 = n_2 \div n_1;$$

that is, the frequencies are inversely proportional to the number of effective turns in stator and in rotor.

Or, since $f_0 = f_1 + f_2$ is the frequency of rotation:

$$f_1 \div f_0 = n_2 \div n_1 + n_2,$$

$$f_1 = \frac{n_2}{n_1 + n_2} f_0.$$

That is, the frequency, f_1, generated by the synchronous-induction machine with commutator, is the frequency of rotation, f_0, times the ratio of rotor turns, n_2, to total turns, $n_1 + n_2$.

Thus, it can be made anything by properly choosing the number of turns in the rotor and in the stator, or, what amounts to the same, interposing between rotor and stator a transformer of the proper ratio of transformation.

The powers generated by the stator and by the rotor, however, are proportional to their respective frequencies, and so are inversely proportional to their respective turns.

$$P_1 \div P_2 = f_1 \div f_2 = n_2 \div n_1;$$

if n_1 and n_2, and therewith the two frequencies, are very different, the two powers, P_1 and P_2, are very different, that is, one of the elements generates very much less power than the other, and since both elements, stator and rotor, have the same active surface, and so can generate approximately the same power, the machine is less economical.

That is, the commutator permits the generation of any desired frequency, f_1, but with best economy only if $f_1 = \frac{f_0}{2}$, or half-synchronous frequency, and the greater the deviation from this frequency, the less is the economy. If one of the frequencies is very small, that is, f_1 is either nearly equal to synchronism f_0, or very low, the low-frequency structure generates very little power.

By shifting the commutator brushes, a component of the rotor current can be made to magnetize and the machine becomes a self-exciting, alternating-current generator.

The use of a commutator on alternating-current machines is in general undesirable, as it imposes limitations on the design, for the purpose of eliminating destructive sparking, as discussed in the chapter on Alternating-Current Commutating Machines.

The synchronous-induction machines have not yet reached a sufficient importance to require a detailed investigation, so only two examples may be considered.

218. (1) *Double Synchronous Alternator.*

Assume the stator and rotor of an induction machine to be wound for the same number of effective turns and phases, and connected in multiple or in series with each other, or, if wound for different number of turns, connected through transformers of such ratios as to give the same effective turns when reduced to the same circuit by the transformer ratio of turns.

Let

$Y_1 = g + jb$ = exciting admittance of the stator,

$Z_1 = r_1 - jx_1$ = self-inductive impedance of the stator,

$Z_2 = r_2 - jx_2$ = self-inductive impedance of the rotor,

and

e = e.m.f. generated in the stator by the mutual inductive magnetic field, that is, by the magnetic flux corresponding to the exciting admittance, Y_1;

and

I = total current, or current supplied to the external circuit,

I_1 = stator current,

I_2 = rotor current.

With series connection of stator and rotor .

$$I = I_1 = I_2,$$

with parallel connection of stator and rotor:

$$I = I_1 + I_2.$$

Using the equations of the general induction machine, the slip of the secondary circuit or rotor is

$$s = -1;$$

the exciting admittance of the rotor is

$$Y_2 = g + jsb = g - jb,$$

and the rotor generated e.m.f.,

$$E_2' = se = -e;$$

that is, the rotor must be connected to the stator in the opposite direction to that in which it would be connected at standstill, or in a stationary transformer.

That is, magnetically, the power components of stator and rotor current neutralize each other. Not so, however, the reactive components, since the reactive component of the rotor current,

$$I_2 = i_2' + ji_2'',$$

in its reaction on the stator is reversed, by the reversed direction of relative rotation, or the slip, $s = -1$, and the effect of the rotor current, I_2, on the stator circuit accordingly corresponds to

$$I_2' = i_2' - ji_2'';$$

hence, the total magnetic effect is

$$I_1 - I_2' = (i_1' - i_2') + j\,(i_1'' + i_2'');$$

and since the total effect must be the exciting current

$$I_0 = i_0' + j_0'',$$

it follows that

$$i_1' - i_2' = i_0' \text{ and } i_1'' + i_2'' = i_0''.$$

Hence, the stator power current and rotor power current, i' and i_2', are equal to each other (when neglecting the small hysteresis power current). The synchronous exciter of the machine must supply in addition to the magnetizing current,

the total reactive current of the load. Or in other words, such a machine requires a synchronous exciter of a volt-ampere capacity equal to the volt-ampere excitation plus the reactive volt-amperes of the load, that is, with an inductive load, a large exciter machine. In this respect, the double-synchronous generator is analogous to the induction generator, and is therefore suited mainly to a load with leading current, as over-excited converters and synchronous motors, in which the reactive component of the load is negative and so compensates for the reactive component of excitation, and thereby reduces the size of the exciter.

This means that the double-synchronous alternator has zero armature reaction for non-inductive load, but a demagnetizing armature reaction for inductive, a magnetizing armature reaction for anti-inductive load, and the excitation, by alternating-reactive current, so has to be varied with the character of the load, in general in a far higher degree than with the synchronous alternator.

219. (2) *Synchronous-Induction Generator with Low-Frequency Excitation.*

Here two cases exist:

(*a*) if the magnetic field of excitation revolves in opposite direction to the mechanical rotation.

(*b*) If it revolves in the same direction.

In the first case, (*a*) the exciter is a low-frequency generator and the machine a frequency converter, calculated by the same equations.

Its voltage regulation is essentially that of a synchronous alternator: with increasing load, at constant voltage impressed upon the rotor or exciter circuit, the voltage drops moderately at non-inductive load, greatly at inductive load, and rises at anti-inductive load. To maintain constant terminal voltage, the excitation has to be changed with a change of load and character of load. With a low-frequency synchronous machine as exciter, this is done by varying the field excitation of the exciter.

At constant field excitation of the synchronous exciter, the

regulation is that due to the impedance between the nominal generated e.m.f. of the exciter, and the terminal voltage of the stator — that is, corresponds to:

$$Z = Z_0 + Z_2 + Z_1.$$

Here Z_0 = synchronous impedance of the exciter, reduced to full frequency f_1,

Z_2 = self-inductive impedance of the rotor, reduced to full frequency f_1,

Z_1 = self-inductive impedance of the stator.

If then E_0 = nominal generated e.m.f. of the exciter generator, that is, corresponding to the field excitation, and,

$I_1 = i + ji_1$ = stator current or output current, the stator terminal voltage is

$$E_1 = E_0 + ZI_1, \text{ or, } E_0 = E + (r - jx)(i + ji_1);$$

and, choosing $E_1 = e_1$ as real axis, and expanding:

$$E_0 = (e_1 + ri + xi_1) - j(xi - ri_1),$$

and the absolute value,

$$e_0^2 = (e_1 + ri + xi_1)^2 + (x_1' - ri_1)^2,$$

$$e_1 = \sqrt{e_0^2 - (xi - ri_1)^2} - (ri + xi_1).$$

220. As an example is shown, in Fig. 136, on p. 359 in dotted lines, with the total current, $I = \sqrt{i^2 + i_1^2}$, as abscissas, the voltage regulation of such a machine, or the terminal voltage, e_1, with a 4-cycle synchronous generator as exciter of the 60-cycle synchronous-induction generator, driven as frequency converter at 56 cycles.

(1) for non-inductive load, or $I_1 = i$. (Curve I.)

(2) for inductive load of 80 per cent power-factor, or $I_1 = I\,(0.8 + 0.6\,j)$. (Curve II.)

(3) for anti-inductive load of 80 per cent power-factor, or

$I_1 = I\,(0.8 - 0.6\,j)$. (Curve III.)

For the constants:

$$e_0 = 2000 \text{ volts}, \qquad Z_2 = 1 - 0.5\,j,$$
$$Z_1 = 0.1 - 0.3\,j, \qquad Z_0 = 0.5 - 0.5\,j;$$

hence

$$Z = 1.6 - 1.3\,j.$$

Then

$$e_1 = \sqrt{4 \times 10^6 - (1.3\,i - 1.6\,i_1)^2} - (1.6\,i + 1.3\,i_1);$$

hence, for non-inductive load, $i_1 = 0$;

$$e_1 = \sqrt{4 \times 10^6 - 1.69\,i^2} - 1.6\,i;$$

for inductive load of 80 per cent power-factor $i_1 = 0.6\,I$, $i = 0.8\,I$:

$$e_1 = \sqrt{4 \times 10^6 - .0064\,I^2} - 2.06\,I;$$

and for anti-inductive load of 80 per cent power-factor $i_1 = -0.6\,I$, $i = 0.8\,I$:

$$e_1 = \sqrt{4 \times 10^6 - 4\,I^2} - 0.5\,I.$$

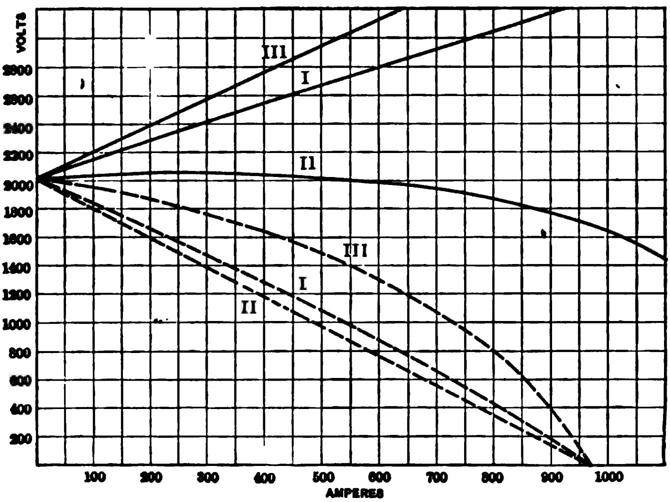


FIG. 136. — Synchronous Induction Generator Regulation Curves.

As seen, due to the internal impedance, and especially the resistance of this machine, the regulation is very poor, and even at the chosen anti-inductive load no rise of voltage occurs.

221. Of more theoretical interest is the case (*b*), where the exciter is a synchronous motor, and the synchronous induction generator produces power in the stator and in the rotor circuit. In this case, the power is produced by the generated e.m.f., E, (e.m.f. of mutual induction, or of the rotating magnetic field) of the induction machine, and energy flows outward in both circuits, in the stator into the receiving circuit, of terminal voltage E_1, in the rotor against the impressed e.m.f. of the synchronous motor exciter E_0. The voltage of one receiving circuit, the stator, therefore, is controlled by a voltage impressed upon another receiving circuit, the rotor, and this results in some interesting effects in voltage regulation.

Assume the voltage, E_0, impressed upon the rotor circuit as the nominal generated e.m.f. of the synchronous-motor exciter, that is, the field corresponding to the exciter field excitation, and assume the field excitation of the exciter, and therewith the voltage, E_0, to be maintained constant.

Reducing all the voltages to the stator circuit by the ratio of their effective turns and the ratio of their respective frequencies, the same e.m.f., E, is generated in the rotor circuit as in the stator circuit of the induction machine.

At no-load, neglecting the exciting current of the induction machine, that is, with no current, we have $E_0 = E = E_1$.

If a load is put on the stator circuit by taking a current I from the same, the terminal voltage, E_1, drops below the generated e.m.f., E, by the drop of voltage in the impedance, Z_1, of the stator circuit. Corresponding to the stator current, I_1, a current, I_2, then exists in the rotor circuit, giving the same ampere-turns as I_1, in opposite direction, and so neutralizing the m.m.f. of the stator (as in any transformer). This current, I_2, exists in the synchronous motor, and the synchronous motor e.m.f., E_0, accordingly drops below the generated e.m.f., E, of the rotor, or, since E_0 is maintained constant, E rises above E_0 with increasing load, by the drop of voltage in the rotor impedance Z_2, and the synchronous impedance, Z_0, of the exciter.

That is, the stator terminal voltage, E_1, drops with increasing load, by the stator impedance drop, and rises with increasing load by the rotor and exciter impedance drop, since the latter causes the generated e.m.f., E, to rise.

If then the impedance drop in the rotor circuit is greater than

that in the stator, with increasing load the terminal voltage E_1, of the machine rises, that is, the machine automatically over-compounds, at constant-exciter field excitation, and if the stator and the rotor impedance drops are equal, the machine compounds for constant voltage.

In such a machine, by properly choosing the stator and rotor impedances, automatic rise, decrease or constancy of the terminal voltage with the load can be produced.

This, however, applies only to non-inductive load. If the current, I, differs in phase from the generated e.m.f., E, the corresponding current, I_2, also differs; but a lagging component of I_1 corresponds to a leading component in I_2, since the stator circuit slips behind, the rotor circuit is driven ahead of the rotating magnetic field, and inversely, a leading component of I_1 gives a lagging component of I_2. The reactance voltage of the lagging current in one circuit is opposite to the reactance voltage of the leading current in the other circuit, therefore does not neutralize it, but adds, that is, instead of compounding, regulates in the wrong direction.

222. The automatic compounding of the synchronous induction generator with low-frequency synchronous-motor excitation so fails if the load is not non-inductive.

Let

$Z_1 = r_1 - jx_1$ = stator self-inductive impedance,

$Z_2 = r_2 - jx_2$ = rotor self-inductive impedance, reduced to the stator circuit by the ratio of the effective turns $t = \frac{n_2}{n_1}$, and the ratio of frequencies $a = \frac{f_2}{f_1}$;

$Z_0 = r_0 - jx_0$ = synchronous impedance of the synchronous-motor exciter;

E_1 = terminal voltage of the stator, chosen as real axis, $= e_1$;

E_0 = nominal generated e.m.f. of the synchronous-motor exciter, reduced to the stator circuit;

E = generated e.m.f. of the synchronous-induction generator stator circuit, or the rotor circuit reduced to the stator circuit.

The actual e.m.f. generated in the rotor circuit then is $E' = taE$, and the actual nominal generated e.m.f. of the synchronous exciter is $E_0' = taE_0$.

Let

$I_1 = i + ji_1$ = current in the stator circuit, or the output current of the machine.

The current in the rotor circuit, in which the direction of rotation is opposite, or ahead of the revolving field, then is, when neglecting the exciter current,

$$I_2 = i - ji_1.$$

(If Y = exciting admittance, the exciting current is $I_0 = EY$, and the total rotor current then $I_0 + I_2$.)

Then in the rotor circuit,

$$E = E_0 + (Z_0 + Z_2)I_2, \tag{1}$$

and in the stator circuit:

$$E = E_1 + Z_1 I_1. \tag{2}$$

Hence,

$$E_1 = E_0 + I_2 (Z_0 + Z_2) - I_1 Z_1, \tag{3}$$

or, substituting for I_1 and I_2,

$$E_1 = E_0 + i (Z_0 + Z_2 - Z_1) - ji_1 (Z_0 + Z_2 + Z_1). \tag{4}$$

Denoting now,

$$\left.\begin{aligned} Z_0 + Z_1 + Z_2 &= Z_3 = r_3 - jx_3, \\ Z_0 + Z_2 - Z_1 &= Z_4 = r_4 - jx_4, \end{aligned}\right\} \tag{5}$$

and substituting,

$$E_1 = E_0 + iZ_4 - ji_1 Z_3, \tag{6}$$

or, since $E_1 = e_1$,

$$\begin{aligned} E_0 &= e_1 - iZ_4 + ji_1 Z_3 \\ &= (e_1 - r_4 i + x_3 i_1) + j (x_4 i + r_3 i_1), \end{aligned} \tag{7}$$

or the absolute value

$$e_0^2 = (e_1 - r_4 i + x_3 i_1)^2 + (x_4 i + r_3 i_1)^2. \tag{8}$$

Hence

$$e_1 = \sqrt{e_0^2 - (x_4 i + r_3 i_1)^2} + r_4 i - x_3 i_1. \tag{9}$$

That is, the terminal voltage, e_1, decreases due to the decrease of the square root, but may increase due to the second term.

At no-load,

$$i = 0, i_1 = 0 \text{ and } e_1 = e_0.$$

At non-inductive load,

$$i_1 = 0 \text{ and } e_1 = \sqrt{e_0^2 - x_4^2 i^2} + r_4 i. \tag{10}$$

e_1 first increases, from its no-load value, e_0, reaches a maximum, and then decreases again.

Since

$$r_4 = r_0 + r_2 - r_1,$$
$$x_4 = x_0 + x_2 - x_1,$$

at

$$r_4 = 0 \text{ and } x_4 = 0, \text{ or,}$$
$$r_1 = r_0 + r_2,$$
$$x_1 = x_0 + x_2,$$

and,

$e_1 = e_0$, that is, in this case the terminal voltage is constant at all non-inductive loads, at constant exciter excitation.

In *general*, or for $I_1 = i + ji_1$,

if i_1 is positive or inductive load, from equation (9) follows that the terminal voltage, e_1, drops with increasing load; while

if i_1 is negative or anti-inductive load, the terminal voltage, e_1, rises with increasing load, ultimately reaches a maximum and then decreases again.

From equation (9) follows, that by changing the impedances, the amount of compounding can be varied. For instance, at non-inductive load, or in equation (10) by increasing the resistance, r_4, the voltage, e_1, increases faster with the load.

That is, the over-compounding of the machine can be increased by inserting resistance in the rotor circuit.

223. As an example is shown, in Fig. 136, in full line, with the total current, $I = \sqrt{i^2 + i_1^2}$, as abscissas, the voltage regulation of such a machine, or the terminal voltage, e_1, with a 4-cycle synchronous motor as exciter of a 60-cycle synchronous-induction generator driven at 64-cycles speed.

(1) For non-inductive load, or $I_1 = i$. (Curve I.)

(2) For inductive load of 80 per cent power-factor; or

$I_1 = I\,(0.8 + 0.6\,j)$. (Curve II.)

(3) For anti-inductive load of 80 per cent power-factor; or

$I_1 = I\,(0.8 - 0.6\,j)$. (Curve III.)

For the constants,

$e_0 = 2000$ volts.
$Z_1 = 0.1 - 0.3\,j$.
$Z_2 = 1 - 0.5\,j$.
$Z_0 = 0.5 - 0.5\,j$.
$a = 0.067$.
$t = 1$, that is, the same number of turns in stator and rotor.

Then,

$$Z_3 = 1.6 - 1.3\,j \text{ and } Z_4 = 1.4 - 0.7\,j.$$

Hence, substituting in equation (9),

$$e_1 = \sqrt{4 \times 10^6 - (0.7 i + 1.6\,i_1)^2} + 1.4\,i - 1.3\,i_1;$$

thus, for non-inductive load, $i_1 = 0$;

$$e_1 = \sqrt{4 \times 10^6 - 0.49\,i^2} + 1.4\,i;$$

for inductive load of 80 per cent power-factor $i_1 = 0.6\,I$; $i = 0.8\,I$;

$$e_1 = \sqrt{4 \times 10^6 - 2.31\,I^2} + .34\,I;$$

and for anti-inductive load of 80 per cent power-factor $i_1 = -\,0.6\,I$; $i = 0.8\,I$;

$$e_1 = \sqrt{4 \times 10^6 - .16\,I^2} + 1.9\,I.$$

Comparing the curves of this example with those of the same machine driven as frequency converter with exciter generator, and shown in dotted lines in the same chart (Fig. 136), it is seen that the voltage is maintained at load far better, and especially at inductive load the machine gives almost perfect regulation of voltage, with the constants assumed here.

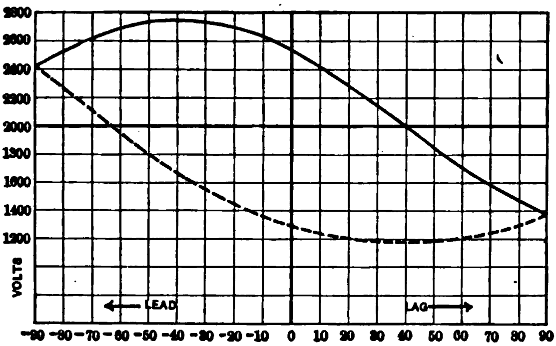


FIG. 137. — Synchronous Induction Generator.

To show the variation of voltage with a change of power-factor, at the same output in current, in Fig. 137, the terminal voltage, e_1, is plotted with the phase angle as abscissas, from wattless anti-inductive load, or 90° lead, to wattless inductive load, or 90° lag, for constant current-output of 400 amperes. As seen, at wattless load both machines give the same voltage but for energy load the type (*b*) gives with the same excitation a higher voltage, or inversely, for the same voltage the type (*a*)

requires a higher excitation. It is, however, seen that with the same current-output, but a change of power-factor, the voltage of type (*a*) is far more constant in the range of inductive load, while that of type (*b*) is more constant on anti-inductive load, and on inductive load very greatly varies with a change of power-factor.

CHAPTER XXI.

ALTERNATING-CURRENT GENERATOR.

224. In the alternating-current generator, e.m.f. is generated in the armature conductors by their relative motion through a constant or approximately constant magnetic field.

When yielding current, two distinctly different m.m.fs. are acting upon the alternator armature — the m.m.f. of the field due to the field-exciting spools, and the m.m.f. of the armature current. The former is constant, or approximately so, while the latter is alternating, and in synchronous motion relatively to the former; hence fixed in space relative to the field m.m.f., or unidirectional, but pulsating in a single-phase alternator. In the polyphase alternator, when evenly loaded or balanced, the resultant m.m.f. of the armature current is more or less constant.

The e.m.f. generated in the armature is due to the magnetic flux passing through and interlinked with the armature

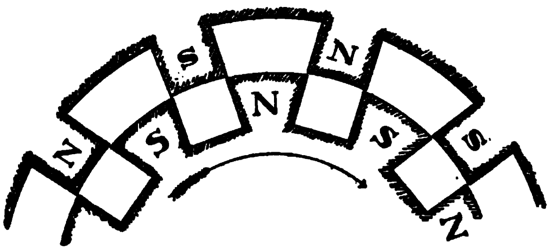


FIG. 138.

conductors. This flux is produced by the resultant of both m.m.fs., that of the field, and that of the armature.

On open-circuit, the m.m.f. of the armature is zero, and the e.m.f. of the armature is due to the m.m.f. of the field-coils only. In this case the e.m.f. is, in general, a maximum at the moment when the armature coil faces the position midway between

adjacent field-coils, as shown in Fig. 138, and thus incloses no magnetism. The e.m.f. wave in this case is, in general, symmetrical.

An exception to this statement may take place only in those types of alternators where the magnetic reluctance of the armature is different in different directions; thereby, during the synchronous rotation of the armature, a pulsation of the magnetic flux passing through it is produced. This pulsation of the magnetic flux generates e.m.f. in the field-spools, and thereby makes the field current pulsating also. Thus, we have, in this case, even on open-circuit, no rotation through a constant magnetic field, but rotation through a pulsating field, which makes the e.m.f. wave unsymmetrical, and shifts the maximum point from its theoretical position midway between the field-poles. In general this secondary reaction can be neglected, and the field m.m.f. be assumed as constant.

The relative position of the armature m.m.f. with respect to the field m.m.f. depends upon the phase relation existing in the electric circuit. Thus, if there is no displacement of phase between current and e.m.f., the current reaches its maximum at the same moment as the e.m.f. or, in the position of the armature shown in Fig. 138, midway between the field-poles. In this case the armature current tends neither to magnetize nor demagnetize the field, but merely distorts it; that is, demagnetizes the trailing-pole corner, *a*, and magnetizes the leading-

FIG. 139.

pole corner, *b*. A change of the total flux, and thereby of the resultant e.m.f., will take place in this case only when the magnetic densities are so near to saturation that the rise of density at the leading-pole corner will be less than the decrease of

density at the trailing-pole corner. Since the internal self-inductive reactance of the alternator itself causes a certain lag of the current behind the generated e.m.f., this condition of no displacement can exist only in a circuit with external negative reactance, as capacity, etc.

If the armature current lags, it reaches the maximum later than the e.m.f.; that is, in a position where the armature-coil partly faces the field-pole which it approaches, as shown in diagram in Fig. 139. Since the armature current is in opposite direction to the current in the following-field pole (in a generator), the armature in this case will tend to demagnetize the field.

If, however, the armature current leads, — that is, reaches its maximum while the armature-coil still partly faces the field-pole which it leaves, as shown in diagram Fig. 140, — it

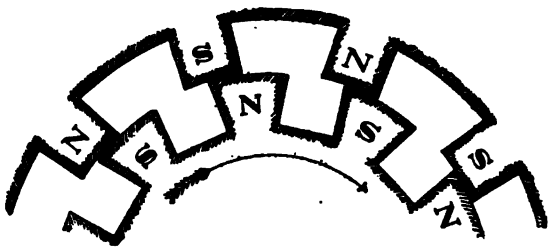


FIG. 140.

tends to magnetize this field-coil, since the armature current is in the same direction as the exciting current of the preceding-field spools.

Thus, with a leading current, the armature reaction of the alternator strengthens the field, and thereby, at constant-field excitation, increases the voltage; with lagging current it weakens the field, and thereby decreases the voltage in a generator. Obviously, the opposite holds for a synchronous motor, in which the armature current is in the opposite direction; and thus a lagging current tends to magnetize, a leading current to demagnetize, the field.

225. The e.m.f. generated in the armature by the resultant magnetic flux, produced by the resultant m.m.f. of the field and

of the armature, is not the terminal voltage of the machine; the terminal voltage is the resultant of this generated e.m.f. and the e.m.f. of self-inductive reactance and the e.m.f. representing the power loss by resistance in the alternator armature. That is, in other words, the armature current not only opposes or assists the field m.m.f. in creating the resultant magnetic flux, but sends a second magnetic flux in a local circuit through the armature, which flux does not pass through the field-spools, and is called the magnetic flux of armature self-inductive reactance.

Thus we have to distinguish in an alternator between armature reaction, or the magnetizing action of the armature upon the field, and armature self-inductive reactance, or the e.m.f. generated in the armature conductors by the current therein. This e.m.f. of self-inductive reactance is (if the magnetic reluctance, and consequently the reactance, of the armature circuit is assumed as constant) in quadrature behind the armature current, and will thus combine with the generated e.m.f. in the proper phase relation. Obviously the e.m.f. of self-inductive reactance and the generated e.m.f. do not in reality combine, but their respective magnetic fluxes combine in the armature-core, where they pass through the same structure. These component e.m.fs. are therefore mathematical fictions, but their resultant is real. This means that, if the armature current lags, the e.m.f. of self-inductive reactance will be more than 90 time degrees behind the generated e.m.f., and therefore in partial opposition, and will tend to reduce the terminal voltage. On the other hand, if the armature current leads, the e.m.f. of self-inductive reactance will be less than 90° behind the generated e.m.f., or in partial conjunction therewith, and increase the terminal voltage. This means that the e.m.f. of self-inductive reactance increases the terminal voltage with a leading, and decreases it with a lagging current, or, in other words, acts in the same manner as the armature reaction. For this reason both actions can be combined in one, and represented by what is called the *synchronous reactance* of the alternator. In the following, we shall represent the total reaction of the armature of the alternator by the one term, *synchronous reactance.* While this is not exact, as stated above, since the reactance should be resolved into the magnetic reaction due to the magnetizing

action of the armature current, and the electric reaction due to the self-induction of the armature current, it is in general sufficiently near for practical purposes, and well suited to explain the phenomena taking place under the various conditions of load. This synchronous reactance, x, is frequently not constant, but is pulsating, owing to the synchronously varying reluctance of the armature magnetic circuit, and the field magnetic circuit; it may, however, be considered in what follows as constant; that is, the e.m.fs. generated thereby may be represented by their equivalent sine waves. A specific discussion of the distortions of the wave shape due to the pulsation of the synchronous reactance is found in Chapter XXVI. The synchronous reactance, x, is not a true reactance in the ordinary sense of the word, but an *equivalent* or *effective* reactance. Sometimes the total effects taking place in the alternator armature are represented by a magnetic reaction, neglecting the self-inductive reactance altogether, or rather replacing it by an increase of the armature reaction or armature m.m.f. to such a value as to include the self-inductive reactance. This assumption is mostly made in the preliminary designs of alternators.

226. Let E_0 = generated e.m.f. of the alternator, or the e.m.f. generated in the armature-coils by their rotation through the constant magnetic-field produced by the current in the field-spools, or the open-circuit voltage, more properly called the "nominal generated e.m.f.," since in reality it does not exist as before stated.

Then

$$E_0 = \sqrt{2}\,\pi n f \Phi\, 10^{-8};$$

where

n = total number of turns in series on the armature,

f = frequency,

Φ = total magnetic flux per field-pole.

Let x_0 = synchronous reactance,

r_0 = internal resistance of the alternator;

then $Z_0 = r_0 - jx_0$ = internal impedance.

If the circuit of the alternator is closed by the external impedance,

$$Z = r - jx,$$

the *current*

$$\dot{I} = \frac{\dot{E}_0}{Z_0 + Z} = \frac{\dot{E}_0}{(r_0 + r) - j(x_0 + x)},$$

or,

$$I = \frac{E_0}{\sqrt{(r_0 + r)^2 + (x_0 + x)^2}};$$

and, the *terminal voltage*,

$$\dot{E} = \dot{I}Z = \dot{E}_0 - \dot{I}Z_0 = \frac{\dot{E}_0(r - jx)}{(r_0 + r) - j(x_0 + x)},$$

or,

$$E = \frac{E_0\sqrt{r^2 + x^2}}{\sqrt{(r_0 + r)^2 + (x_0 + x)^2}}$$

$$= E_0 \frac{1}{\sqrt{1 + 2\frac{r_0 r + x_0 x}{r^2 + x^2} + \frac{r_0^2 + x_0^2}{r^2 + x^2}}};$$

or, expanded in a series,

$$E = E_0\left\{1 - \frac{r_0 r + x_0 x}{r^2 + x^2} + \frac{2(r_0 r + x_0 x) - (r_0 x + x_0 r)}{2(r^2 + x^2)} \pm \dots\right\}.$$

As shown, the terminal voltage varies with the conditions of the external circuit.

227. As an example in Figs. 141–146, at constant generated e.m.f.,

$$E_0 = 2500;$$

and the values of the internal impedance,

$$Z_0 = r_0 - jx_0 = 1 - 10j.$$

With the current, I, as abscissas, the terminal voltages, E, as ordinates in full line, and the kilowatts output, $= I^2r$, in

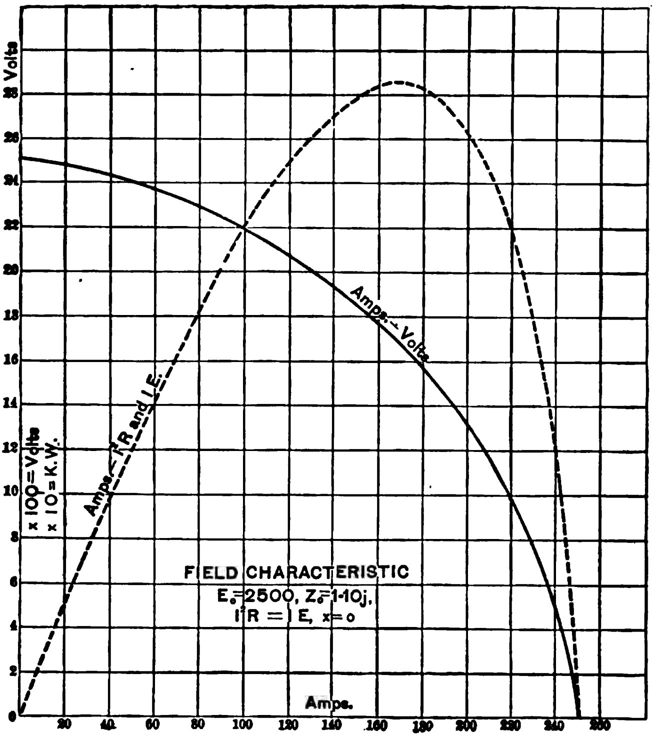


Fig. 141.—Field Characteristic of Alternator on Non-Inductive Load.

dotted lines, the kilovolt-amperes output, $= IE$, in dash-dotted lines, we have, for the following conditions of external circuit:

In Fig. 141, non-inductive external circuit, $x = 0$.

In Fig. 142, inductive external circuit, of the condition, $\frac{r}{x} =$ $+0.75$, with a power-factor, 0.6.

In Fig. 143, inductive external circuit, of the condition, $r = 0$, with a power-factor, 0.

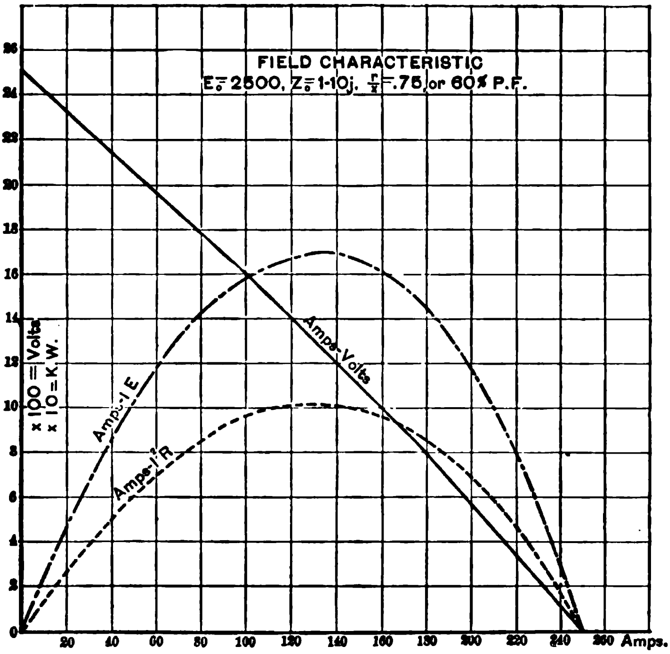


Fig. 142.—Field Characteristic of Alternator at 60 per cent Power-Factor on Inductive Load.

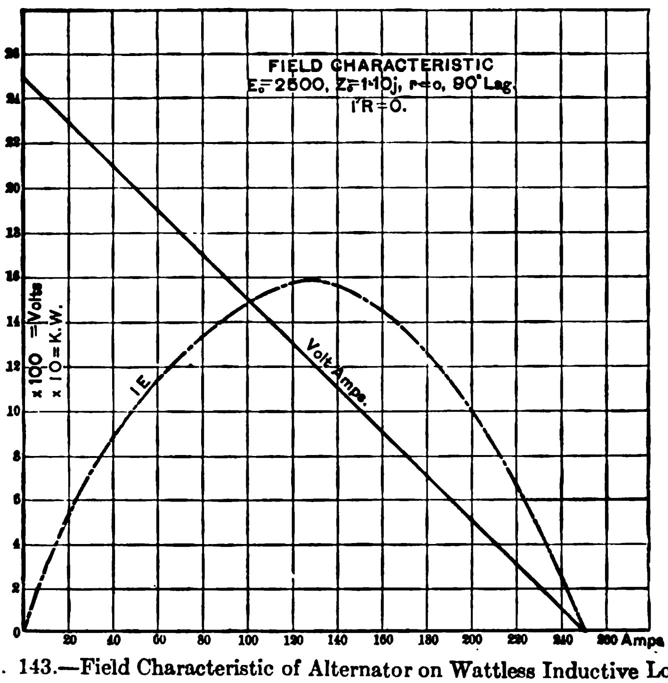


Fig. 143.—Field Characteristic of Alternator on Wattless Inductive Load.

In Fig. 144, external circuit with leading current, of the condition, $\frac{r}{x} = -0.75$, with a power-factor, 0.6.

In Fig. 145, external circuit with leading current, of the condition, $r = 0$, with a power-factor, 0.

In Fig. 146, all the volt-ampere curves are shown together as complete ellipses, giving also the negative or synchronous motor part of the curves.

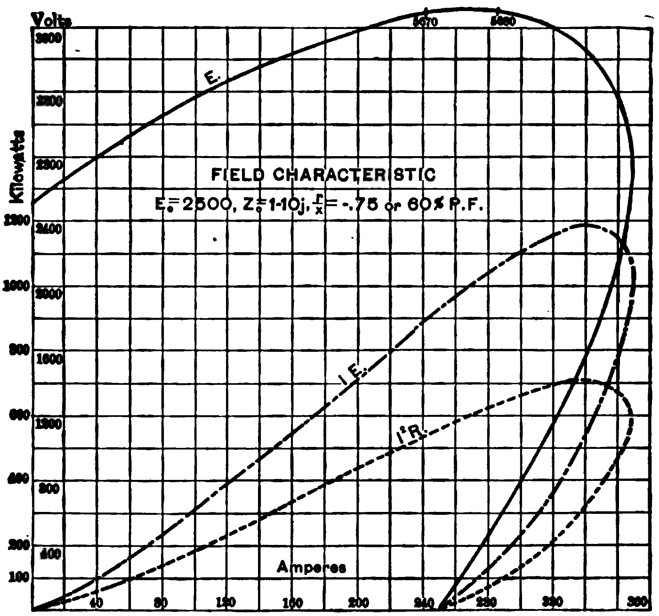


FIG. 144.—Field Characteristic of Alternator at 60 per cent Power-Factor on Condenser Load.

Such a curve is called a *field characteristic.*

As shown, the e.m.f. curve at non-inductive load is nearly horizontal at open-circuit, nearly vertical at short-circuit, and is similar to an arc of an ellipse.

With reactive load the curves are more nearly straight lines.

The voltage drops on inductive load and rises on capacity load.

The output increases from zero at open-circuit to a maximum, and then decreases again to zero at short-circuit.

228. The dependence of the terminal voltage, E, upon the phase relation of the external circuit is shown in Fig. 147, which gives, at impressed e.m.f., $E_0 = 2500$ volts, and the

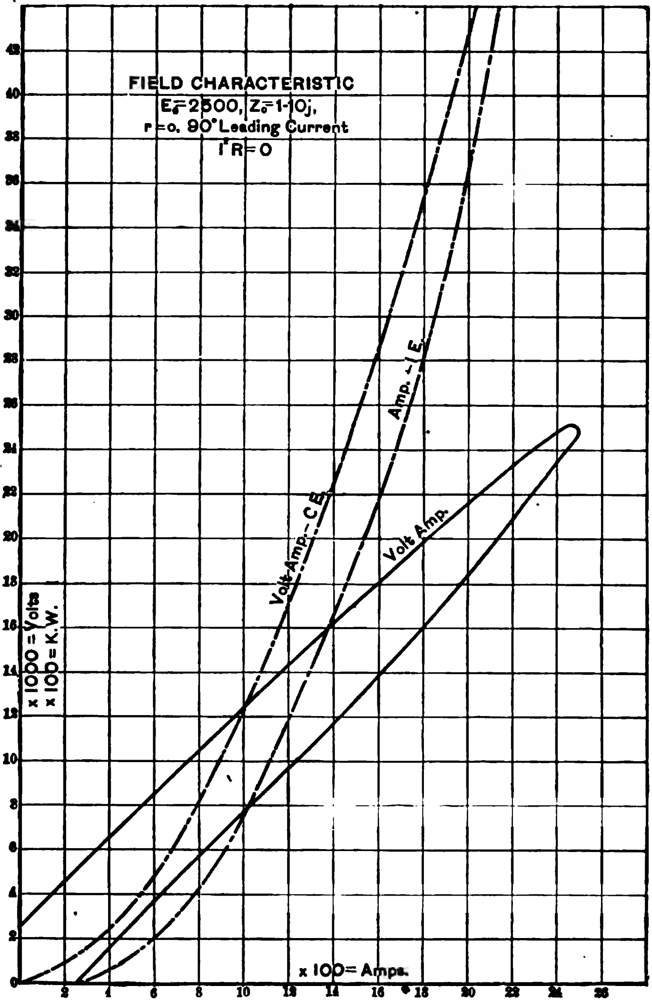


Fig. 145.—Field Characteristic of Alternator on Wattless Condenser Load.

currents, $I = 50, 100, 150, 200, 250$ amperes, the terminal voltages, E, as ordinates, with the inductance factor of the external circuit, $\frac{x}{\sqrt{r^2 + x^2}}$, as abscissas.

229. If the internal impedance is negligible compared with the external impedance, then, approximately,

$$E = \frac{E_0\sqrt{r^2 + x^2}}{\sqrt{(r_0 + r)^2 + (x_0 + x)^2}} = E_0;$$

that is, *an alternator with small internal resistance and synchronous reactance tends to regulate for constant-terminal voltage.*

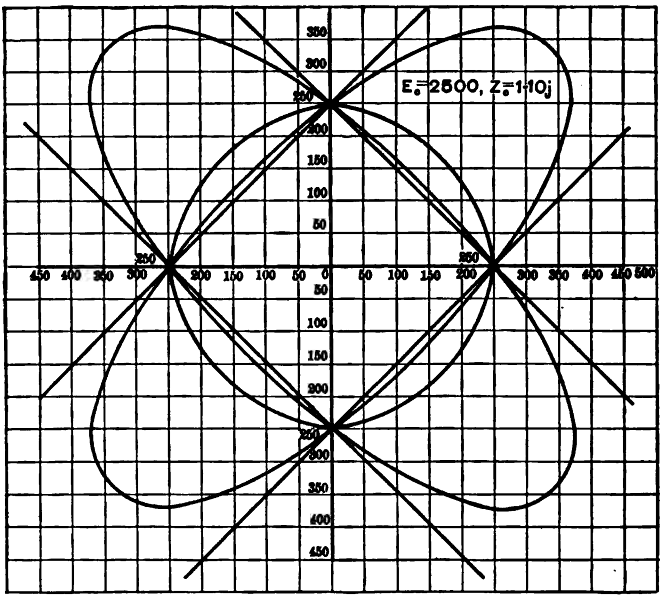


Fig. 146.—Field Characteristic of Alternator.

Every alternator does this near open-circuit, especially on non-inductive load.

Even if the synchronous reactance, x_0, is not quite negligible, this regulation takes place, to a certain extent, on non-inductive circuit, since for $x = 0$,

$$E = \frac{E_0}{\sqrt{1 + 2\frac{r_0}{r} + \frac{x_0^2}{r^2}}};$$

and thus the expression of the terminal voltage, E, contains the synchronous reactance, x_0, only as a term of second order in the denominator.

On inductive circuit, however, x_0 appears in the denominator as a term of first order, and therefore constant-potential regulation does not take place as well.

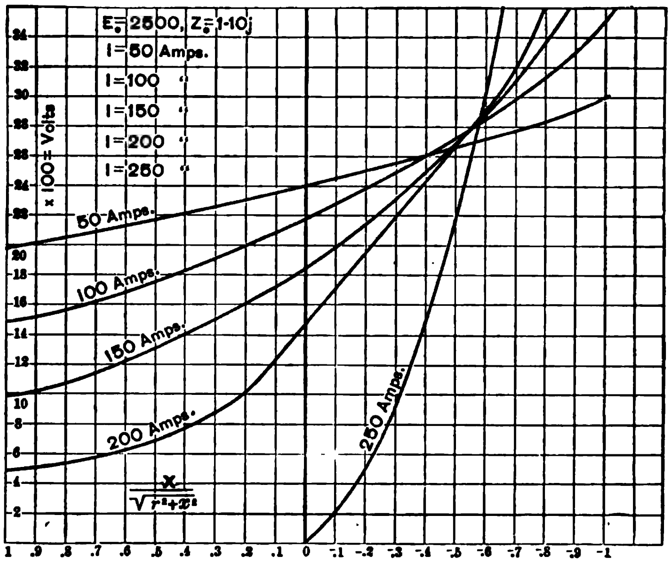


FIG. 147.—Regulation of Alternator on Various Loads.

With a non-inductive external circuit, if the synchronous reactance, x_0, of the alternator is very large compared with the external resistance, r,

$$\text{current} \qquad I = \frac{E_0}{x_0} \frac{1}{\sqrt{1 + \left(\frac{r_0 + r}{x_0}\right)^2}} = \frac{E_0}{x_0},$$

approximately, or constant; or, if the external circuit contains the reactance, x,

$$I = \frac{E_0}{x_0 + x} \frac{1}{\sqrt{1 + \left(\frac{r_0 + r}{x_0 + x}\right)^2}} = \frac{E_0}{x_0 + x},$$

approximately, or constant.

In this case, the terminal voltage of a non-inductive circuit is

$$E = \frac{E_0}{x_0} r,$$

approximately, or proportional to the external resistance,

in an inductive circuit,

$$E = \frac{E_0}{x_0 + x} \sqrt{r^2 + x^2},$$

approximately, or proportional to the external impedance.

230. That is, on a non-inductive external circuit, an alternator with very low synchronous reactance regulates for constant-terminal voltage, as a constant-potential machine; an alternator with a very high synchronous reactance regulates for a terminal voltage proportional to the external resistance, as a constant-current machine.

Thus, every alternator acts as a constant-potential machine near open-circuit, and as a constant-current machine near short-circuit. Between these conditions, there is a range where the alternator regulates approximately as a constant-power machine, that is, current and e.m.f. vary in inverse proportion, as between 130 and 200 amperes in Fig. 141.

The modern alternators are generally more or less machines of the first class; the old alternators, as built by Jablockkoff, Gramme, etc., were machines of the second class, used for arc lighting, where constant-current regulation is an advantage.

Obviously, large external reactances cause the same regulation for constant current independently of the resistance, r, as a large internal reactance, x_0.

On non-inductive circuit, if

$$I = \frac{E_0}{\sqrt{(r + r_0)^2 + x_0^2}},$$

and

$$E = \frac{E_0 r}{\sqrt{(r + r_0)^2 + x_0^2}},$$

the output is

$$P = IE = \frac{E_0^2 r}{(r + r_0)^2 + x_0^2};$$

and

$$\frac{dP}{dr} = \frac{x_0^2 - r^2 + r_0^2}{\{(r + r_0)^2 + x_0^2\}^2} E_0^2;$$

Hence, if $$x_0 = \sqrt{r^2 - r_0^2},$$

or $$r = \sqrt{r_0^2 + x_0^2} = z_0;$$

$$\frac{dP}{dr} = 0.$$

the power is a maximum, and

$$P = \frac{E_0^2}{2\{z_0 + r_0\}},$$

$$E = \frac{E_0}{\sqrt{2\left\{1 + \frac{r_0}{z_0}\right\}}},$$

and $$I = \frac{E_0}{\sqrt{2 z_0 \{z_0 + r_0\}}}.$$

Therefore, with an external resistance equal to the internal impedance, or, $r = z_0 = \sqrt{r_0^2 + x_0^2}$, the output of an alternator is a maximum, and near this point it regulates for constant output; that is, an increase of current causes a proportional decrease of terminal voltage, and inversely.

The field characteristic of the alternator shows this effect plainly.

CHAPTER XXII.

ARMATURE REACTIONS OF ALTERNATORS.

231. The change of the terminal voltage of an alternating current generator, resulting from a change of load at constant field-excitation, is due to the combined effect of armature reaction and armature self-induction. The counter m.m.f. of the armature current, or armature reaction, combines with the impressed m.m.f. or field excitation to the resultant m.m.f., which produces the resultant magnetic field in the field-poles, and generates in the armature an e.m.f. called the "virtual generated e.m.f.," since it has no actual existence, but is merely a mathematical fiction. The counter e.m.f. of self-induction of the armature current, that is, e.m.f. generated by the armature current, by a local magnetic flux, combines with the virtual generated e.m.f. to the actual generated e.m.f. of the armature, which corresponds to the magnetic flux in the armature core. This combined with the e.m.f. consumed by the armature resistance, gives the terminal voltage.

In most cases, the effect of armature reaction and of self-induction are the same in character, and so both effects usually are contracted in one constant; for purposes of design, frequently the self-induction is represented by an increase of the armature reaction, that is, an effective armature reaction used which combines the effect of the true armature reaction, and the armature self-induction. That is, instead of the counter e.m.f. of self-induction, a counter m.m.f. is used, which would produce the magnetic flux which would generate the e.m.f. of self-induction. For theoretical investigations usually the armature reaction is represented by an effective self-induction, that is, instead of the counter m.m.f.. of the armature reaction, the e.m.f. considered, which would be generated by the magnetic flux, which the armature reaction would produce. That is, both effects are combined in an effective reactance, the "synchronous reactance."

While armature reaction and self-inductance are similar in

effect, in some cases they differ in their action; the e.m.f. of self-inductance is instantaneous, that is, appears and disappears with the current to which it is due. The effect of the armature reaction, however, requires time; the change of the magnetic field resulting from the combination of the counter m.m.f. of armature reaction with the impressed m.m.f. of field excitation occurs gradually, since the magnetic field flux interlinks with the field winding, and any sudden change of the field generates an e.m.f. in the field circuit, which temporarily increases or decreases the field current, and so retards the change of the field flux. So for instance, a sudden increase of load results in a simultaneous increase of the counter e.m.f. of self-induction and counter m.m.f. of armature reaction. With the armature reaction demagnetizing the field, the field flux begins to decrease, and thus generates an e.m.f. in the field-exciting circuit, which increases the field current, and retards the decrease of field flux, so that the field flux adjusts itself only gradually to the change of circuit conditions, at a rate of speed depending upon the constants of the field-exciting circuit, etc.

The extreme case hereof takes place when suddenly short-circuiting an alternator; at the first moment, the short-circuit current is limited only by the self-inductance, and the magnetic field still has full strength, the field-exciting current has greatly increased by the e.m.f. generated in the field circuit by the armature reaction. Gradually the field-exciting current and therewith the field magnetism, die down to the values corresponding to the short-circuit condition. Thus the momentary short-circuit current of an alternator is far greater than the permanent short-circuit current; many times in a machine of low self-induction and high armature reaction, as a low-frequency, high-speed alternator of large capacity, relatively little in a machine of low armature-reaction and high self-induction, as a high-frequency unitooth alternator.

232. Graphically, the internal reactions of the alternating-current generator can be represented as follows:

Let the impressed m.m.f., or field excitation, $\mathfrak{F}_0$, be represented by the vector, $\overline{OF_0}$, in Fig. 148, chosen for convenience as negative vertical axis. Let the armature current, I, be represented by vector $\overline{OI}$. This current, I, gives armature

reaction $\mathfrak{F}_1 = nI$, where n = number of effective turns of the armature, and is represented by the vector, $\overline{OF_1}$, with the two quadrature components, $\overline{OF_1'}$, in line with the field m.m.f., $\overline{OF_0}$, – and usually opposite thereto — and $\overline{OF''}$, in quadrature with $\overline{OF_0}$.

$\overline{OF_0}$ combined with $\overline{OF_1}$ gives the resultant m.m.f., $\overline{OF}$, with the quadrature components, $\overline{OF'} = \overline{OF_0} - \overline{OF_1'}$, and $\overline{OF''}$.

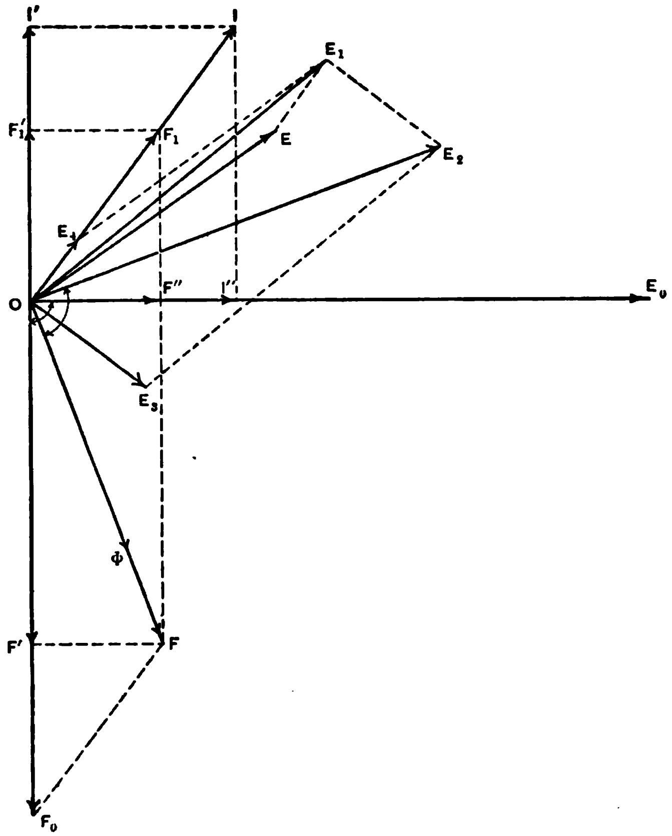


FIG. 148.

The m.m.f., $\overline{OF}$, produces a magnetic flux, $\overline{O\Phi}$, and this generates an e.m.f., $\overline{OE_2}$, in the armature circuit, 90 degrees behind $\overline{OF}$ in phase, the virtual generated e.m.f

The armature self-induction consumes an e.m.f., $\overline{OE_3}$, 90 time-degrees ahead of the current, thus, subtracted vectorially from $\overline{OE_2}$, gives the actual generated e.m.f., $\overline{OE_1}$.

The armature resistance, r, consumes an e.m.f., $\overline{OE_4}$, in phase with the current, which subtracts vectorially from the actual generated e.m.f., and thus gives the terminal voltage, $\overline{OE}$.

233. Analytically, these reactions are best calculated by the symbolic method.

Let the impressed m.m.f., or field-excitation, $\mathfrak{F}_0$, be chosen as the negative imaginary axis, hence represented by

$$\dot{\mathfrak{F}}_0 = -jf_0 \tag{1}$$

Let

$$\dot{I} = i_1 + ji_2 = \text{armature current.} \tag{2}$$

The m.m.f. of the armature then is

$$\dot{\mathfrak{F}}_1 = n\dot{I} = n\,(i_1 + ji_2) \tag{3}$$

where

$$n = \text{number of effective armature turns,}$$

and the resultant m.m.f. then is

$$\dot{\mathfrak{F}} = \dot{\mathfrak{F}}_0 + \dot{\mathfrak{F}}_1 = -j\,(f_0 - ni_2) + ni_1. \tag{4}$$

If, then,

$\mathcal{P}$ = magnetic permeance of the structure, that is, magnetic flux divided by the ampere-turns m.m.f. producing it,

$$\mathcal{P} = \frac{\Phi}{\mathfrak{F}}, \text{ or, } \dot{\Phi} = \mathcal{P}\mathfrak{F} = -j\mathcal{P}\,(f_0 - ni_2) + \mathcal{P}ni_1. \tag{5}$$

The e.m.f. generated by the magnetic flux Φ in the armature is

$$e_0 = 2\,\pi f n\Phi\, 10^{-8}, \tag{6}$$

where f = frequency.

Denoting $2\,\pi f n\, 10^{-8}$ by a we have, (7)

$$e_2 = a,\Phi \tag{8}$$

and since the generated e.m.f. is 90 time-degrees behind the generating flux, in symbolic expression,

$$\dot{E}_2 = ja\Phi; \tag{9}$$

hence, substituting (5) in (9),

$$\dot{E}_2 = a\mathcal{P}\,(f_0 - ni_2) + ja\mathcal{P}ni_1, \tag{10}$$

the *virtual generated e.m.f.*

The e.m.f. consumed by the self-inductive reactance of the armature circuit is,

$$\dot{E}_3 = -jx\dot{I} = -jxi_1 + xi_2; \tag{11}$$

and therefore, the *actual generated e.m.f.*

$$\begin{aligned}\dot{E}_1 &= \dot{E}_2 - \dot{E}_3\\ &= \left\{a\mathcal{P}f_0 - (a\mathcal{P}n + x)i_2\right\} + ji_1\,(a\mathcal{P}n + x).\end{aligned} \tag{12}$$

The e.m.f. consumed by the armature resistance r is

$$\dot{E}_4 = r\dot{I} = ri_1 + jri_2; \tag{13}$$

hence, the *terminal voltage,*

$$\begin{aligned}\dot{E} &= \dot{E}_1 - \dot{E}_4\\ &= \left\{a\mathcal{P}f_0 - (a\mathcal{P}n + x)\,i_2 - ri_1\right\} + j\left\{i_1\,(a\mathcal{P}n + x) - ri_2\right\}.\end{aligned} \tag{14}$$

234. It is

f_0 = field m.m.f.; hence

$\Phi_0 = \mathcal{P}f_0$ = magnetic flux, which would be produced by the field excitation, f_0, if the magnetic permeance at this m.m.f., f_0, were the same, $\mathcal{P}$, as at the m.m.f., $\mathfrak{F}$ — that is, if the magnetic characteristic would not bend between f_0 and $\mathfrak{F}$, due to magnetic saturation, or in other words, when neglecting saturation, and therefore $e_0 = a\mathcal{P}f_0$ (15) = e.m.f. generated in the armature by the field excitation, when neglecting magnetic saturation, or assuming a straight line saturation curve.

e_0 is called the "*nominal generated e.m.f.* of the machine."

ni = armature m.m.f; therefore,

$\mathcal{P}ni$ = magnetic flux produced thereby, and,

$a\mathcal{P}ni$ = e.m.f. generated in the armature by the magnetic flux of armature reaction; hence,

$$a\mathcal{P}n = x_1$$

= effective reactance, representing the armature reaction,

$$\text{and } x_0 = a\mathcal{P}n + x \tag{16}$$

= synchronous reactance, that is, the effective reactance representing the combined effect of armature self-induction and armature reaction.

Substituting (15) and (16) in (14) gives,

$$\dot{E} = (e_0 - x_0 i_2 - r i_1) + j\,(x_0 i_1 - r i_2). \tag{17}$$

It follows herefrom:

In an alternating-current generator, the combined effect of armature reaction and self-induction can be represented by an effective reactance, the *synchronous reactance*, x_0, which consists of the two components:

$$x_0 = x + x_1 \tag{18}$$

where,

x = true self-inductive reactance of the armature circuit.

$$x_1 = a\mathcal{P}n = \text{effective reactance of armature reaction,} \tag{19}$$

and the *nominal generated e.m.f.*,

$$e_0 = a\mathcal{P}f_0; \tag{15}$$

where,

n = number of armature turns, effective,

f_0 = field excitation, in ampere-turns,

$$a = 2\,\pi f n\, 10^{-8}. \tag{7}$$

$\mathcal{P}$ = magnetic permeance of the field structure at a magnetic flux in the field-poles corresponding to the virtual generated e.m.f., E_2.

The introduction of the term "synchronous reactance," x_0, and "nominal generated e.m.f.," e_0, is hereby justified, when

dealing with the permanent condition of the electric circuit. The case of the transient phenomena of momentary short-circuit currents, etc., is discussed in a chapter on "Transient Phenomena and Oscillations," section I.

It must be understood that the "nominal generated e.m.f.", e_0, in an actual machine, in which the magnetic characteristic bends due to the approach to magnetic saturation, is not the voltage generated by the field excitation f_0 at open-circuit, but is the voltage which would be generated, if at excitation, f_0, the magnetic permeance, $\mathcal{P} = \frac{\Phi}{\mathcal{F}}$ were the same as at the actual flux existing in the machine — that is, if the magnetic characteristic would continue in a straight line passing through the origin when prolonged.

The equation (17) may also be written

$$\dot{E} = e_0 - Z_0 \dot{I}; \tag{20}$$

where,

$$Z_0 = r - jx_0 = \text{synchronous impedance of the alternator.}$$

$$\dot{I} = i_1 + ji_2,$$

or, more generally

$$\dot{E} = \dot{E}_0 - Z_0 \dot{I}, \tag{22}$$

and so is the equation of a circuit, supplied by the e.m.f., $\dot{E}_0$, with the current, $\dot{I}$, over the impedance, Z_0, as has been discussed in the chapter on resistance, inductive reactance and condensive reactance.

An alternator so is equivalent to an e.m.f., $\dot{E}_0$, the nominal generated e.m.f., supplying current over an impedance, Z_0, the synchronous impedance.

235. In theoretical investigations of alternators, the synchronous reactance, x_0, is usually assumed as constant, and has been assumed so in the preceding.

In reality, however, this is not exactly, and frequently not even approximately correct, but the synchronous reactance is different in different positions of the armature with regard to the field. Since the relative position of the armature to the field

varies with the armature current, and with the phase angle of the armature current, the regulation curve of the alternator, and other characteristic curves, when calculated under the assumption of constant synchronous reactance, may differ considerably from the observed curves, in machines in which the synchronous reactance varies with the position of the armature.

The two components of the synchronous reactance are the self-inductive reactance, and the effective reactance of armature reaction. The self-inductive reactance represents the e.m.f. generated in the armature by the local field produced in the armature by the armature current. The magnetic reluctance of the self-inductive field of the armature coil, however, is, in general, different when this coil stands in front of a field-pole, and when it stands midway between two field-poles, and the self-inductive reactance so periodically varies, between two extreme values, representing respectively the positions of the armature coils in front of, and midway between the field-poles, that is, pulsates with double frequency, between a value, x', corresponding to the position in front, and a value, x'', corresponding to a position midway between the field-poles. Depending upon the structure of the machine, as the angle of the pole arc, that is, the angle covered by the pole face, either x' or x'' may be the larger one.

The effective reactance of armature reaction, x_1, corresponds to the magnetic flux, which the armature would produce in the field-circuit. With the armature coil facing the field-pole, that is, in a nearly closed magnetic field-circuit, x_1 therefore is usually far greater than with the armature coil facing midway between the field-poles, in a more or less open magnetic circuit. Hence, x_1 also varies between two extreme values, x_1' and x_1'', corresponding respectively to the position in line with, and in quadrature with, the field-poles. In this case, usually x_1' is larger than x_1''.

Since $x_1 = a\mathcal{P}n$, where $\mathcal{P}$ = magnetic permeance, $\mathcal{P}$ varies between $\mathcal{P}'$, corresponding to the position of the armature coil opposite the field-poles, and $\mathcal{P}''$, corresponding to the position of the armature coil midway between the field-poles. Usually $\mathcal{P}'$ is far larger.

This means that the two components of the resultant m.m.f., $\mathfrak{F}$, $\mathfrak{F}_1'$ in line with, and $\mathfrak{F}''$ in quadrature with, the field-poles,

act upon magnetic circuits of very different permeance, $\mathcal{P}'$ and $\mathcal{P}''$, and the components of magnetic flux, due to $\mathcal{F}'$ and $\mathcal{F}''$ respectively, are

$$\Phi' = \mathcal{P}'\mathcal{F}'$$
$$\Phi'' = \mathcal{P}''\mathcal{F}''.$$

The two components of magnetic flux, Φ' and Φ'', therefore, are in general, not proportional to their respective m.m.fs. $\mathcal{F}'$ and $\mathcal{F}''$, and the resultant flux, Φ, accordingly is not in line with the resultant m.m.f., $\mathcal{F}$, but differs therefrom in direction, being usually nearer to the center line of the field-poles. That is, the resultant magnetic flux, Φ, is more nearly in line with the impressed m.m.f. of field excitation, $\mathcal{F}_0$, than the resultant m.m.f., $\mathcal{F}$, is — or in other words — the magnetic flux is shifted by the armature reaction less than the resultant m.m.f. is shifted.

236. To consider, in the investigation of the armature reactions of an alternator, the difference of the magnetic reluctance of the structure in the different directions with regard to the field, that is, the effect of the polar construction of the field, or the use of definite polar projections in the magnetic field, the reactions of the machine must be resolved into two components, one in line and the other in quadrature with the center line of the field-poles, or the direction of the impressed m.m.f. or field-excitation, $\mathcal{F}_0$.

Denoting then the components in line with the field-poles, or parallel with the field-excitation, $\mathcal{F}_0$, by *prime*, as I', $\mathcal{F}'$, etc., and the components facing midway between the field-poles, or in quadrature position with the field-excitation, $\mathcal{F}_0$, by *second*, as I'', $\mathcal{F}''$, the diagram of the alternator reactions is modified from that given in Fig. 148.

Choosing again, in Fig. 149, the impressed m.m.f. or field-excitation, $\mathcal{F}_0$, as negative vertical vector $\overline{OF_0}$, the current, $\overline{OI}$, consists of the component, $\overline{OI'}$, in line with $\mathcal{F}_0$, or vertical, and $\overline{OI''}$ in quadrature with $\mathcal{F}_0$, or horizontal. The armature reaction, $\overline{OF_1}$, gives the components, $\overline{OF_1'}$ and $\overline{OF_1''}$, and the resultant m.m.f. therefore consists of two components, $\overline{OF'} = \overline{OF_0} - \overline{OF_1'}$, and $\overline{OF''} = \overline{OF_1''}$.

Let now,

$\mathcal{P}' =$ permeance of the field magnetic circuit; (23)

$\mathcal{P}'' =$ permeance of the magnetic circuit through the armature in quadrature position to the field-poles; (24)

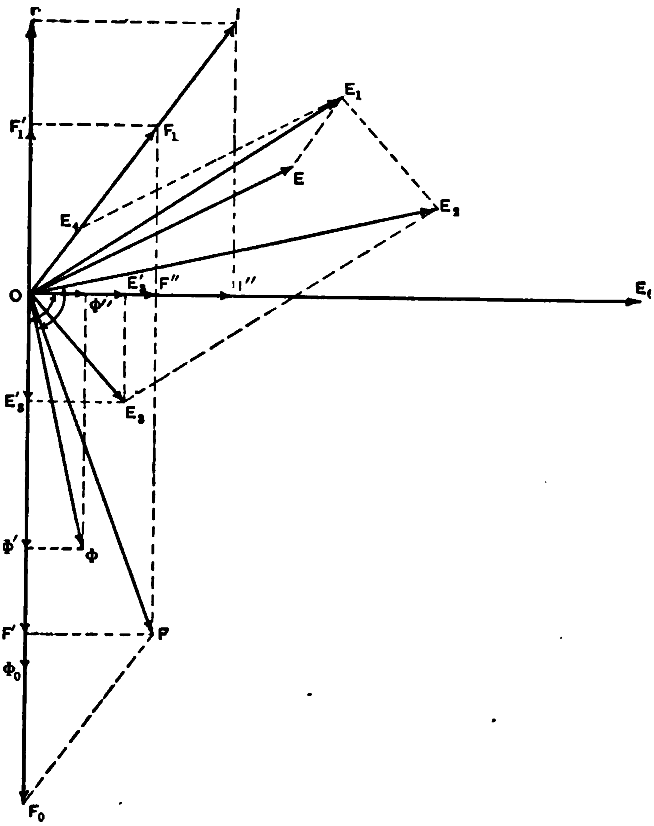


FIG. 149.

the components of the resultant magnetic flux are,

$\Phi' = \mathcal{P}'\mathcal{F}'$, represented by $\overline{O\Phi'}$; and $\Phi'' = \mathcal{P}''\mathcal{F}''$, represented by $\overline{O\Phi''}$,

and the resultant magnetic flux, by combination of $\overline{O\Phi'}$ and $\overline{O\Phi''}$, is $\overline{O\Phi}$, and is not in line with $\overline{OF}$, but differs therefrom, being usually nearer to $\overline{OF_0}$.

The virtual generated e.m.f. is

$$E_2 = a\Phi,$$

and represented by $\overline{OE_2}$, 90 degrees behind $O\Phi$.

Let now

x' = self-inductive reactance of the armature when facing the field-poles, and thus corresponding to the component, I', of the current, (25)

and

x'' = self-inductive reactance of the armature when facing midway between the field-poles, and thus corresponding to the component, I'', of the current. (26)

Then

$E_3' = x'I'$ = e.m.f. consumed by the self-induction of the current component, I',

and

$E_3'' = x'' I''$ = e.m.f. consumed by the self-induction of the current component, I''.

E_3' is represented by vector $\overline{OE_3'}$, 90 degrees ahead of $\overline{OI'}$, and E_3'' is represented by vector $\overline{OE_3''}$, 90 degrees ahead of $\overline{OI''}$. The resultant e.m.f. of self-induction then is given by the combination of $\overline{OE_3'}$ and $\overline{OE_3''}$, as $\overline{OE_3}$. It is not 90 degrees ahead of $\overline{OI}$, but either more or less. In the former case, the self-induction consumes power, in the latter case, it produces power. That is, in such an armature revolving in the structure of non-uniform reluctance, the e.m.f. of self-induction is not wattless, but may represent consumption, or production of power.

Subtracting vectorially $\overline{OE_3}$ from the virtual generated e.m.f. $\overline{OE_2}$, gives the actual generated e.m.f., $\overline{OE_1}$, and subtracting therefrom the e.m.f. consumed by the armature resistance, $\overline{OE_4}$, in phase with the current, $\overline{OI}$, gives the terminal voltage, $\overline{OE}$.

237. Here the diagram has been constructed graphically, by starting with the field-excitation, $\mathfrak{F}_0$, the armature current, I,

and the phase angle between the armature current, I, and the field-excitation, $\mathfrak{F}_0$ — that is, the angle between the position in which the armature current reaches its maximum, and the direction of the field-poles. This angle, however, is unknown. Usually the terminal voltage, $\overline{OE}$, the current, $\overline{OI}$, and the angle, EOI, between current and terminal voltage are given. From these latter quantities, however, the diagram cannot be constructed, since the position of the field-excitation, $\mathfrak{F}_0$, and so the

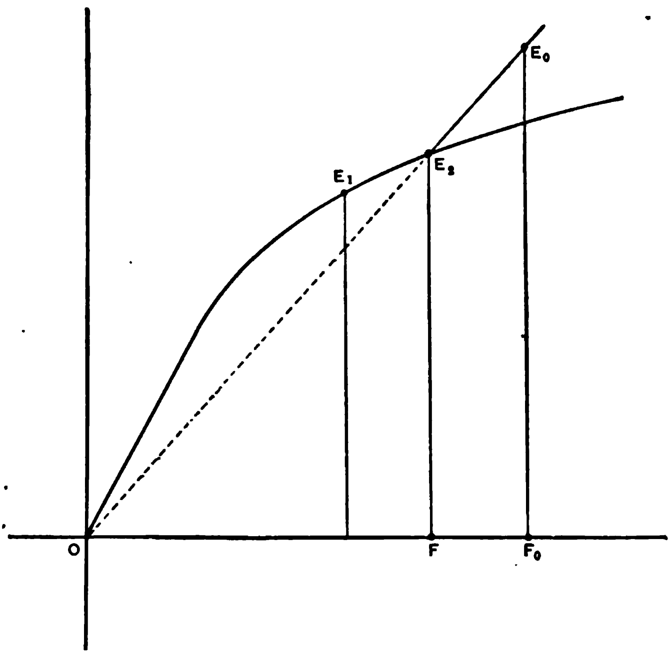


FIG. 150.

directions, in which the electric quantities have to be resolved into components, are still unknown, when starting the construction of the diagram.

That is, as usually, the graphical representation affords an insight into the inner relations of the phenomena, but not a method for their numerical calculations, and for the latter purpose, the symbolic method is required.

Let

E_0 = nominal generated e.m.f., or e.m.f. corresponding to the field-excitation, $\mathfrak{F}_0$, on a straight line continuation of the magnetic characteristic from the actual value of the field onwards, — as shown by Fig. 150.

The impressed m.m.f., or field excitation, is then given by

$$-j\dot{\mathfrak{F}}_0. \tag{27}$$

Let

$$\dot{I} = \dot{I}' + \dot{I}'' = \text{armature current,} \tag{28}$$

where the component, $\dot{I}'$, is in line, the component, $\dot{I}''$, in quadrature with: $-j\dot{\mathfrak{F}}_0$.

If n = number of effective armature turns, the m.m.f. of the armature current, $\dot{I}$, or the armature reaction, then is

$$\dot{\mathfrak{F}}_1 = n\dot{I}, \tag{29}$$

with its components, in phase and in quadrature with the field;

$$\left.\begin{aligned} \dot{\mathfrak{F}}_1' &= n\dot{I}', \\ \dot{\mathfrak{F}}_1'' &= n\dot{I}''; \end{aligned}\right\} \tag{30}$$

and the components of the resultant m.m.f. then are

$$\left.\begin{aligned} \dot{\mathfrak{F}}' &= -j\dot{\mathfrak{F}}_0 + n\dot{I}', \\ \dot{\mathfrak{F}}'' &= n\dot{I}''; \end{aligned}\right\} \tag{31}$$

and the resultant

$$\dot{\mathfrak{F}} = -j\dot{\mathfrak{F}}_0 + n\dot{I}' + n\dot{I}''. \tag{32}$$

The components of the magnetic flux, in line and in quadrature with $-j\dot{\mathfrak{F}}_0$, then are

$$\begin{aligned} \dot{\Phi}' &= \mathcal{P}'\dot{\mathfrak{F}}' \\ &= \mathcal{P}'(-j\dot{\mathfrak{F}}_0 + n\dot{I}'); \end{aligned} \tag{33}$$

$$\begin{aligned} \dot{\Phi}'' &= \mathcal{P}''\dot{\mathfrak{F}}'' \\ &= \mathcal{P}''n\dot{I}''; \end{aligned} \tag{34}$$

hence, the resultant magnetic flux

$$\begin{aligned} \dot{\Phi} &= \dot{\Phi}' + \dot{\Phi}'' \\ &= \mathcal{P}'(-j\dot{\mathfrak{F}}_0 + n\dot{I}') + \mathcal{P}''n\dot{I}'' \end{aligned} \tag{35}$$

The e.m.f. generated by this magnetic flux, Φ, or the virtual generated e.m.f. is

$$E_2 = ja\Phi$$
$$= a\mathcal{P}' (\mathfrak{F}_0 + jnI') + ja\mathcal{P}''nI''. \qquad (36)$$

The e.m.f. consumed by the self-inductive reactance, x', of the current component, I', is,

$$E_3' = -jx'I', \qquad (37)$$

the e.m.f. consumed by the self-inductive reactance, x'', of the current component, I'', is

$$E_3'' = -jx''I'', \qquad (38)$$

and the total e.m.f. consumed by self-induction thus is

$$E_3 = -j(x'I' + x''I''); \qquad (39)$$

hence, the actual generated e.m.f.

$$E_1 = E_2 - E_3$$
$$= a\mathcal{P}'\mathfrak{F}_0 + jI'(a\mathcal{P}'n + x') + jI''(a\mathcal{P}''n + x''). \qquad (40$$

The e.m.f. consumed by the resistance, r, is

$$E_4 = rI$$
$$= rI' + rI''; \qquad (41)$$

hence, the terminal voltage of the machine is

$$E = E_1 - E_4$$
$$= a\mathcal{P}'\mathfrak{F}_0 - I'\{r - j(a\mathcal{P}'n + x')\} - I''\{r - j(a\mathcal{P}''n + x'')\}. \qquad (42)$$

In this equation of the terminal voltage,

$$\left.\begin{aligned} x_0' &= a\mathcal{P}'n + x', \\ x_0'' &= a\mathcal{P}''n + x'', \end{aligned}\right\} \qquad (43)$$

are effective reactances, corresponding to the two quadrature positions; that is

x_0' = synchronous reactance corresponding to the position of the armature circuit parallel to the field circuit; (44*a*)

x_0'' = synchronous reactance corresponding to the position of the armature circuit in quadrature with the field circuit; (44*b*)

$a\mathcal{P}'\underset{\cdot}{\mathcal{F}}_0$ is the e.m.f. which would be generated by the field excitation, $\underset{\cdot}{\mathcal{F}}_0$, with the permeance, $\mathcal{P}'$, in the direction in which the field excitation, $\underset{\cdot}{\mathcal{F}}_0$, acts, that is

$$\underset{\cdot}{E}_0 = a\mathcal{P}'\underset{\cdot}{\mathcal{F}}_0 = \textit{nominal generated e.m.f.} \tag{45}$$

and it is: *terminal voltage,*

$$\underset{\cdot}{E} = \underset{\cdot}{E}_0 - \underset{\cdot}{I}'(r - jx_0') - \underset{\cdot}{I}''(r - jx_0''). \tag{46}$$

That is, even with an heteroform structure, as a machine with definite polar projections, the armature reaction and armature self-induction can be combined by the introduction of the terms "nominal generated e.m.f." and "synchronous reactance," as defined above, except that in this case the synchronous reactance, x_0, has two different values, x_0' and x_0'', corresponding respectively to the two main axes of the magnetic structure, in line and in quadrature with the field-poles.

238. In the equation (46), $\underset{\cdot}{E}$, $\underset{\cdot}{E}_0$, $\underset{\cdot}{I}'$ and $\underset{\cdot}{I}''$ are complex quantities, and

$\underset{\cdot}{I}''$ is in phase with $\underset{\cdot}{E}_0$,

$\underset{\cdot}{I}'$ is in quadrature behind $\underset{\cdot}{E}_0$, and so behind $\underset{\cdot}{I}''$:

hence, $\underset{\cdot}{I}'$ can be represented by

$$\underset{\cdot}{I}' = jt\underset{\cdot}{I}'', \tag{47}$$

where t = ratio of numerical values of $\underset{\cdot}{I}''$ and $\underset{\cdot}{I}'$, that is

$$t = \frac{I'}{I''} = \tan\theta. \tag{48}$$

and

θ = angle of lag of current $\dot{I}$ behind nominal generated e.m.f., $\dot{E}_0$. Then

$$\dot{I} = \dot{I}' + \dot{I}'' = \dot{I}''(1 + jt), \tag{49}$$

or

$$\dot{I}'' = \frac{\dot{I}}{1 + jt} \quad \text{and} \quad \dot{I}' = \frac{jt\dot{I}}{1 + jt}. \tag{50}$$

Substituting these values (50) in equation (46) gives

$$\dot{E} = \dot{E}_0 - \frac{\dot{I}}{1 + jt}\left\{(r - jx_0'') + jt(r - jx_0')\right\}. \tag{51}$$

In this equation, $\dot{E}_0$ leads $\dot{I}$ by angle θ.
Hence, choosing the current $\dot{I}$ as zero vector,

$$\dot{I} = i, \tag{52}$$

the e.m.f., $\dot{E}_0$, which leads i by angle θ, can be represented by

$$\dot{E}_0 = e_0(\cos\theta - j\sin\theta), \tag{53}$$

or, since by equation (48),

$$\sin\theta = \frac{t}{\sqrt{1 + t^2}} \text{ and } \cos\theta = \frac{1}{\sqrt{1 + t^2}}, \tag{54}$$

$$\dot{E}_0 = \frac{e_0}{\sqrt{1 + t^2}}(1 - jt) = \frac{e_0\sqrt{1 + t^2}}{1 + jt}. \tag{55}$$

Substituting (52) and (55) in equation (51), gives

$$\dot{E} = \frac{e_0\sqrt{1 + t^2} - i\left\{(r - jx_0'') + jt(r - jx_0')\right\}}{1 + jt}. \tag{56}$$

Let

$$\dot{E} = e_1 - je_2, \tag{57}$$

where

$$\frac{e_2}{e_1} = \tan\theta', \tag{58}$$

and

$$\theta' = \text{angle of lag of current } i \text{ behind terminal voltage } \dot{E}, \qquad (59)$$

substituting (57) in (56) and transposing,

$$e_0\sqrt{1+t^2}-(e_1-je_2)(1+jt)-i\left\{(r-jx_0'')+jt(r-jx_0')\right\}=0, \qquad (60)$$

or, expanded,

$$\left\{e_0\sqrt{1+t^2}-e_1-te_2-i(r+tx_0')\right\}-j\left\{te_1-e_2+i(tr-x_0'')\right\}=0. \qquad (61)$$

As the left side is a complex quantity, and equals zero, the real part as well as the imaginary part must be zero, and equation (61) so resolves into the two equations

$$e_0\sqrt{1+t^2}-e_1-te_2-i(r+tx_0')=0, \qquad (62)$$

$$te_1-e_2+i(tr-x_0'')=0. \qquad (63)$$

From equation (63) follows

$$t=\frac{e_2+x_0''i}{e_1+ri}. \qquad (64)$$

Substituting (64) in (62), and expanding, gives

$$e_0=\frac{(e_1+ri)^2+(e_2+x_0'i)(e_2+x_0''i)}{\sqrt{(e_1+ri)^2+(e_2+x_0''i)^2}} \qquad (65)$$

That is, if

x_0' = synchronous reactance in the direction of the field excitation,
x_0'' = synchronous reactance in quadrature with the field excitation,
r = armature resistance,
} (66)

i = armature current,
$\dot{E} = e_1 - je_2 = e(\cos\theta' - j\sin\theta')$ = terminal voltage, that is,
$\tan\theta' = \dfrac{e_2}{e_1}$ = angle of lag of current i behind terminal voltage e,
} (67)

the nominal generated e.m.f. of the machine is

$$e_0 = \frac{(e_1 + ri)^2 + (e_2 + x_0'i)(e_2 + x_0''i)}{\sqrt{(e_1 + ri)^2 + (e_2 + x_0''i)^2}}$$

$$= \frac{(e\cos\theta' + ri)^2 + (e\sin\theta' + x_0'i)(e\sin\theta' + x_0''i)}{\sqrt{(e\cos\theta' + ri)^2 + (e\sin\theta', + x_0''i)^2}} \qquad (68)$$

and the field excitation, f_0, required to give terminal voltage, e, at current, i, and angle of lag, θ', is, therefore

$$f_0 = \frac{e_0}{a\mathcal{P}'n} = \frac{e_0 10^8}{2\pi f n^2 \mathcal{P}'}, \qquad (69)$$

and the position angle, θ, between the field-excitation, f_0, and the armature current, i, that is, between the direction of the field-poles and the direction in which the armature current reaches its maximum, is

$$\tan\theta = t = \frac{e_2 + x_0''i}{e_1 + ri} = \frac{e\sin\theta' + x_0''i}{e\cos\theta' + ri}. \qquad (70)$$

239. At non-inductive load,

$$e_1 = e \text{ and } e_2 = 0 \qquad (71)$$

from (68),

$$e_0 = \frac{(e + ri)^2 + x_0'x_0''i^2}{\sqrt{(e + ri)^2 + x_0''^2 i^2}}. \qquad (72)$$

If

$$x_0' = x_0'' = x_0, \qquad (73)$$

that is, the synchronous reactance of the machine is constant in all positions of the armature, or in other words, the magnetic permeance, $\mathcal{P}$, and the self-inductive reactance, x, do not vary with the position of the armature in the field, equation (68) assumes the form

$$e_0 = \sqrt{(e_1 + ri)^2 + (e_2 + x_0 i)^2}, \qquad (74)$$

and this is the absolute value of the equation (22)

$$\dot{E}_0 = \dot{E} + Z_0 \dot{I}, \qquad (22)$$

derived in § 234 for the case of uniform synchronous impedance.

Substituting in (22),

$$\dot{I} = i, \text{ and } \dot{E} = e_1 - je_2,$$

and expanding, gives

$$\begin{aligned}\dot{E}_0 &= (e_1 - je_2) + i(r - jx_0) \\ &= (e_1 + ri) - j(e_2 + x_0 i);\end{aligned}$$

thus, the absolute value,

$$e_0^2 = (e_1 + ri)^2 + (e_2 + x_0 i)^2. \tag{74}$$

240. At short-circuit, and approximately, near short-circuit,

$$e_1 = 0 \text{ and } e_2 = 0, \tag{75}$$

equation (68) assumes the form

$$e_0 = \frac{r^2 + x_0' x_0''}{\sqrt{r^2 + x_0''^2}} \dot{i}_0, \tag{76}$$

or the short-circuit current,

$$\dot{i}_0 = \frac{e_0 \sqrt{r^2 + x_0''^2}}{r^2 + x_0' x_0''}. \tag{77}$$

Since x_0' and x_0'' usually are large, compared with r, r can be neglected in equation (77), and (77) so assumes the form

$$\dot{i}_0 = \frac{e_0}{x_0'}, \tag{78}$$

that is, the short-circuit current of an alternator,

$$\dot{i}_0 = \frac{e_0}{x_0'},$$

depends only upon the synchronous reactance of the armature in the direction of the field excitation, x_0', but not upon the synchronous reactance of the armature in quadrature position to the field excitation, x_0''.

Near open-circuit, that is, in the range where the machine regulates approximately for constant potential, and ix_0 and especially ir are small compared with e, we have, for non-inductive load, from equation (72),

$$e_0 = \frac{(e + ri)^2 + x_0' x_0'' i^2}{\sqrt{(e + ri)^2 + x_0''^2 i^2}}$$

$$= (e + ri) \frac{1 + \frac{x_0' x_0'' i^2}{(e + ri)^2}}{\sqrt{1 + \frac{x_0''^2 i^2}{(e + ri)^2}}},$$

or, approximately,

$$e_0 = (e + ri)\left(1 + \frac{x_0' x_0'' i^2}{e^2}\right)\left(1 + \left(\frac{x_0'' i}{e}\right)^2\right)^{-\frac{1}{2}};$$

hence, expanded by the binomial series,

$$e_0 = (e + ri)\left(1 + \frac{x_0' x_0'' i^2}{e^2}\right)\left(1 - \frac{1}{2}\left(\frac{x_0'' i}{e}\right)^2 - + \ldots\right),$$

and, dropping terms of higher order,

$$e_0 = e + ri + \frac{x_0' x_0'' i^2}{e} - \frac{x_0''^2 i^2}{2e},$$

or,

$$e_0 = e + ri + \frac{x_0'' (2 x_0' - x_0'')}{2} \frac{i^2}{e} \tag{79}$$

For $x_0' = x_0'' = x_0$, this equation (79) assumes the usual form,

$$e_0 = e + ri + \frac{x_0^2}{2} \frac{i^2}{e}. \tag{80}$$

In the range near open-circuit, for non-inductive load, the regulation of the machine accordingly depends not upon the synchronous reactance, x_0', nor upon x_0'', but upon the equivalent synchronous reactance,

$$x_0''' = \sqrt{x_0'' (2 x_0' - x_0'')}. \tag{81}$$

That is, in an alternator with non-uniform synchronous reactance, the short-circuit current and the regulation of the machine near short-circuit, depend upon the value of the synchronous reactance, corresponding to the position of the armature coils parallel, or coaxial with the field-poles, x_0', while the regulation of the machine for non-inductive load, in the range where the machine tends to regulate for approximately constant potential, that is, near open-circuit, depends upon the value of the synchronous reactance, $x_0''' = \sqrt{x_0''(2x_0' - x_0'')}$, where x_0' and x_0'' are the two quadrature components of the synchronous reactance.

That is, the regulation of such an alternator of variable synchronous reactance cannot be calculated from open-circuit test and short-circuit test, or from the magnetic characteristic of the machine at open-circuit, or nominal generated e.m.f., and the synchronous reactance, as given by the machine at short-circuit.

For instance, if

$$x_0' = 10 \text{ and } x_0'' = 4,$$

the effective synchronous reactance near short-circuit,

$$x_0' = 10;$$

and the effective synchronous reactance near open-circuit,

$$x_0''' = 8.$$

The regulation for non-inductive load thus is better than corresponds to the short-circuit impedance.

From equation (68), by solving for the terminal voltage, e, the variation of the terminal voltage, e, with change of load, i, at constant field-excitation, f_0, and so constant nominal generated e.m.f., e_0, that is, the regulation curve of the machine, is calculated.

For instance, for non-inductive load, or $\theta' = 0$, equation (68), solved for e, gives

$$e = \sqrt{\frac{e_0^2}{2} - x_0'x_0''i^2 + e_0\sqrt{\frac{e_0^2}{4} + x_0''^2 i^2 (x_0'' - x_1')^2} - ri}. \qquad (82)$$

241. As illustrations are shown, in Fig. 151, the regulation curves, with the terminal voltage, e, as ordinates, and the current, i, as abscissas, at constant field-excitation, that is, constant nominal generated e.m.f., e_0, for the constants

$$e_0 = 2500 \text{ volts}; \qquad x_0' = 10 \text{ ohms};$$

$$r = 1 \text{ ohm}; \qquad x_0'' = 4 \text{ ohms};$$

for non-inductive load $E = e$, (Curve I.)

and for inductive load of 60 per cent power-factor, $E = e\,(0.6 - 0.8j.)$ (Curve II.)

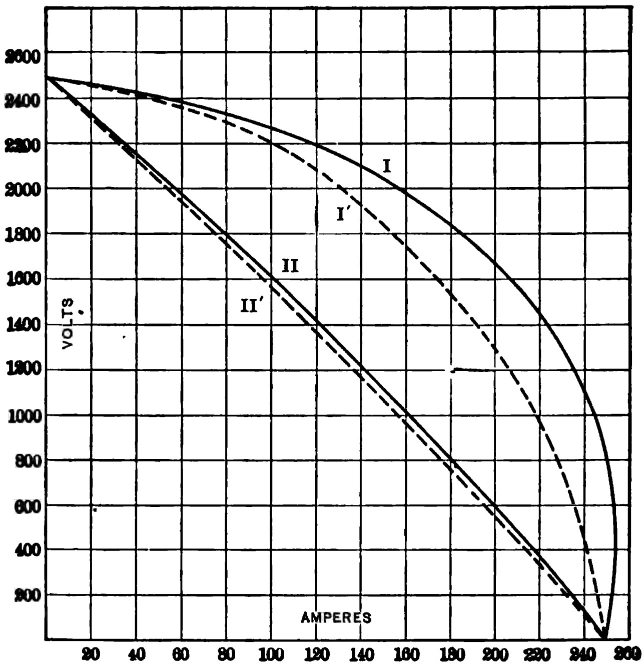


FIG. 151.

For comparison are plotted in the same figure, in dotted lines, the regulation curves for constant synchronous reactance

$$x_0 = 10 \text{ ohms},$$

that is, for the same open-circuit voltage and same short-circuit current.

As seen from Fig. 151, the difference between the two regulation curves, for variable and for constant synchronous reactance, is quite considerable at non-inductive load, but practically negligible at highly inductive load. This is to be expected, since at inductive load the armature current reaches its maximum nearly in opposition to the field-poles, and in this direction the synchronous reactance is the same, x_0', as at short-circuit.

In the preceding discussion of the alternator with variable synchronous reactance, e.m.f. and current are assumed as sine waves. The periodic variation of reactance produces, however, a distortion of wave-shape, consisting mainly of a third harmonic which superimposes on the fundamental, as discussed in Chapter XXVIII. The preceding, therefore, applies to the equivalent sine wave, which represents approximately the actual distorted wave.

As the intensity, and the phase difference between the third harmonic and the fundamental changes with the load, in such an alternator of pulsating synchronous reactance, the wave-shape of the machine changes more or less with the load and the character of the load.

CHAPTER XXIII.

SYNCHRONIZING ALTERNATORS.

242. All alternators, when brought to synchronism with each other, operate in parallel more or less satisfactorily. This is due to the reversibility of the alternating-current machine; that is, its ability to operate as synchronous motor. In consequence thereof, if the driving power of one of several parallel-operating generators is withdrawn, this generator will keep revolving in synchronism as a synchronous motor; and the power with which it tends to remain in synchronism is the maximum power which it can furnish as synchronous motor under the conditions of running.

243. The principal and foremost condition of parallel operation of alternators is equality of frequency; that is, the transmission of power from the prime movers to the alternators must be such as to allow them to run at the same frequency without slippage or excessive strains on the belts or transmission devices.

Rigid mechanical connection of the alternators cannot be considered as synchronizing, since it allows no flexibility or phase adjustment between the alternators, but makes them essentially one machine. If connected in parallel, a difference in the field-excitation, and thus the generated e.m.f. of the machines, must cause large cross-current; since it cannot be taken care of by phase adjustment of the machines.

Thus rigid mechanical connection is not desirable for parallel operation of alternators.

244. The second important condition of parallel operation is uniformity of speed; that is, constancy of frequency. If, for instance, two alternators are driven by independent single-cylinder engines, and the cranks of the engines happen to be crossed, the one engine will pull, while the other is near the dead-point, and conversely. Consequently, alternately the one

alternator will tend to speed up and the other slow down, then the other speed up and the first slow down. This effect, if not taken care of by fly wheel capacity, causes a "hunting" or pumping action; that is, a fluctuation of the voltage with the period of the engine revolution, due to the alternating transfer of the load from one engine to the other, which may even become so excessive as to throw the machines out of step, especially when by an approximate coincidence of the period of engine impulses (or a multiple thereof), with the natural period of oscillation of the revolving structure, the effect is made cumulative. This difficulty as a rule does not exist with turbine or water-wheel driving.

245. In synchronizing alternators, we have to distinguish the phenomena taking place when throwing the machines in parallel or out of parallel, and the phenomena when running in synchronism.

When connecting alternators in parallel, they are first brought approximately to the same frequency and same voltage; and then, at the moment of approximate equality of phase, as shown by a phase-lamp or other device, they are thrown in parallel.

Equality of voltage is much less important with modern alternators than equality of frequency, and equality of phase is usually of importance only in avoiding an instantaneous flickering of the light of lamps connected to the system. When two alternators are thrown together, currents exist between the machines, which accelerate the one and retard the other machine until equal frequency and proper phase relation are reached.

With modern ironclad alternators, this interchange of mechanical power is usually, even without very careful adjustment before synchronizing, sufficiently limited not to endanger the machines mechanically, since the cross-currents, and thus the interchange of power, are limited by self-induction and armature reaction.

In machines of very low armature-reaction, that is, machines of "very good constant-potential regulation," much greater care has to be exerted in the adjustment to equality of frequency, voltage, and phase, or the interchange of current may become so large as to destroy the machine by the mechanical shock; and sometimes the machines are so sensitive in this respect that it is difficult to operate them in parallel. The same applies in getting out of step.

246. When running in synchronism, nearly all types of machines will operate satisfactorily; a medium amount of armature reaction is preferable, however, such as is given by modern alternators — not too high to reduce the synchronizing power too much, nor too low to make the machine unsafe in case of accident, such as falling out of step, etc.

If the armature reaction is very low, an accident, — such as a short-circuit, falling out of step, opening of the field-circuit, etc., — may destroy the machine. If the armature reaction is very high, the driving-power has to be adjusted very carefully to constancy; since the synchronizing power of the alternators is too weak to hold them in step, and carry them over irregularities of the driving-power.

247. Series operation of alternators is possible only by rigid mechanical connection, or by some means whereby the machines, with regard to their synchronizing power, act essentially in parallel; as, for instance, by the arrangement shown in Fig. 152, where the two alternators, A_1, A_2, are connected in series, but

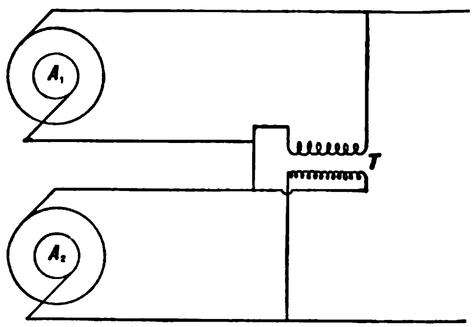


FIG. 152.

interlinked by the two coils of a transformer, T, of which the one is connected across the terminals of one alternator, and the other across the terminals of the other alternator in such a way that, when operating in series, the coils of the transformer will be without current. In this case, by interchange of power through the transformers, the series connection will be maintained stable.

248. In two parallel operating alternators, as shown in Fig. 153, let the voltage at the common busbars be assumed as zero line, or real axis of coordinates of the complex representation; and let

e = difference of potential at the common busbars of the two alternators;

$Z = r - jx$ = impedance of the external circuit;

$Y = g + jb$ = admittance of the external circuit;

hence, the current in the external circuit is

$$\dot{I} = \frac{e}{r - jx} = e\,(g + jb).$$

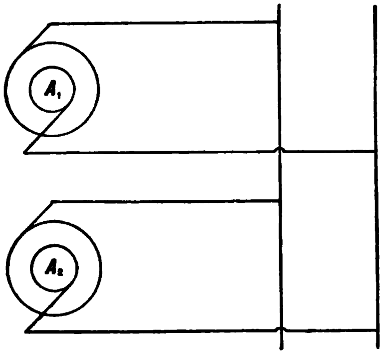


FIG. 153.

Let

$\dot{E}_1 = e_1 - je_1' = a_1\,(\cos\theta_1 - j\sin\theta_1)$ = generated e.m.f.of first machine;

$\dot{E}_2 = e_2 - je_2' = a_2\,(\cos\theta_2 - j\sin\theta_2)$ = generated e.m.f. of second machine;

$\dot{I}_1 = i_1 + ji_1'$ = current of the first machine;

$\dot{I}_2 = i_2 + ji_2'$ = current of the second machine;

$Z_1 = r_1 - jx_1$ = internal impedance, and $Y_1 = g_1 + jb_1$ = internal admittance of the first machine;

$Z_2 = r_2 - jx_2$ = internal impedance, and $Y_2 = g_2 + jb_2$ = internal admittance of the second machine.

Then,

$$e_1^2 + e_1'^2 = a_1^2;$$
$$e_2^2 + e_2'^2 = a_2^2;$$
$$\dot{E}_1 = e + \dot{I}_1 Z_1, \text{ or } e_1 - je_1' = (e + i_1 r_1 + i_1' x_1) - j(i_1 x_1 - i_1' r_1);$$
$$\dot{E}_2 = e + \dot{I}_2 Z_2, \text{ or } e_2 - je_2' = (e + i_2 r_2 + i_2' x_2) - j(i_2 x_2 - i_2' r_2);$$
$$\dot{I} = \dot{I}_1 + \dot{I}_2, \text{ or } eg + jeb = (i_1 + i_2) + j(i_1' + i_2').$$

This gives the equations,

$$\begin{aligned} e_1 &= e + i_1 r_1 + i_1' x_1; \\ e_2 &= e + i_2 r_2 + i_2' x_2; \\ e_1' &= i_1 x_1 - i_1' r_1; \\ e_2' &= i_2 x_2 - i_2' r_2; \\ eg &= i_1 + i_2; \\ eb &= i_1' + i_2'; \\ e_1^2 + e_1'^2 &= a_1^2; \\ e_2^2 + e_2'^2 &= a_2^2; \end{aligned}$$

or eight equations with nine variables, $e_1, e_1', e_2, e_2', i_1, i_1', i_2, i_2', e$.

Combining these equations by twos,

$$e_1 r_1 + e_1' x_1 = er_1 + i_1 z_1^2;$$
$$e_2 r_2 + e_2' x_2 = er_2 + i_2 z_2^2;$$

substituting in

$$i_1 + i_2 = eg,$$

we have

$$e_1 g_1 + e_1' b_1 + e_2 g_2 + e_2' b_2 = e(g_1 + g_2 + g);$$

and analogously,

$$e_1 b_1 - e_1' g_1 + e_2 b_2 - e_2' g_2 = e(b_1 + b_2 + b):$$

dividing,

$$\frac{g + g_1 + g_2}{b + b_1 + b_2} = \frac{e_1 g_1 + e_2 g_2 + e_1' b_1 + e_2' b_2}{e_1 b_1 + e_2 b_2 - e_1' g_1 - e_2' g_2};$$

substituting

$$g = y\cos\alpha \qquad e_1 = a_1\cos\theta_1 \qquad e_2 = a_2\cos\theta_2$$
$$b = y\sin\alpha \qquad e_1' = a_1\sin\theta_1 \qquad e_2' = a_2\sin\theta_2$$

gives

$$\frac{g + g_1 + g_2}{b + b_1 + b_2} = \frac{a_1 y_1 \cos(\alpha_1 - \theta_1) + a_2 y_2 \cos(\alpha_2 - \theta_2)}{a_1 y_1 \sin(\alpha_1 - \theta_1) + a_2 y_2 \sin(\alpha_2 - \theta_2)}$$

as the equation between the phase displacement angles, θ_1 and θ_2, in parallel operation.

The power supplied to the external circuit is

$$p = e^2 g,$$

of which that supplied by the first machine is,

$$p_1 = e\dot{i}_1;$$

by the second machine,

$$p_2 = e\dot{i}_2.$$

The total electrical power of both machines is,

$$P = P_1 + P_2,$$

of which that of the first machine is,

$$P_1 = e_1 i_1 - e_1' i_1';$$

and that of the second machine,

$$P_2 = e_2 i_2 - e_2' i_2'.$$

The difference of output of the two machines is,

$$\Delta P = P_1 - P_2 = e(i_1 - i_2);$$

denoting

$$\frac{\theta_1 + \theta_2}{2} = \epsilon \qquad \frac{\theta_1 - \theta_2}{2} = \delta.$$

$\frac{\Delta P}{\Delta\delta}$ may be called the synchronizing power of the machines, or the power which is transferred from one machine to the other by a change of the relative phase angle.

249. Special Case. — *Two equal alternators of equal excitation.*

$$a_1 = a_2 = a,$$
$$Z_1 = Z_2 = Z_0.$$

Substituting this in the eight initial equations, these assume the form,

$$e_1 = e + i_1 r_0 + i_1' x_0,$$
$$e_2 = e + i_2 r_0 + i_2' x_0,$$
$$e_1' = i_1 x_0 - i_1' r_0',$$
$$e_2' = i_2 x_0 - i_2 r_0,$$
$$eg = i_1 + i_2,$$
$$eb = i_1' + i_2',$$
$$e_1^2 + e_1'^2 = e_2^2 + e_2'^2 = a^2.$$

Combining these equations by twos,

$$e_1 + e_2 = 2\,e + e\,(r_0 g + x_0 b),$$
$$e_1' + e_2' = e\,(x_0 g - r_0 b);$$

substituting
$$e_1 = a \cos \theta_1,$$
$$e_1' = a \sin \theta_1,$$
$$e_2 = a \cos \theta_2,$$
$$e_2' = a \sin \theta_2,$$

we have
$$a\,(\cos \theta_1 + \cos \theta_2) = e\,(2 + r_0 g + x_0 b),$$
$$a\,(\sin \theta_1 + \sin \theta_2) = e\,(x_0 g - r_0 b);$$

expanding and substituting

$$\epsilon = \frac{\theta_1 + \theta_2}{2},$$

$$\delta = \frac{\theta_1 - \theta_2}{2};$$

gives
$$a \cos \epsilon \cos \delta = e\left(1 + \frac{r_0 g + x_0 b}{2}\right),$$

$$a \sin \epsilon \cos \delta = e\,\frac{x_0 g - r_0 b}{2};$$

hence
$$\tan \varepsilon = \frac{x_0 g - r_0 b}{2 + r_0 g + x_0 b} = \text{constant.}$$

That is
$$\theta_1 + \theta_2 = \text{constant};$$

and
$$\cos \delta = \frac{e}{a}\sqrt{\left(1 + \frac{r_0 g + x_0 b}{2}\right)^2 + \left(\frac{x_0 g - r_0 b}{2}\right)^2};$$

or,
$$e = \frac{a \cos \delta}{\sqrt{\left(1 + \frac{r_0 g + x_0 b}{2}\right)^2 + \left(\frac{x_0 g - r_0 b}{2}\right)^2}};$$

at no-phase displacement between the alternators, or,
$$\delta = \frac{\theta_1 - \theta_2}{2} = 0\cdot$$

we have
$$e = \frac{a}{\sqrt{\left(1 + \frac{r_0 g + x_0 b}{2}\right)^2 + \left(\frac{x_0 g - r_0 b}{2}\right)^2}}.$$

From the eight initial equations we get, by combination —
$$e_1 r_0 + e_1' x_0 = e_0 r_0 + i_1 (r_0^2 + x_0^2),$$
$$e_2 r_0 + e_2' x_0 = e_0 r_0 + i_2 (r_0^2 + x_0^2);$$

subtracted and expanded,
$$i_1 - i_2 = \frac{r_0 (e_1 - e_2) + x_0 (e_1' - e_2')}{z_0^2};$$

or, since
$$e_1 - e_2 = a (\cos \theta_1 - \cos \theta_2) = -2 a \sin \varepsilon \sin \delta;$$
$$e_1' - e_2' = a (\sin \theta_1 - \sin \theta_2) = 2 a \cos \varepsilon \sin \delta;$$

we have
$$i_1 - i_2 = \frac{2 a \sin \delta}{z_0^2} \{x_0 \cos \varepsilon - r_0 \sin \varepsilon\}$$
$$= 2 a y_0 \sin \delta \sin (\alpha - \varepsilon),$$

where
$$\tan \alpha = \frac{x_0}{r_0}\cdot$$

The difference of output of the two alternators is

$$\Delta P = P_1 - P_2 = e\,(i_1 - i_2);$$

hence, substituting,

$$\Delta P = \frac{2\,ae\sin\delta}{z_0^2}\left\{x_0\cos\epsilon - r_0\sin\epsilon\right\};$$

substituting,

$$e = \frac{a\cos\delta}{\sqrt{\left(1+\frac{r_0g+x_0b}{2}\right)^2+\left(\frac{x_0g-r_0b}{2}\right)^2}},$$

$$\sin\epsilon = \frac{\frac{x_0g-r_0b}{2}}{\sqrt{\left(1+\frac{r_0g+x_0b}{2}\right)^2+\left(\frac{x_0g-r_0b}{2}\right)^2}},$$

$$\cos\epsilon = \frac{1+\frac{r_0g+x_0b}{2}}{\sqrt{\left(1+\frac{r_0g+x_0b}{2}\right)^2+\left(\frac{x_0g-r_0b}{2}\right)^2}},$$

we have,

$$\Delta P = \frac{2\,a^2\sin\delta\cos\delta\left\{x_0\left(1+\frac{r_0g+x_0b}{2}\right)-r_0\left(\frac{x_0g-r_0b}{2}\right)\right\}}{z_0^2\left\{\left(1+\frac{r_0g+x_0b}{2}\right)^2+\left(\frac{x_0g-r_0b}{2}\right)^2\right\}};$$

expanding,

$$\Delta P = \frac{a^2\sin 2\delta\left\{x_0+\frac{bz_0^2}{2}\right\}}{z_0^2\left\{1+r_0g+x_0b+\frac{1}{4}Z_0^2y^2\right\}};$$

or

$$\Delta P = \frac{a^2\sin 2\delta\left\{b_0+\frac{b}{2}\right\}}{y_0^2+gg_0+bb_0+y^2};$$

$$\frac{\Delta P}{\Delta\delta} = \frac{2\,a^2\cos 2\delta\left\{b_0+\frac{b}{2}\right\}}{y_0^2+gg_0+bb_0+y^2}.$$

Hence, the transfer of power between the alternators, $P\Delta$, is a maximum, if $\delta = 45°$; or $\theta_1 - \theta_2 = 90$ degrees; that is, when the alternators are in quadrature.

The synchronizing power, $\frac{\Delta P}{\Delta \delta}$, is a maximum if $\delta = 0$; that is, the alternators are in phase with each other.

250. As an instance, curves may be plotted for,

$a = 2500$,

$Z_0 = r_0 - jx_0 = 1 - 10j$; or $Y_0 = g_0 + jb_0 = 0.01 + 0.1j$,

with the angle, $\delta = \frac{\theta_1 - \theta_2}{2}$, as abscissas, giving

the value of terminal voltage, e;
the value of current in the external circuit, $i = ey$;
the value of interchange of current between the alternators $\dot{i}_1 - \dot{i}_2$;
the value of interchange of power between the alternators, $\Delta P = P_1 - P_2$;
the value of synchronizing power, $\frac{\Delta P}{\Delta \delta}$.

For the condition of external circuit,

$g = 0$,	$b = 0$,	$y = 0$,
0.05,	0,	0.05,
0.08,	0,	0.08,
0.03,	+ 0.04,	0.05,
0.03,	− 0.04,	0.05.

CHAPTER XXIV.

SYNCHRONOUS MOTOR.

251. In the chapter on synchronizing alternators we have seen that when an alternator running in synchronism is connected with a system of given e.m.f., the work done by the alternator can be either positive or negative. In the latter case the alternator consumes electrical, and consequently produces mechanical, power; that is, runs as a synchronous motor, so that the investigation of the synchronous motor is already contained essentially in the equations of parallel-running alternators.

Since in the foregoing we have made use mostly of the symbolic method, we may in the following, as an example of the graphical method, treat the action of the synchronous motor graphically.

Let an alternator of the e.m.f., E_1, be connected as synchronous motor with a supply circuit of e.m.f., E_0, by a circuit of the impedance, Z.

If E_0 is the e.m.f. impressed upon the motor terminals, Z is the impedance of the motor of generated e.m.f., E_1. If E_0 is the e.m.f. at the generator terminals, Z is the impedance of motor and line, including transformers and other intermediate apparatus. If E_0 is the generated e.m.f. of the generator, Z is the sum of the impedances of motor, line, and generator, and thus we have the problem, generator of generated e.m.f. E_0, and motor of generated e.m.f. E_1; or, more general, two alternators of generated e.m.fs., E_0, E_1, connected together into a circuit of total impedance, Z.

Since in this case several e.m.fs. are acting in circuit with the same current, it is convenient to use the current, I, as zero line $\overline{OI}$ of the polar diagram. (Fig. 154.)

If $I = i$ = current, and Z = impedance, r = effective resistance, x = effective reactance, and $z = \sqrt{r^2 + x^2}$ = absolute value of impedance, then the e.m.f. consumed by the resistance

is $E_{11} = ri$, and is in phase with the current, hence represented by vector $\overline{OE}_{11}$; and the e.m.f. consumed by the reactance is $E_2 = xi$, and 90 degrees ahead of the current, hence the e.m.f. consumed by the impedance is $E = \sqrt{(E_{11})^2 + (E_2)^2}$, or $= i\sqrt{r^2 + x^2} = iz$, and ahead of the current by the angle, δ, where $\tan\delta = \frac{x}{r}$.

We have now acting in circuit the e.m.fs., E, E_1, E_0; or E_1 and E are components of E_0, that is, E_0 is the diagonal of a parallelogram, with E_1 and E as sides.

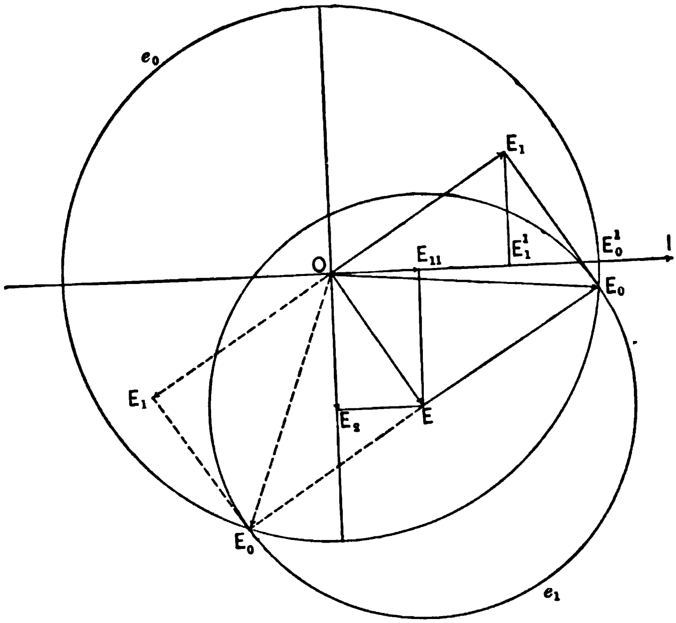


FIG. 154.

Since the e.m.fs. E_1, E_0, E, are represented in the diagram, Fig. 154, by the vectors $\overline{OE}_1$, $\overline{OE}_0$, $\overline{OE}$, to get the parallelogram of E_0, E_1, E, we draw arcs of circles around 0 with E_0, and around E with E_1. Their point of intersection gives the impressed e.m.f., $\overline{OE}_0 = E_0$, and completing the parallelogram OEE_0E_1 we get, $\overline{OE}_1 = E_1$, the generated e.m.f. of the motor.

$\measuredangle IOE_0$ is the difference of phase between current and impressed e.m.f., or generated e.m.f. of the generator.

$\measuredangle IOE_1$ is the difference of phase between current and generated e.m.f. of the motor.

And the power is the current i times the projection of the e.m.f. upon the current, or the zero line, $\overline{OI}$.

Hence, dropping perpendiculars, $\overline{E_0E_0^1}$ and $\overline{E_1E_1^1}$, from E_0 and E_1 upon $\overline{OI}$, it is —

$P_0 = i \times \overline{OE_0^1}$ = power supplied by generator e.m.f. of generator;

$P_1 = i \times \overline{OE_1^1}$ = electric power transformed into mechanical power by the motor;

$P = i \times \overline{OE_{11}}$ = power consumed in the circuit by effective resistance.

Obviously $P_0 = P_1 + P$.

Since the circles drawn with E_0 and E_1 around O and E, respectively, intersect twice, two diagrams exist. In general, in one of these diagrams shown in Fig. 154 in full lines, current and e.m.f. are in the same direction, representing mechanical work done by the machine as motor. In the other, shown in dotted lines, current and e.m.f. are in opposite direction, representing mechanical work consumed by the machine as generator.

Under certain conditions, however, E_0 is in the same, E_1 in opposite direction, with the current; that is, both machines are generators.

252. It is seen that in these diagrams the e.m.fs. are considered from the point of view of the motor; that is, work done as synchronous motor is considered as positive, work done as generator is negative. In the chapter on synchronizing generators we took the opposite view, from the generator side.

In a single unit-power transmission, that is, one generator supplying one synchronous motor over a line, the e.m.f. consumed by the impedance, $E = \overline{OE}$, Figs. 155 to 157, consists three components; the e.m.f., $\overline{OE_2^1} = E_2$, consumed by the

impedance of the motor, the e.m.f., $\overline{E_2{}^1E_3{}^1} = E_3$ consumed by the impedance of the line, and the e.m.f., $\overline{E_3{}^1E} = E_4$, consumed by the impedance of the generator. Hence, dividing the opposite side of the parallelogram, $\overline{E_1E_0}$, in the same way, we have:

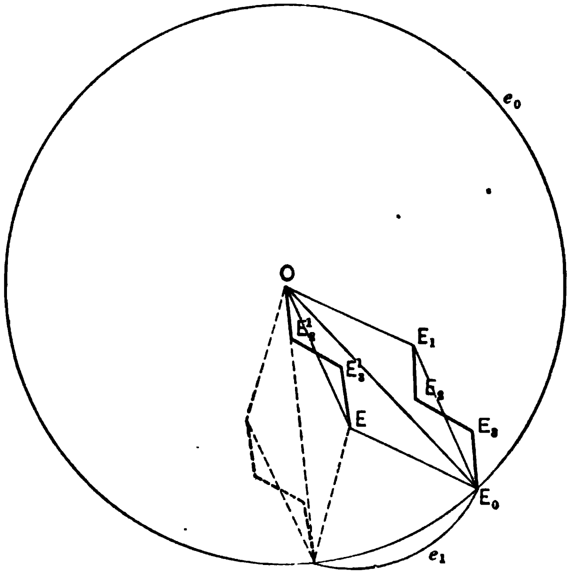


FIG. 155.

$\overline{OE_1} = E_1$ = generated e.m.f. of the motor, $\overline{OE_2} = E_2$ = e.m.f. at motor terminals or at end of line, $\overline{OE_3} = E_3$ = e.m.f. at generator terminals, or at beginning of line, $\overline{OE_0} = E_0$ = generated e.m.f. of generator.

The phase relation of the current with the e.m.fs. E_1, E_0, depends upon the current strength and the e.m.fs., E_1 and E_0.

252. Figs. 155 to 157 show several such diagrams for different values of E_1, but the same value of I and E_0. The motor diagram being given in drawn line, the generator diagram in dotted line.

As seen, for small values of E_1 the potential drops in the alternator and in the line. For the value of $E_1 = E_0$ the potential rises in the generator, drops in the line, and rises again in the motor. For larger values of E_1, the potential rises in the alter-

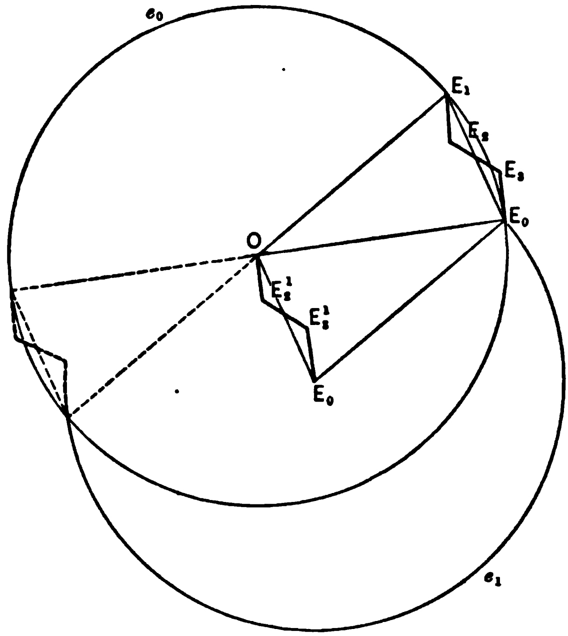


FIG. 156.

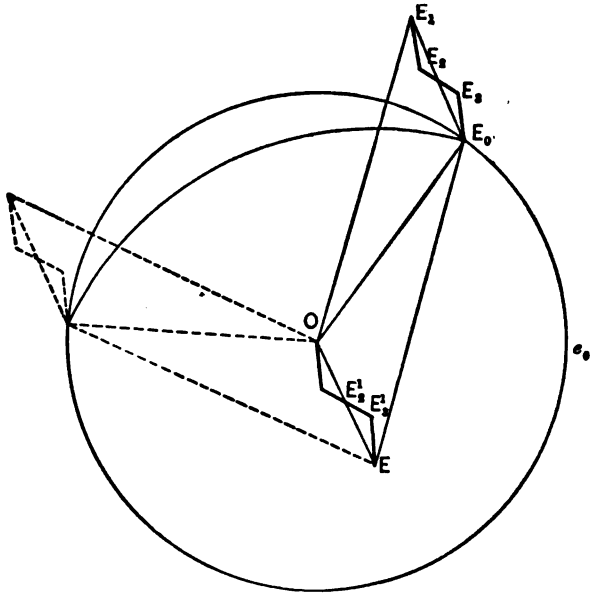


FIG. 157.

nator as well as in the line, so that the highest potential is the generated e.m.f. of the motor, the lowest potential the generated e.m.f. of the generator.

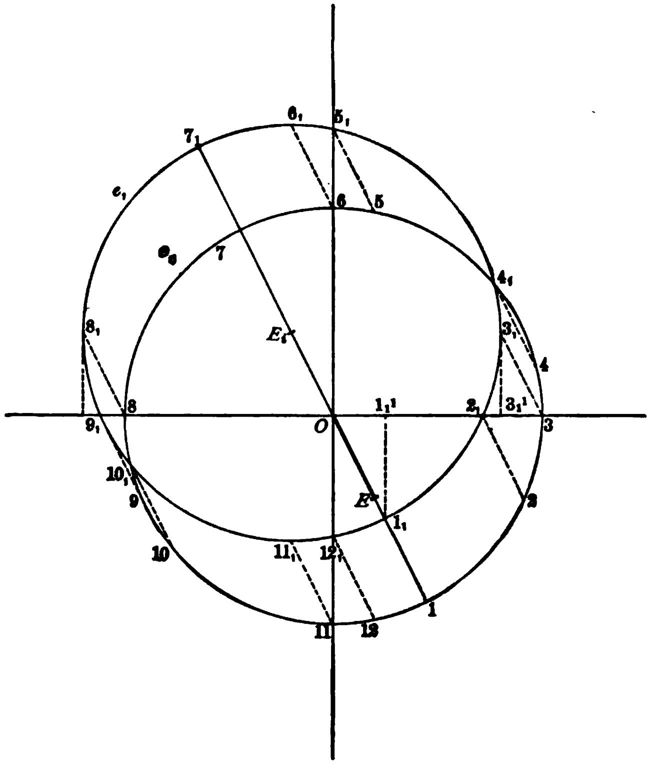


FIG. 158.

It is of interest now to investigate how the values of these quantities change with a change of the constants.

254. A. — *Constant impressed e.m.f.* E_0, *constant-current strength* $I = i$, *variable motor excitation* E_1. (Fig. 158.)

If the current is constant, $= i$; $\overline{OE}$, the e.m.f. consumed by the impedance, and therefore point E, are constant. Since the intensity, but not the phase of E_0 is constant, E_0 lies on a circle e_0 with E_0 as radius. From the parallelogram, OEE_0E_1 follows, since $\overline{E_1E_0}$ parallel and $= \overline{OE}$, that E_1 lies on a circle, e_1, congruent to the circle, e_0, but with E_i, the image of E, as center; $\overline{OE_i} = \overline{OE}$.

We can construct now the variation of the diagram with the variation of E_1; in the parallelogram, OEE_0E_1, O, and E are fixed, and E_0 and E_1 move on the circles, e_0 and e_1, so that $\overline{E_0E_1}$ is parallel to $\overline{OE}$.

The smallest value of F_1 consistent with current strength I is $\overline{01_1} = E_1$, $\overline{01} = E_0$. In this case the power of the motor is $01_1^1 \times I$, hence already considerable. Increasing E_1 to $\overline{02_1}$, $\overline{03_1}$, etc., the impressed e.m.fs. move to $\overline{02}$, $\overline{03}$, etc., the power is $I \times \overline{02_1^1}$, $I \times \overline{03_1^1}$, etc., increases first, reaches the maximum at the point 3_1, 3, the most extreme point at the right, with the impressed e.m.f. in phase with the current, and then decreases again, while the generated e.m.f. of the motor, E_1, increases and becomes $= E_0$ at 4_1, 4. At 5_1, 5, the power becomes zero, and further on negative; that is, the motor has changed to a dynamo, and produces electrical energy, while the impressed e.m.f., e_0, still furnishes electrical energy — that is, both machines as generators feed into the line, until at 6_1, 6, the power of the impressed e.m.f., E_0, becomes zero, and further on energy begins to flow back; that is, the motor is changed to a generator and the generator to a motor, and we are on the generator side of the diagram. At 7_1, 7, the maximum value of E_1, consistent with the current, I, has been reached, and passing still further the e.m.f., E_1 decreases again, while the power still increases up to the maximum at 8_1, 8, and then decreases again, but still E_1 remaining generator, E_0 motor, until at 11_1, 11, the power of E_0 becomes zero; that is, E_0 changes again to a generator, and both machines are generators, up to 12_1, 12, where the power of E_1 is zero, E_1 changes from generator to motor, and we come again to the motor side of the diagram, and the power of the motor increases while E_1 still decreases, until 1_1, 1, is reached.

Hence, there are two regions, for very large E_1 from 5 to 6 and for very small E_1 from 11 to 12, where both machines are generators; otherwise the one is generator, the other motor.

For small values of E_1 the current is lagging, begins, however, at 2 to lead the generated e.m.f. of the motor, E_1, at 3 the generated e.m.f. of the generator, E_0.

It is of interest to note that at the smallest possible value of E_1, 1_1, the power is already considerable. Hence, the motor can run under these conditions only at a certain load. If this

load is thrown off, the motor cannot run with the same current, but the current must increase. We have here the curious condition that loading the motor reduces, unloading increases, the current within the range between 1 and 12.

The condition of maximum output is 3, current in phase with impressed e.m.f. Since at constant current the loss is constant, this is at the same time the condition of maximum efficiency: no displacement of phase of the impressed e.m.f., or self-induction of the circuit compensated by the effect of the lead of the motor current. This condition of maximum efficiency of a circuit we have found already in the Chapter XI.

255. B. *E_0 and E_1 constant, I variable.*

Obviously E_0 lies again on the circle e_0 with E_0 as radius and O as center.

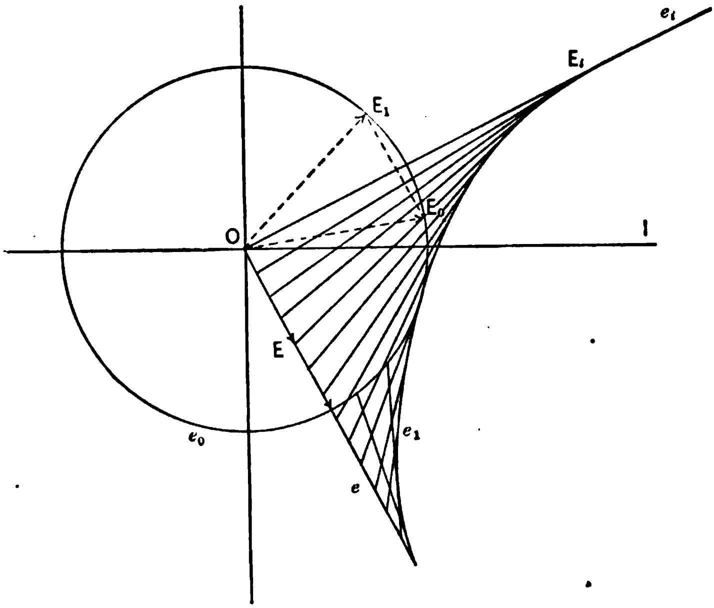


FIG. 159.

E lies on a straight line, *e*, passing through the origin.

Since in the parallelogram, OEE_0E_1, $\overline{EE_0} = E_1$, we derive E_0 by laying a line, $\overline{EE_0} = E_1$, from any point, E, in the circle, e_0, and complete the parallelogram.

All these lines, $\overline{EE_0}$, envelop a certain curve, e_1, which can be considered as the characteristic curve of this problem, just as circle e_1 in the former problem.

These curves are drawn in Figs. 159, 160, 161, for the three cases: 1st, $E_1 = E_0$; 2d, $E_1 < E_0$; 3d, $E_1 > E_0$.

In the first case, $E_1 = E_0$ (Fig. 159), we see that at very small current, that is very small $\overline{OE}$, the current, I, leads the impressed e.m.f., E_0, by an angle, $E_0OI = \theta_0$. This lead decreases with increasing current, becomes zero, and afterwards for larger current, the current lags. Taking now any pair of correspond-

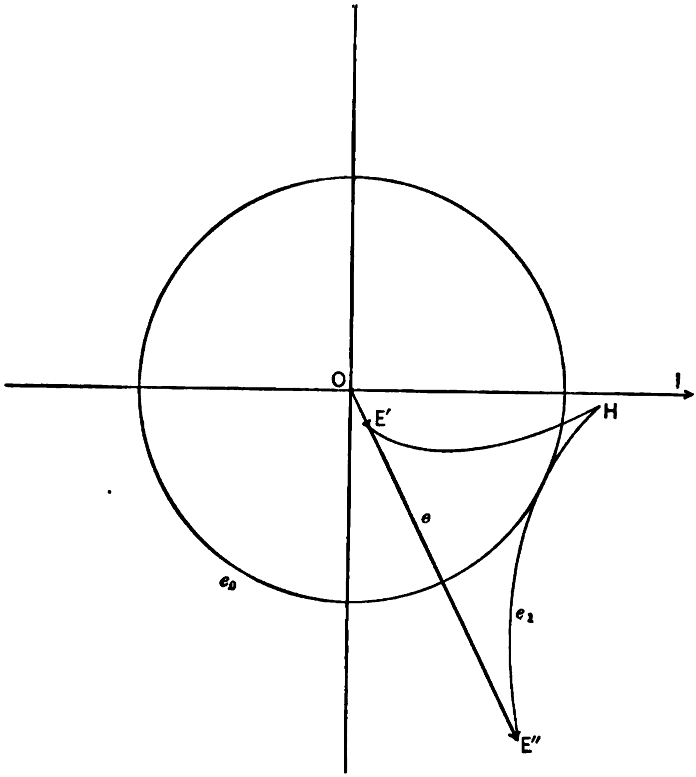


FIG. 160.

ing points, E, E_0, and producing $\overline{EE_0}$ until it intersects e_i, in E_i, we have $\measuredangle\, E_i OE = 90°$, $\overline{E_1} = \overline{E_0}$, thus: $\overline{OE_1} = \overline{EE_0} = \overline{OE_0} = \overline{E_0E_i}$; that is, $\overline{EE_i} = 2\, E_0$. That means the characteristic curve, e_1, is the envelope of lines $\overline{EE_i}$, of constant lengths $2\, E_0$,

sliding between the legs of the right angle, E_tOE; hence, it is the sextic hypocycloid osculating circle, e_0, which has the general equation, with e, e_t as axes of coordinates,

$$\sqrt[3]{x^2} + \sqrt[3]{y^2} = \sqrt[3]{4 E_0^2}.$$

In the next case, $E_1 < E_0$ (Fig. 160), we see first, that the current can never become zero like in the first case, $E_1 = E_0$, but has a minimum value corresponding to the minimum value of $\overline{OE_1}$: $I_1' = \frac{E_0 - E_1}{z}$, and a maximum value: $I_1'' = \frac{E_0 + E_1}{z}$. Furthermore, the current may never lead the impressed e.m.f., E_0, but always lags. The minimum lag is at the point H. The locus, e_1, as envelope of the lines, $\overline{EE_0}$, is a finite sextic curve, shown in Fig. 160.

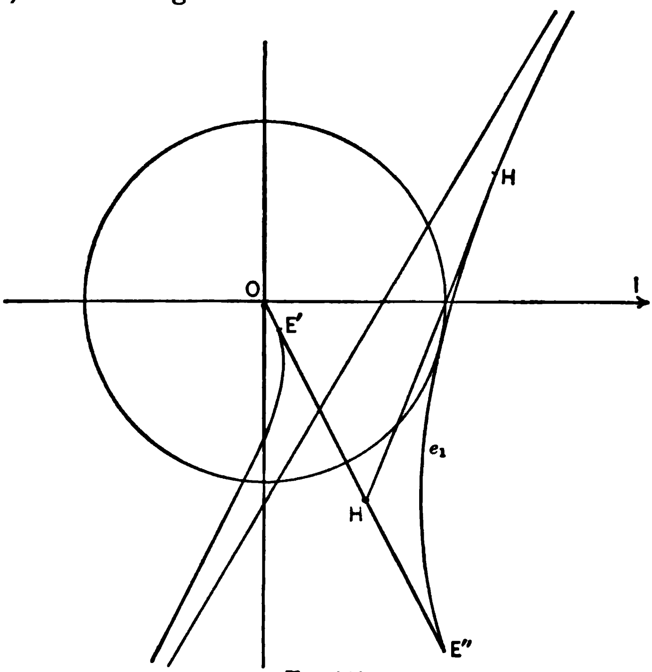


FIG. 161.

If $E_1 < E_0$, at small $E_0 - E_1$, H can be above the zero line, and a range of leading current exist between two ranges of lagging currents.

In the case, $E_1 > E_0$ (Fig. 161), the current cannot equal

zero either, but begins at a finite value, I_1', corresponding to the minimum value of $\overline{OE_0}$, $I_1' = \frac{E_1 - E_0}{z}$. At this value, however, the alternator, E_1, is still generator and changes to a motor, its power passing through zero, at the point corresponding to the vertical tangent, upon e_1, with a very large lead of the impressed e.m.f. against the current. At H the lead changes to lag.

The minimum and maximum values of current in the three conditions are given by:

	Minimum	*Maximum*
1st.	$I = 0,$	$I = \frac{2 E_0}{z}.$
2d.	$I = \frac{E_0 - E_1}{z},$	$I = \frac{E_0 + E_1}{z}.$
3d.	$I = \frac{E_1 - E_0}{z},$	$I = \frac{E_0 + E_1}{z}.$

Since the current in the line at $E_1 = O$, that is, when the motor stands still, is $I_0 = \frac{E_0}{z}$, we see that in such a synchronous motor-plant, when running at synchronism, the current can rise far beyond the value it has at standstill of the motor, to twice this value at 1, somewhat less at 2, but more at 3.

256. C. E_0 = *constant*, E_1 *varied so that the efficiency is a maximum for all currents.* (Fig. 162).

Since we have seen that the output at a given current strength, that is, a given loss, is a maximum, and therefore the efficiency a maximum, when the current is in phase with the generated e.m.f., E_0, of the generator, we have as the locus of E_0 the point, E_0, (Fig. 162), and when E with increasing current varies on e, E_1 must vary on the straight line, e_1, parallel to e.

Hence, at no-load or zero current, $E_1 = E_0$, decreases with increasing load, reaches a minimum at $\overline{OE_1^1}$ perpendicular to e_1, and then increases again, reaches once more $E_1 = E_0$ at E_1^2, and then increases beyond E_0. The current is always ahead of the generated e.m.f., E_1, of the motor, and by its lead compen-

sates for the self-induction of the system, making the total circuit non-inductive.

The power is a maximum at E_1^3, where $\overline{OE_1^4} = \overline{E_1^4 E_0} = 0.5 \times \overline{OE_0}$, and is then $= I \times \frac{\overline{E_0}}{2}$. Hence, since $OE_1^4 = Ir = \frac{E_0}{2}$, $I = \frac{E_0}{2r}$ and $P = \frac{E_0^2}{4r}$, hence = the maximum power which,

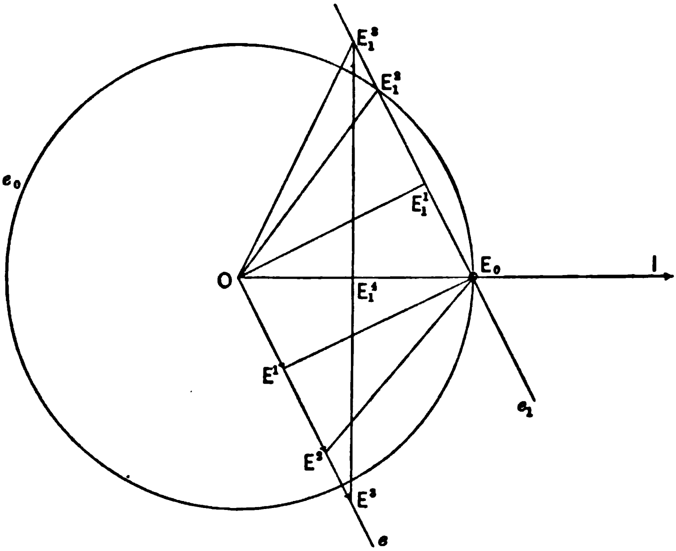


FIG. 162.

over a non-inductive line of resistance r can be transmitted, at 50 per cent efficiency, into a non-inductive circuit.

In this case,

$$E_1^3 = \frac{z}{r} \times \frac{E_0}{2} = \frac{E_0}{2}\sqrt{1 + \left(\frac{x}{r}\right)^2}.$$

In general, it is, taken from the diagram, at the condition of maximum efficiency,

$$E_1 = \sqrt{(E_0 - Ir)^2 + I^2 x^2}.$$

Comparing these results with those in Chapter XI on Inductive and Condensive Reactance, we see that the condition of

maximum efficiency of the synchronous motor system is the same as in a system containing only inductive and condensive reactance, the lead of the current against the generated e.m.f. E_1, here acting in the same way as the condenser capacity in Chapter XI.

257. D. E_0 = *constant*; P_1 = *constant.*

If the power of a synchronous motor remains constant, we have (Fig. 163) $I \times \overline{OE_1^1}$ = constant, or, since $\overline{OE^1} = Ir$, $I = \frac{\overline{OE^1}}{r}$ and $\overline{OE^1} \times \overline{OE_1^1} = \overline{OE^1} \times \overline{E^1E_0^1}$ = constant.

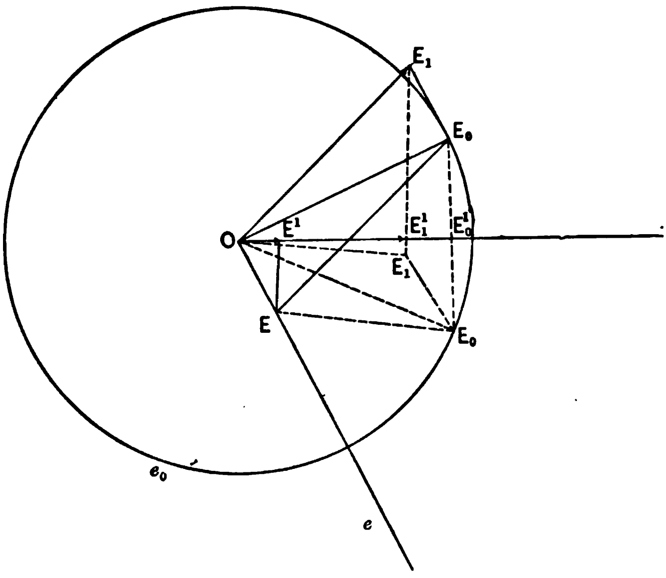


FIG. 163.

Hence we get the diagram for any value of the current I, at constant power P_1, by making $\overline{OE}^1 = Ir$, $\overline{E^1E_0^1} = \frac{P_1}{I}$ erecting in E_0^1 a perpendicular, which gives two points of intersection with circle e_0, E_0, one leading, the other lagging. Hence, at a given impressed e.m.f., E_0, the same power, P_1, can be trans-

mitted by the same current, I, with two different generated e.m.fs., E_1, of the motor; one, $\overline{OE_1} = \overline{EE_0}$ small, corresponding to a lagging current; and the other, $\overline{OE_1} = \overline{EE_0}$ large, corresponding to a leading current. The former is shown in dotted lines, the latter in full lines, in the diagram, Fig. 163.

Hence a synchronous motor can work with a given output, at the same current with two different counter e.m.fs. E_1. In one of the cases the current is leading, in the other lagging.

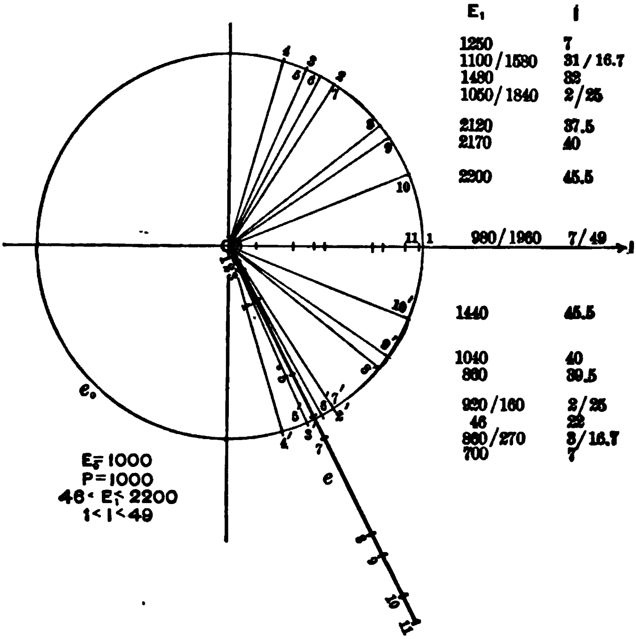


Fig. 164.

In Figs. 164 to 167 are shown diagrams, giving the points

E_0 = impressed e.m.f., assumed as constant = 1000 volts,

E = e.m.f. consumed by impedance,

E^1 = e.m.f. consumed by resistance (not numbered.

The counter e.m.f. of the motor, E_1, is $\overline{OE_1}$, equal and parallel $\overline{EE_0}$, but not shown in the diagrams, to avoid complication.

The four diagrams correspond to the values of power, or motor output,

$P =$ 1,000, 6,000, 9,000, 12,000 watts, and give:

$P = 1,000$	$46 < E_1 < 2,200,$	$1 < I < 49$	Fig. 164.
$P = 6,000$	$340 < E_1 < 1,920,$	$7 < I < 43$	Fig. 165.
$P = 9,000$	$540 < E_1 < 1,750,$	$11.8 < I < 38.2$	Fig. 166.
$P = 12,000$	$920 < E_1 < 1,320,$	$20 < I < 30$	Fig. 167.

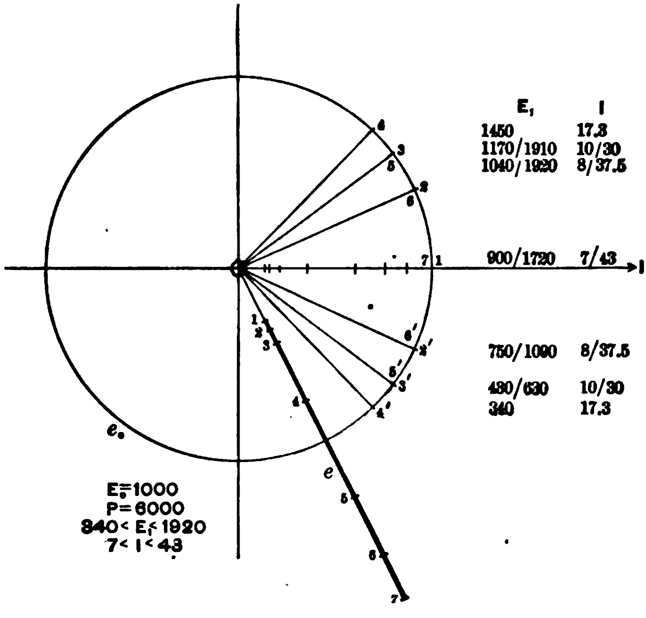


FIG. 165.

As seen, the permissible value of counter e.m.f., E_1, and of current I, becomes narrower with increasing output.

In the diagrams; different points of E_0 are marked with 1, 2, 3 . . . , when corresponding to leading current, with 2^1, 3^1, . . . , when corresponding to lagging current.

The values of counter e.m.f. E_1 and of current I are noted on the diagrams, opposite to the corresponding points E_0.

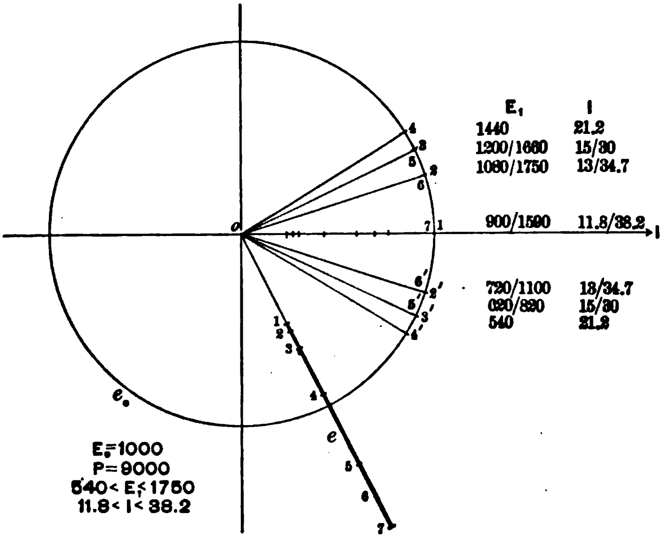


FIG. 166.

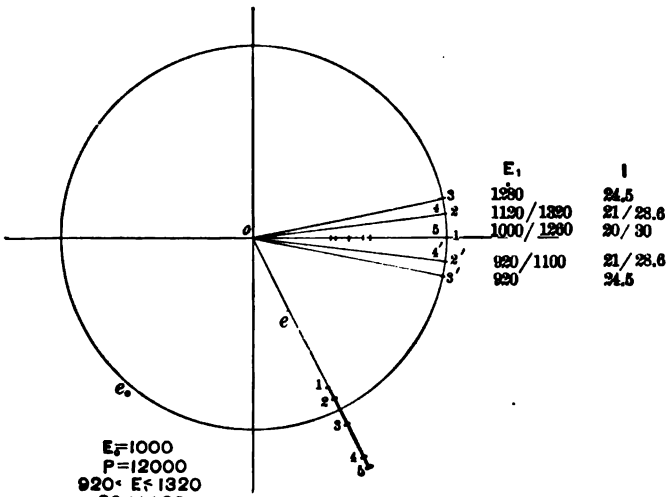


FIG. 167.

In this condition it is interesting to plot the current as function of the generated e.m.f. E_1 of the motor, for constant power P_1. Such curves are given in Fig. 171 and explained in the following on page 430.

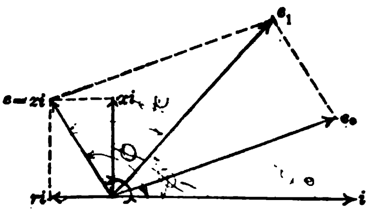


FIG. 168.

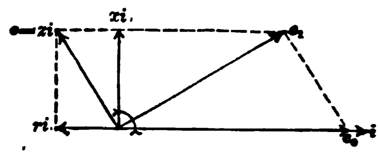


FIG. 169.

258. While the graphic method is very convenient to get a clear insight into the interdependence of the different quantities, for numerical calculation it is preferable to express the diagrams analytically.

For this purpose,

Let $z = \sqrt{r^2 + x^2}$ = impedance of the circuit of (equivalent) resistance, r, and (equivalent) reactance, $x = 2\,\pi f L$, containing the impressed e.m.f., e_0,* and the counter e.m.f., e_1, of the synchronous motor; that is, the e.m.f. generated in the motor armature by its rotation through the (resultant) magnetic field.

Let i = current in the circuit (effective values).

The mechanical power delivered by the synchronous motor (including friction and core loss) is the electric power consumed by the counter e.m.f., e_1; hence,

$$p = ie_1 \cos(i, e_1); \qquad (1)$$

thus,

$$\left.\begin{aligned} \cos(i, e_1) &= \frac{p}{ie_1}, \\ \sin(i, e_1) &= \sqrt{1 - \left(\frac{p}{ie_1}\right)^2}. \end{aligned}\right\} \qquad (2)$$

* If e_0 = e.m.f. at motor terminals, z = internal impedance of the motor; if e_0 = terminal voltage of the generator, z = total impedance of line and motor; if e_0 = e.m.f. of generator, that is, e.m.f. generated in generator armature by its rotation through the magnetic field, z includes the generator impedance also.

The displacement of phase between current i and e.m.f. $e = zi$ consumed by the impedance, z, is

$$\left.\begin{aligned} \cos(i, e) &= \frac{r}{z} \\ \sin(i, e) &= \frac{x}{z} \end{aligned}\right\} \qquad (3)$$

Since the three e.m.fs. acting in the closed circuit,

e_0 = e.m.f. of generator,

e_1 = counter e.m.f. of synchronous motor,

$e = zi$ = e.m.f. consumed by impedance,

form a triangle, that is, e_1 and e are components of e_0, it is (Figs. 168 and 169),

$$e_0^2 = e_1^2 + e^2 + 2\, ee_1 \cos(e_1, e), \qquad (4)$$

$$\text{hence, } \cos(e_1, e) = \frac{e_0^2 - e_1^2 - e^2}{2\, e_1 e} = \frac{e_0^2 - e_1^2 - z^2 i^2}{2\, zie_1}, \qquad (5)$$

since, however, by diagram,

$$\begin{aligned} \cos(e_1, e) &= \cos(i, e - i, e_1) \\ &= \cos(i, e)\cos(i, e_1) + \sin(i, e)\sin(i, e_1) \qquad (6) \end{aligned}$$

substitution of (2), (3) and (5) in (6) gives, after some transposition,

$$e_0^2 - e_1^2 - z^2 i^2 - 2\, rp = 2\, x \sqrt{i^2 e_1^2 - p^2}, \qquad (7)$$

the *fundamental equation of the synchronous motor*, relating impressed e.m.f., e_0; counter e.m.f., e_1; current i; power, p, and resistance, r; reactance, x; impedance, z.

This equation shows that, at given impressed e.m.f. e_0, and given impedance $z = \sqrt{r^2 + x^2}$, three variables are left, e_1, i, p, of which two are independent. Hence, at given e_0 and z, the current, i, is not determined by the load, p, only, but also by the excitation, and thus the same current, i, can represent widely different loads, p, according to the excitation; and with the

same load, the current, i, can be varied in a wide range, by varying the field excitation, e_1.

The meaning of equation (7) is made more perspicuous by some transformations, which separate e_1 and i, as function of p and of an angular parameter, ϕ.

Substituting in (7) the new coordinates;

$$\left.\begin{aligned} \alpha &= \frac{e_1^2 + z^2 i^2}{\sqrt{2}}, \\ \beta &= \frac{e_1^2 - z^2 i^2}{\sqrt{2}}, \end{aligned}\right\} \quad \text{or,} \quad \left\{\begin{aligned} e_1^2 &= \frac{\alpha + \beta}{\sqrt{2}}, \\ z^2 i^2 &= \frac{\alpha - \beta}{\sqrt{2}}, \end{aligned}\right\} \tag{8}$$

we get

$$e_0^2 - \alpha\sqrt{2} - 2\,rp = 2\frac{x}{z}\sqrt{\frac{\alpha^2 - \beta^2}{2} - z^2 p^2}; \tag{9}$$

substituting again,

$$\left.\begin{aligned} e_0^2 &= a \\ 2\,zp &= b \\ r &= \epsilon z, \end{aligned}\right.$$

hence,

$$\left.\begin{aligned} x &= z\sqrt{1 - \epsilon^2} \\ 2\,rp &= \epsilon b, \end{aligned}\right\} \tag{10}$$

we get

$$a - \alpha\sqrt{2} - \epsilon b = \sqrt{(1 - \epsilon^2)(2\,\alpha^2 - 2\,\beta^2 - b^2)}; \tag{11}$$

and, squared,

$$\epsilon^2\alpha^2 + (1 - \epsilon^2)\,\beta^2 - \alpha\sqrt{2}\,(a - \epsilon b) + \frac{b^2\,(1 - \epsilon^2)}{2} + \frac{(a - \epsilon b)^2}{2} = 0; \tag{12}$$

substituting

$$\left.\begin{aligned} \epsilon\alpha - \frac{(a - \epsilon b)\sqrt{2}}{2\,\epsilon} &= v, \\ \beta\sqrt{1 - \epsilon^2} &= w, \end{aligned}\right\} \tag{13}$$

gives, after some transposition,

$$v^2 + w^2 = \frac{(1 - \epsilon^2)}{2\epsilon^2}\,a\,(a - 2\,\epsilon\,b), \tag{14}$$

hence, if

$$R = \sqrt{\frac{(1 - \epsilon^2)(a - 2\,\epsilon b)\,a}{2\,\epsilon^2}}, \tag{15}$$

it is

$$v^2 + w^2 = R^2 \tag{16}$$

the equation of a circle with radius R.

Substituting now backwards, we get, with some transpositions,

$$\left\{r^2 (e_1^2 + z^2 i^2) - z^2 (e_0^2 - 2\,rp)\right\}^2 + \left\{rx\,(e_1^2 + z^2 i^2)\right\}^2 = x^2 z^2 e_0^2 (e_0^2 - 4\,rp) \tag{17}$$

the *fundamental equation of the synchronous motor* in a modified form.

The separation of e_1 and i can be effected by the introduction of a parameter, ϕ, by the equations

$$\left.\begin{aligned} r^2(e_1^2 + z^2 i^2) - z^2 (e_0^2 - 2\,rp) &= xze_0 \sqrt{e_0^2 - 4\,rp}\cos\phi \\ rx\,(e_1^2 - z^2 i^2) &= xze_0 \sqrt{e_0^2 - 4\,rp}\sin\phi \end{aligned}\right\} \tag{18}$$

These equations (18), transposed, give

$$\begin{aligned} e_1 &= \sqrt{\frac{1}{2}\left\{\frac{z^2}{r^2}(e_0^2 - 2\,rp) + \frac{ze_0}{r}\left(\frac{x}{r}\cos\phi + \sin\phi\right)\sqrt{e_0^2 - 4\,rp}\right\}} \\ &= \frac{e_0 z}{r}\sqrt{\frac{1}{2}\left\{\left(1 - \frac{2\,rp}{e_0^2}\right) + \frac{x}{z}\cos\phi + \frac{r}{z}\sin\phi\right)\sqrt{1 - \frac{4\,rp}{e_0^2}}\right\}}\,; \end{aligned} \tag{19}$$

$$\begin{aligned} i &= \sqrt{\frac{1}{2}\left\{\left(\frac{1}{r^2}(e_0^2 - 2\,rp) + \frac{e_0}{rz}\left(\frac{x}{r}\cos\phi - \sin\phi\right)\sqrt{e_0^2 - 4\,rp}\right\}} \\ &= \frac{e_0}{r}\sqrt{\frac{1}{2}\left\{\left(1 - \frac{2\,rp}{e_0^2}\right) + \left(\frac{x}{z}\cos\phi - \frac{r}{z}\sin\phi\right)\sqrt{1 - \frac{4\,rp}{e_0^2}}\right\}}. \end{aligned} \tag{20}$$

The parameter, ϕ, has no direct physical meaning, apparently.

These equations (19) and (20), by giving the values of e_1 and i as functions of p and the parameter, ϕ, enable us to construct the *power characteristics of the synchronous motor*, as the

curves relating e_1 and i, for a given power, p, by attributing to ϕ all different values.

Since the variables, v and w, in the equation of the circle (16) are quadratic functions of e_1 and i, the *power characteristics of the synchronous motor are quartic curves.*

They represent the action of the synchronous motor under all conditions of load and excitation, as an element of power transmission even including the line, etc.

Before discussing further these power characteristics, some special conditions may be considered.

259. A. *Maximum Output.*

Since the expression of e_1 and i [equations (19) and (20)] contain the square root, $\sqrt{e_0^2 - 4\,rp}$, it is obvious that the maximum value of p corresponds to the moment where this square root disappears by passing from real to imaginary; that is,

$$e_0^2 - 4\,rp = 0,$$

or,

$$p = \frac{e_0^2}{4\,r}. \tag{21}$$

This is the same value which represents the maximum power transmissible by e.m.f., e_0, over a non-inductive line of resistance, r; or, more generally, the maximum power which can be transmitted over a line of impedance,

$$z = \sqrt{r^2 + x^2},$$

into any circuit, shunted by a condenser of suitable capacity.

Substituting (21) in (19) and (20), we get,

$$\left.\begin{aligned} e_1 &= \frac{z}{2\,r}\,e_0, \\ i &= \frac{e_0}{2\,r}, \end{aligned}\right\} \tag{22}$$

and the displacement of phase in the synchronous motor,

$$\cos\,(e_1, i) = \frac{p}{ie_1} = \frac{r}{z};$$

hence,

$$\tan(e_1, i) = -\frac{x}{r}, \tag{23}$$

that is, the angle of internal displacement in the synchronous motor is equal, but opposite to, the angle of displacement of line impedance,

$$\begin{aligned}(e_1, i) &= -(e, i), \\ &= -(z, r),\end{aligned} \tag{24}$$

and consequently,

$$(e_0, i) = 0; \tag{25}$$

that is, the current, i, is in phase with the impressed e.m.f., e_0.

If $z < 2r$, $e_1 < e_0$; that is, motor e.m.f. < generator e.m.f.

If $z = 2r$, $e_1 = e_0$; that is, motor e.m.f. = generator e.m.f.

If $z > 2r$, $e_1 > e_0$; that is, motor e.m.f. > generator e.m.f.

In either case, the current in the synchronous motor is leading.

260. B. *Running Light, $p = 0$.*

When running light, or for $p = 0$, we get, by substituting in (19) and (20),

$$\left.\begin{aligned} e_1 &= \frac{e_0 z}{r}\sqrt{\frac{1}{2}\left\{1 + \frac{x}{z}\cos\phi + \frac{r}{z}\sin\phi,\right\}} \\ i &= \frac{e_0}{r}\sqrt{\frac{1}{2}\left\{1 + \frac{x}{z}\cos\phi - \frac{r}{z}\sin\phi.\right\}} \end{aligned}\right\} \tag{26}$$

Obviously this condition cannot well be fulfilled, since p must at least equal the power consumed by friction, etc.; and thus the true no-load curve merely approaches the curve $p = 0$, being, however, rounded off, where curve (26) gives sharp corners.

Substituting $p = 0$ into equation (7) gives, after squaring and transposing,

$$e_1^4 + e_0^4 + z^4 i^4 - 2e_1^2 e_0^2 - 2z^2 i^2 e_0^2 + 2z^2 i^2 e_1^2 - 4x^2 i^2 e_1^2 = 0. \tag{27}$$

This quartic equation can be resolved into the product of two quadratic equations,

$$\left.\begin{array}{l} e_1^2 + z^2 i^2 - e_0^2 + 2\,x i e_1 = 0. \\ e_1^2 + z^2 i^2 - e_0^2 - 2\,x i e_1 = 0. \end{array}\right\} \qquad (28)$$

which are the equations of two ellipses, the one the image of the other, both inclined with their axes.

The minimum value of counter e.m.f., e_1, is $e_1 = 0$ at $i = \frac{e_0}{z}$ (29)

The minimum value of current, i, is $i = 0$ at $e_1 = e_0$. (30)

The maximum value of e.m.f., e_1, is given by equation (28),

$$f = e_1^2 + z^2 i^2 - e_0^2 \pm 2\,x i e_1 = 0;$$

by the condition,

$$\frac{de_1}{di} = -\frac{df/di}{df/de_1} = 0, \text{ as } z^2 i \mp x e_1 = 0,$$

hence,

$$i = e_0 \frac{x}{rz}, \quad e_1 = \pm\, e_0 \frac{z}{r}\,. \qquad (31)$$

The maximum value of current, i, is given by equation (28) by

$$\frac{di}{de_1} = 0,$$

as

$$i = \frac{e_0}{r} e_1 = \mp\, e_0 \frac{x}{r}\,. \qquad (32)$$

If, as abscissas, e_1, and as ordinates, zi, are chosen, the axes of these ellipses pass through the points of maximum power given by equation (22).

It is obvious thus, that in the V-shaped curves of synchronous motors running light, the two sides of the curves are not straight lines, as usually assumed, but arcs of ellipses, the one of concave, the other of convex, curvature.

These two ellipses are shown in Fig. 170, and divide the whole space into six parts — the two parts, A and A', whose areas

contain the quartic curves (19) (20) of the synchronous motor, the two parts, *B* and *B′*, whose areas contain the quartic curves of the generator, the interior space, *C*, and exterior space, *D*, whose points do not represent any actual condition of the alternator circuit, but make e_1 and i imaginary. Some of the quartic curves however may overlap into space *C*.

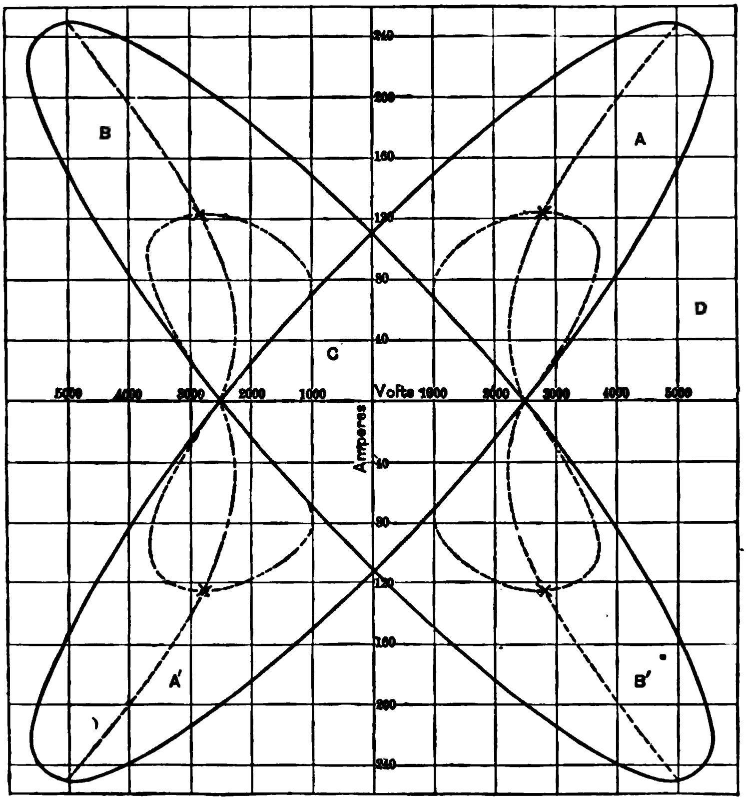


FIG. 170.

A and *A′* and the same *B* and *B′*, are identical conditions of the alternator circuit, differing merely by a simultaneous reversal of current and e.m.f., that is, differing by the time of a half-period.

Each of the spaces *A* and *B* contains one point of equation (22), representing the condition of maximum output as generator, viz., synchronous motor.

261. C. *Minimum Current at Given Power.*

The condition of minimum current, i, at given power, p, is determined by the absence of a phase displacement at the impressed e.m.f., e_0,

$$(e_0, i) = 0.$$

This gives from diagram Fig. 169,

$$e_1^2 = e_0^2 + i^2z^2 - 2\,ie_0r, \tag{33}$$

or, transposed,

$$e_1 = \sqrt{(e_0 - ir)^2 + i^2x^2}. \tag{34}$$

This quadratic curve passes through the point of zero current and zero power,

$$i = 0, \qquad e_1 = e_0,$$

through the point of maximum power (22),

$$i = \frac{e_0}{2\,r}, \qquad e_1 = \frac{e_0z}{2\,r},$$

and through the point of maximum current and zero power,

$$i = \frac{e_0}{r}, \qquad e_1 = \frac{e_0x}{r}, \tag{35}$$

and divides each of the quartic curves or power characteristics into two sections, one with leading, the other with lagging, current, which sections are separated by the two points of equation 34, the one corresponding to minimum, the other to maximum, current.

It is interesting to note that at the latter point the current can be many times larger than the current which would pass through the motor while at rest, which latter current is,

$$i = \frac{e_0}{z}, \tag{36}$$

while at no-load, the current can reach the maximum value,

$$i = \frac{e_0}{r}, \tag{35}$$

the same value as would exist in a non-inductive circuit of the same resistance.

The minimum value at counter e.m.f., e_1, at which coincidence of phase $(e_0, i) = 0$, can still be reached, is determined from equation (34) by,

$$\frac{de_1}{di} = 0;$$

as

$$i = e_0 \frac{r}{z^2}, \quad e_1 = e_0 \frac{x}{z} \cdot \tag{37}$$

The curve of no-displacement, or of minimum current, is shown in Figs. 170 and 171 in dotted lines.*

262. D. *Maximum Displacement of Phase.*

$$(e_0, i) = \text{maximum}.$$

At a given power, p, the input is,

$$p_0 = p + i^2 r = e_0 i \cos (e_0, i); \tag{38}$$

hence,

$$\cos (e_0, i) = \frac{p + i^2 r}{e_0 i} \cdot \tag{39}$$

At a given power, p, this value, as function of the current, i, is a maximum when

$$\frac{d}{di}\left(\frac{p + i^2 r}{e_0 i}\right) = 0,$$

this gives,

$$p = i^2 r; \tag{40}$$

or,

$$i = \sqrt{\frac{p}{r}} \cdot \tag{41}$$

* It is interesting to note that the equation (34) is similar to the value, $e_1 = \sqrt{(e_0 - ir)^2 - i^2 x^2}$, which represents the output transmitted over an inductive line of impedance, $z = \sqrt{r^2 + x^2}$, into a non-inductive circuit.

Equation (34) is identical with the equation giving the maximum voltage, e_1, at current, i, which can be produced by shunting the receiving circuit with a condenser; that is, the condition of "complete resonance" of the line, $z = \sqrt{r^2 + x^2}$, with current, i. Hence, referring to equation (35), $e_1 = e_0 \frac{x}{r}$ is the maximum resonance voltage of the line reached when closed by a condenser of reactance, $- x$.

That is, the displacement of phase, lead or lag, is a maximum, when the power of the motor equals the power consumed by the resistance; that is, at the electrical efficiency of 50 per cent.

Substituting (40) in equation (7) gives, after squaring and transposing, the quartic equation of maximum displacement,

$$(e_0^2 - e_1^2)^2 + i^4 z^2 (z^2 + 8\,r^2) + 2\,i^2 e_1^2 (5\,r^2 - z^2) - 2\,i^2 e_0^2 (z^2 + 3\,r^2) = 0. \quad (42)$$

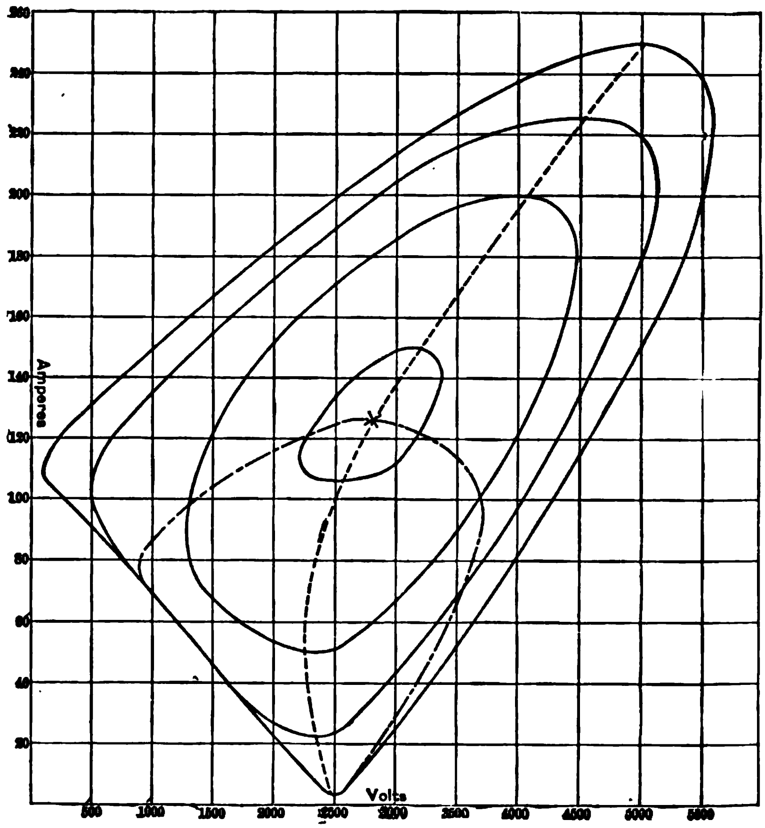


FIG. 171.

The curve of maximum displacement is shown in dash-dotted lines in Figs. 170 and 171. It passes through the point of zero current — as singular or nodal point — and through the point of maximum power, where the maximum displacement is zero, and it intersects the curve of zero displacement.

263. E. *Constant Counter e.m.f.*

At constant counter e.m.f., e_1 = constant, if,

$$e_1 < e_0 \sqrt{1 - \frac{r^2 x^2}{z^4}},$$

the current at no-load is not a minimum, and is lagging. With increasing load, the lag decreases, reaches a minimum, and then increases again, until the motor falls out of step, without ever coming into coincidence of phase.

If $$e_0 \sqrt{1 - \frac{r^2 x^2}{z^4}} < e_1 < e_0,$$

the current is lagging at no-load; with increasing load the lag decreases, the current comes into coincidence of phase with e_0, then becomes leading, reaches a maximum lead; then the lead decreases again, the current comes again into coincidence of phase, and becomes lagging, until the motor falls out of step.

If $e_0 < e_1$, the current is leading at no-load, and the lead first increases, reaches a maximum, then decreases; and whether the current ever comes into coincidence of phase, and then becomes lagging, or whether the motor falls out of step while the current is still leading, depends, whether the counter e.m.f. at the point of maximum output is $> e_0$ or $< e_0$.

264. F. *Numerical Example.*

Figs. 170 and 171 show the characteristics of a 100-kilowatt motor, supplied from a 2500-volt generator over a distance of 5 miles, the line consisting of two wires, No. 2 B. & S., 18 inches apart.

In this case we have,

$$\left.\begin{aligned} e_0 &= 2500 \text{ volts constant at generator terminals;} \\ r &= 10 \text{ ohms, including line and motor;} \\ x &= 20 \text{ ohms, including line and motor:} \\ \text{hence } z &= 22.36 \text{ ohms.} \end{aligned}\right\} \qquad (43)$$

Substituting these values, we get,

$$2500^2 - e_1^2 - 500\, i^2 - 20\, p = 40 \sqrt{i^2 e_1^2 - p^2} \qquad (7)$$

$$\{e_1^2 + 500\, i^2 - 31.25 \times 10^6 + 100\, p\}^2 + \{2\, e_1^2 - 1000\, i^2\}^2 = 7.8125 \times 10^{15} - 5 + 10^9\, p. \qquad (17)$$

$$e_1 = 5590 \times \sqrt{\tfrac{1}{2}\{(1 - 3.2 \times 10^{-6} p) + (.894 \cos\phi + .447 \sin\phi) \sqrt{1 - 6.4 \times 10^{-6} p}\}}. \qquad (19)$$

$$i = 2.50 \times \sqrt{\tfrac{1}{2}\{(1 - 3.2 \times 10^{-6} p) + (.894 \cos\phi - .447 \sin\phi) \sqrt{16.4 \times 10^{-6} p}\}}. \qquad (20)$$

Maximum output,

$$p = 156.25 \text{ kilowatts} \qquad (21)$$

at

$$\left.\begin{aligned} e_1 &= 2{,}795 \text{ volts} \\ i &= 125 \text{ amperes} \end{aligned}\right\} \qquad (22)$$

Running light,

$$\left.\begin{aligned} e_1^2 + 500\, i^2 - 6.25 \times 10^4 \mp 40\, i e_1 &= 0 \\ e_1 &= 20\, i \pm \sqrt{6.25 \times 10^4 - 100\, i^2} \end{aligned}\right\} \qquad (28)$$

At the minimum value of counter e.m.f., $e_1 = 0$ is $i = 112$ (29)

At the minimum value of current, $i = 0$ is $e_1 = 2500$ (30)

At the maximum value of counter e.m.f., $e_1 = 5590$ is $i = 223.5$ (31)

At the maximum value of current, $i = 250$ is $e_1 = 5000$. (32)

Curve of zero displacement of phase,

$$\begin{aligned} e_1 &= 10 \sqrt{(250 - i)^2 + 4\, i^2} \\ &= 10 \sqrt{6.25 \times 10^4 - 500\, i + 5\, i^2}. \end{aligned} \qquad (34)$$

Minimum counter e.m.f. point of this curve,

$$i = 50, \qquad e_1 = 2240. \qquad (35)$$

Curve of maximum displacement of phase,

$$p = 10\, i^2 \qquad (40)$$

$$(6.25 \times 10^6 - e_1^2)^2 + .65 \times 10^6\, i^4 - 10^{10}\, i^2 = 0. \qquad (42)$$

Fig. 170 gives the two ellipses of zero power, in full lines, with the curves of zero displacement in dotted, the curves of maximum displacement in dash-dotted lines, and the points of maximum power as crosses.

Fig. 171 gives the motor-power characteristics, for $p = 10$ kilowatts; $p = 50$ kilowatts; $p = 100$ kilowatts; $p = 150$ kilowatts, and $p = 156.25$ kilowatts, together with the curves of zero displacement, and of maximum displacement.

265. G. *Discussion of Results.*

The characteristic curves of the synchronous motor, as shown in Fig. 171, have been observed frequently, with their essential features, the V-shaped curve of no-load, with the point rounded off and the two legs slightly curved, the one concave, the other convex; the increased rounding off and contraction of the curves with increasing load; and the gradual shifting of the point of minimum current with increasing load, first towards lower, then towards higher, values of counter e.m.f., e_1.

The upper parts of the curves, however, I have never been able to observe completely and consider it as probable that they correspond to a condition of synchronous motor-running, which is unstable. The experimental observations usually extend about over that part of the curves of Fig. 171 which is reproduced in Fig. 172, and in trying to extend the curves further to either side, the motor is thrown out of synchronism.

It must be understood, however, that these power characteristics of the synchronous motor in Fig. 171 can be considered as approximations only, since a number of assumptions are made which are not, or only partly, fulfilled in practice. The foremost of these are:

1. It is assumed that e_1 can be varied unrestrictedly, while in reality the possible increase of e_1 is limited by magnetic saturation. Thus in Fig. 171, at an impressed e.m.f., $e_0 =$ 2,500 volts, e_1 rises up to 5,590 volts, which may or may not be beyond that which can be produced by the motor, but certainly is beyond that which can be constantly given by the motor.

2. The reactance, x, is assumed as constant. While the reactance of the line is practically constant, that of the motor is not, but varies more or less with the saturation, decreasing

for higher values. This decrease of x increases the current, i, corresponding to higher values of e_1, and thereby bends the curves upwards at a lower value of e_1 than represented in Fig. 171.

It must be understood that the motor reactance is not a simple quantity, but represents the combined effect of self-induction, that is, the e.m.f. generated in the armature conductor by the current therein and armature reaction, or the variation of the counter e.m.f. of the motor by the change of the resultant field, due to the superposition of the m.m.f. of the armature current upon the field excitation; that is, it is the "synchronous reactance."

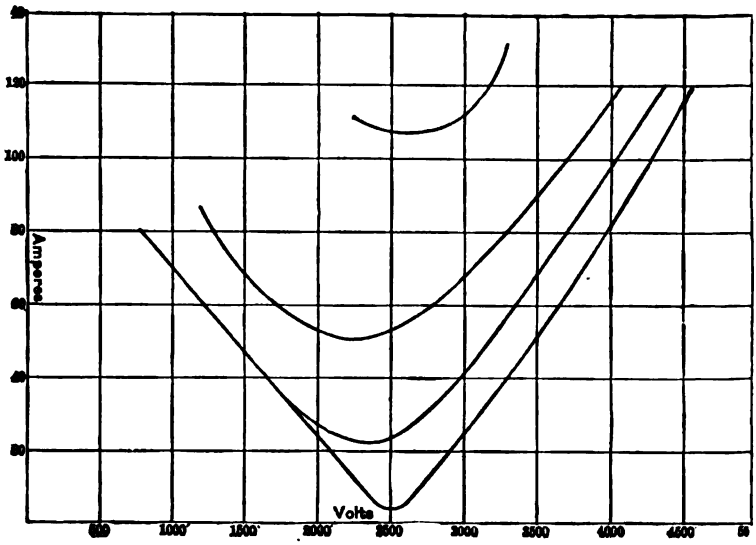


FIG. 172.

3. Furthermore, this synchronous reactance usually is not a constant quantity even at constant induced e.m.f., but varies with the position of the armature with regard to the field, that is, varies with the current and its phase angle, as discussed in the chapter on the armature reactions of alternators. While in most cases the synchronous reactance can be assumed as constant, with sufficient approximation, sometimes a more complete investigation is necessary, consisting in a resolution of the synchronous impedance in two components, in phase and in quadrature respectively with the field-poles.

Especially is this the case at low power-factors. So by gradually decreasing the excitation and thereby the e.m.f., e, the curves may, especially at light load, occasionally be extended below zero, into negative values of e, or onto the part of the curve, B, in Fig. 170, while the power still remains constant and positive, as synchronous motor. In other words, the motor keeps in step even if the field excitation is reversed: the lagging component of the armature reaction magnetizes the field, in opposition to the demagnetizing action of the reversed field excitation.

4. These curves in Fig. 171 represent the conditions of constant electric power of the motor, thus including the mechanical and the magnetic friction (core loss). While the mechanical friction can be considered as approximately constant, the magnetic friction is not, but increases with the magnetic induction; that is, with e_1, and the same holds for the power consumed for field excitation.

Hence the useful mechanical output of the motor will on the same curve, p = const., be larger at points of lower counter e.m.f., e_1, than at points of higher e_1; and if the curves are plotted for constant useful mechanical output, the whole system of curves will be shifted somewhat towards lower values of e_1; hence the points of maximum output of the motor correspond to a lower e.m.f. also.

It is obvious that the true mechanical power-characteristics of the synchronous motor can be determined only in the case of the particular conditions of the installation under consideration.

266. H. *Phase Characteristics of the Synchronous Motor.*

While an induction motor at constant impressed voltage is fully determined as regards to current, power-factor, efficiency, etc., by one independent variable, the load or output, in the synchronous motor two independent variables exist, load and field excitation. That is, at constant impressed voltage the current, power-factor, etc., of a synchronous motor can at the same power output be varied over a wide range by varying the field excitation, that is, the counter e.m.f. or "nominal generated e.m.f." Hence the synchronous motor can be utilized to fulfill two independent functions, to carry a certain load and to

produce a certain wattless current, lagging by under-excitation, leading by over-excitation. Synchronous motors are, therefore, to a considerable extent used to control the phase relation and thereby the voltage, in addition to producing mechanical power.

The same applies to synchronous converters.

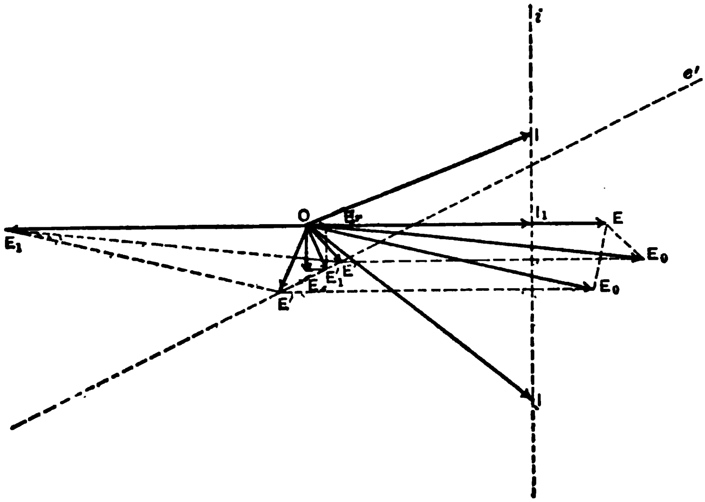


Fig. 173.

With given impressed e.m.f., field excitation or nominal generated e.m.f. corresponding thereto, and load, determine all the quantities of the synchronous motor, as current, power-factor, etc. Thus if in diagram Fig. 173, $\overline{OE} = e =$ e.m.f. consumed by the counter e.m.f. or nominal generated e.m.f. of the synchronous motor, and if $P_0 =$ output of motor (exclusive of friction and core loss and, if the exciter is driven by the motor, power consumed by the exciter), $i_1 = \frac{P_0}{e} =$ power component of current, represented by $\overline{OI_1}$, and the current vector therefore must terminate on a line, i, perpendicular to $\overline{OI_1}$. If, then, $r =$ resistance and $x =$ reactance of the circuit between counter e.m.f. e and impressed e.m.f. e_0, $\overline{OE_r} = i_1 r =$ e.m.f. consumed by resistance, $\overline{OE_x} = i_1 x =$ e.m.f. consumed by reactance of the power component of the current i_1, hence $\overline{OE'_1} =$ e.m.f. consumed by

impedance of the power component of the current i_1, and the impedance voltage of the total current lies on the perpendicular e' on $\overline{OE_1'}$. Producing $\overline{OE_1} = \overline{OE}$, and drawing an arc with the impressed e.m.f., e_0, as radius and E_1 as center, the point of intersection with e' gives the impedance voltage, $\overline{OE'}$, and corresponding thereto the current $\overline{OI} = i$; and completing the parallelogram, OEE_0E', gives the impressed e.m.f., $\overline{OE_0}$.

Hence, by impressed e.m.f. e_0, counter e.m.f. e, and load P_0, the polar diagram is determined, and thereby the vectors, OI = current, $\overline{OE_0}$ = impressed e.m.f., $\overline{OE}$ = counter e.m.f., and their phase relation.

Or, in symbolic representation, let

$\dot{E}_0 = e_0' + je_0''$ = impressed e.m.f.;

$$e_0 = \sqrt{e_0'^2 + e_0''^2}; \tag{1}$$

$\dot{E} = e' + je''$ = e.m.f. consumed by counter e.m.f.;

$$e = \sqrt{e'^2 + e''^2}; \tag{2}$$

$\dot{I} = i$ = current, assumed as zero vector;

$Z = r - jx$ = impedance of circuit between e_0 and e.

Z is the synchronous impedance of the motor, if e_0 is its terminal voltage. It is the impedance of transmission line with transformers and motor, if e_0 is terminal voltage of generator, and Z is synchronous impedance of motor and generator, plus impedance of line and transformers, if e_0 is the nominal generated e.m.f. of the generator (corresponding to its field excitation).

It is, then,

$$\dot{E}_0 = \dot{E} + iZ, \tag{3}$$

or,

$$e_0' + je_0'' = e' + je'' + ir - jix, \tag{4}$$

and, resolved,

$$\begin{cases} e_0' = e' + ir; & (5) \\ e_0'' = e'' - ix. & (6) \end{cases}$$

The power output of the motor (inclusive of friction and core loss, and if the exciter is driven by the motor, power consumed by exciter) is current times power component of generated e.m.f., or

$$P_0 = e'i. \tag{7}$$

Hence, the calculation of the motor, from power output P_0, occurs by the equations:

Chosen: i = current.

$$\left.\begin{aligned} (7)\ e' &= \frac{P_0}{i}, \\ (5)\ e_0' &= e' + ir, \\ (1)\ e_0'' &= \pm\sqrt{e_0^2 - e_0'^2}, \\ (6)\ e'' &= e_0'' + ix \\ (2)\ e &= \sqrt{e'^2 + e''^2}. \end{aligned}\right\} \tag{8}$$

That is, at given power P_0, to every value of current i correspond two values of the counter e.m.f. e (and hence the field excitation).

Solving equations (8) for i and P_0, that is, eliminating e', e_0', e_0'', e'', gives as the nominal generated e.m.f.,

$$e = \sqrt{e_0^2 - r^2i^2 + x^2i^2 - 2\,rP_0 \pm 2\,xi\sqrt{e_0^2 - \left(\frac{P_0}{i} + ri\right)^2}}, \tag{9}$$

and the power-factor of the motor is,

$$\cos\theta = \frac{e'}{e} = \frac{P_0}{ei}, \tag{10}$$

The power-factor of the supply is

$$\cos\theta_0 = \frac{e_0'}{e_0} = \frac{\frac{P_0}{i} + ir}{e_0} = \frac{P_0 + ri^2}{e_0 i}. \tag{11}$$

From equation (9), by solving for i, i can now be expressed as function of P_0 and e, that is, of power output and field excitation.

267. As illustration are plotted, in Fig. 174, curves giving the current i as function of the counter or nominal generated e.mf., e, at constant power P_0. Such curves are called "phase characteristics of the synchronous motor."

They are given for the values

$$e_0 = 2{,}200 \text{ volts},$$
$$Z = 1 - 4j \text{ ohms},$$

and

$$P_0 = 20, 200, 400, 600, 800, 1000 \text{ kw. output}.$$

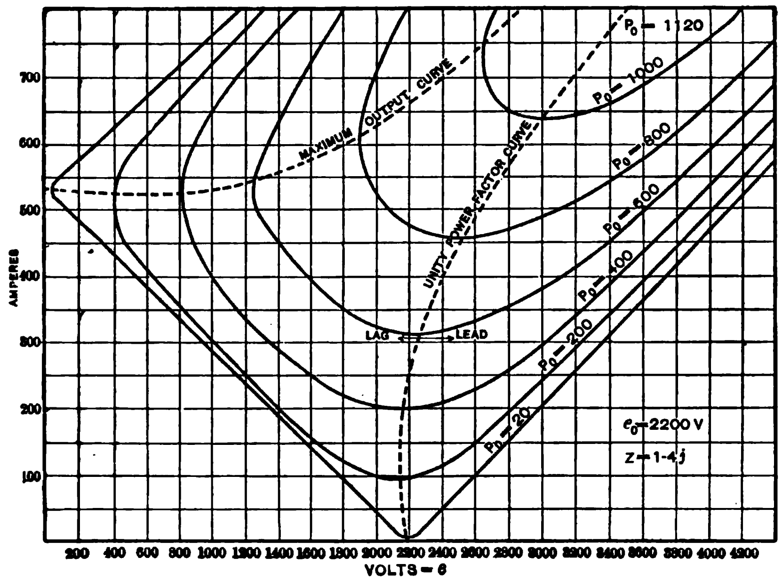


FIG. 174.

The five equations of the synchronous motor,

$$(1)\ e_0^2 = e_0'^2 + e_0''^2,$$
$$(2)\ e^2 = e'^2 + e''^2,$$
$$(7)\ P_0 = e'i,$$
$$(5)\ e_0' = e' + ir,$$
$$(6)\ e_0'' = e'' - ix,$$

determine the five quantities, e_0', e_0'', e', e'', e, as functions of P_0 and i.

The condition of zero phase displacement, or unity power-factor at the impressed e.m.f., e_0, is

$$e_0'' = 0;$$

hence $e_0' = e_0,$

and (6) $e'' = ix,$

(5) $e' = e_0 - ir;$

hence,

$$e^2 = (e_0 - ir)^2 + i^2x^2, \tag{12}$$

a quadratic equation, the hyperbola of unity power-factor, shown as dotted line in Fig. 174.

In this case, the power is found by substituting $e' = e_0 - ir$ in $P_0 = e' i$, as

$$P_0 = e_0 i - i^2 r, \tag{13}$$

or,

$$i = \frac{e_0}{2r}\left\{1 \pm \sqrt{1 - \frac{4rP_0}{e_0^2}}\right\}. \tag{14}$$

The maximum output of the synchronous motor follows herefrom, by the condition,

$$\sqrt{1 - \frac{4rp_0}{e_0^2}} = 0$$

as

$$P_m = \frac{e_0^2}{4r} \tag{15}$$

in above example

$$P_m = 1{,}210 \text{ kw. at } i = 1{,}100 \text{ amp.}$$

The curve of unity power-factor (12) divides the synchronous motor-phase characteristics into two sections, one, for lower e, with lagging, the other with leading current.

The study of these "phase characteristics," Fig. 174, gives the best insight into the behavior of the synchronous motor under conditions of steady operation.

268. I. *Load Curves of Synchronous Motor.*

Of special interest are the "load curves" of the synchronous motor, or curves giving, at constant excitation, e = constant, the current, power-factor, efficiency and apparent efficiency as function of the load or output $P = P_0$ − (friction + core loss + excitation). Such load curves are represented in Figs. 175 to

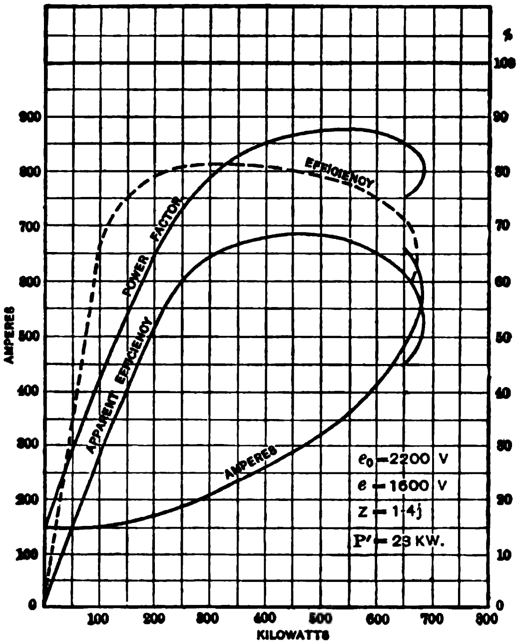


FIG. 175.

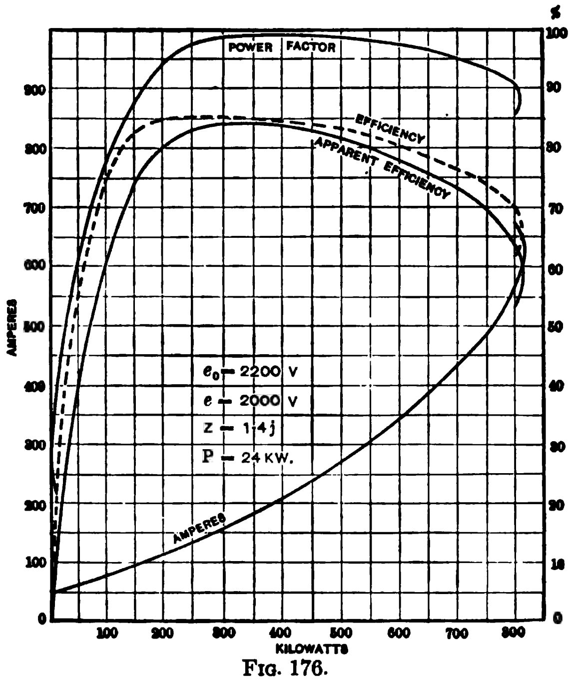


FIG. 176.

179, for e = 1600, 2000, 2180, 2400, 2800 volts. They can be derived from Fig. 174 as the intersection of the curves P_0 = constant with the vertical lines e = constant.

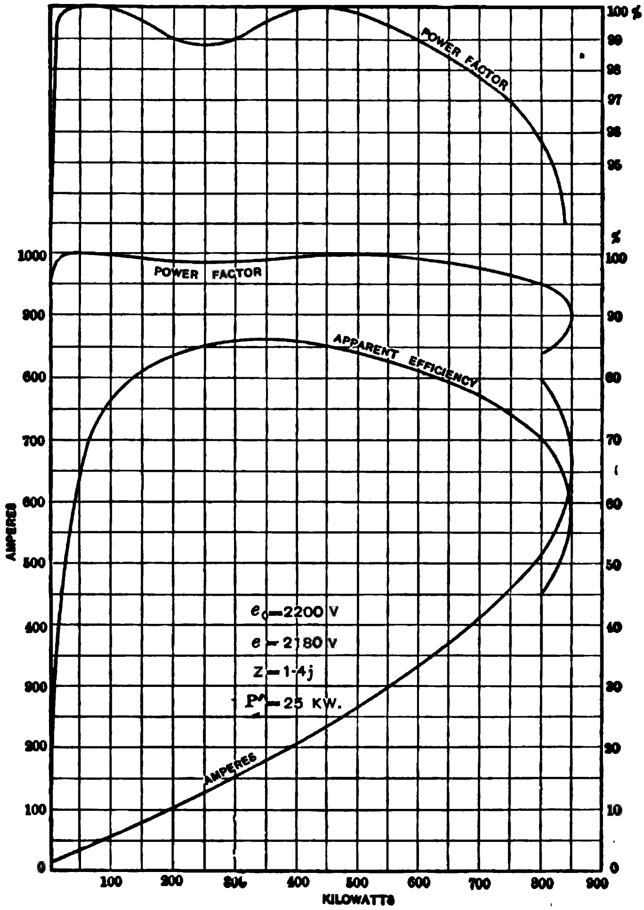


FIG. 177.

Hence, while an induction motor has one load curve only, a synchronous motor has an infinite series of load curves, depending upon the value of e.

For low values of e (e = 1,600, under excitation, Fig. 175), the load curves are similar to those of an induction motor.

The current is lagging, the power-factor rises from a low initial value to a maximum, and then falls again. With increasing excitation ($e = 2{,}000$, Fig. 176) the power-factor curve rises to values beyond those available in induction motors, approaches and ultimately touches unity, and with still higher excitation ($e = 2{,}180$, Fig. 177) two points of unity power-factor exist, at $P = 20$ and $P = 450$ kw. output, which are separated by a range with leading current, while at very low and very high load

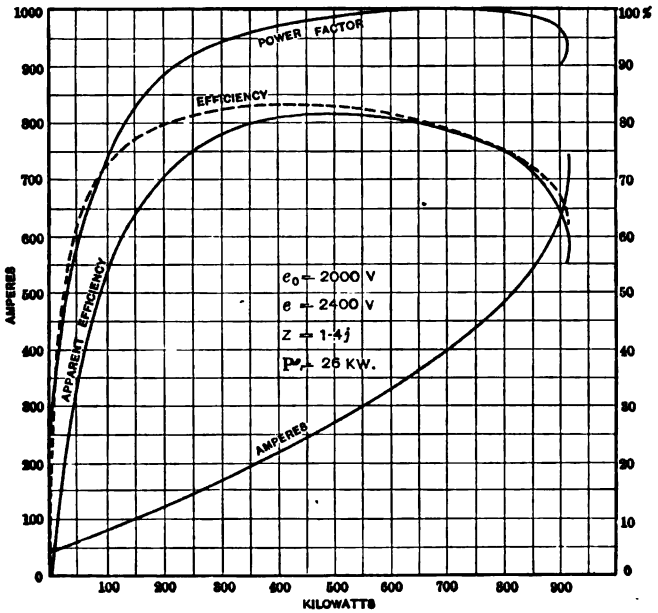


FIG. 178.

the current is lagging. The first point of unity power-factor moves towards $P = O$, and then disappears, that is, the current becomes leading already at no-load, and the second point of unity power-factor moves with increasing excitation toward higher loads, from $P = 450$ kw. at $e = 2{,}180$ in Fig. 177, to $P = 700$ kw. at $e = 2{,}400$, Fig. 178, and $P = 900$ kw. at $e = 2{,}800$, Fig. 179, while the power-factor and thereby the apparent efficiency decrease at light loads. The maximum output increases with the increase of excitation and almost proportionally thereto.

It is interesting that at $e = 2{,}180$, the power-factor is practically unity over the whole range of load up to near the maximum output. It is shown once more in Fig. 177 with increased scale of the ordinates. A synchronous motor at constant excitation can, therefore, give practically unity power-factor for all loads.

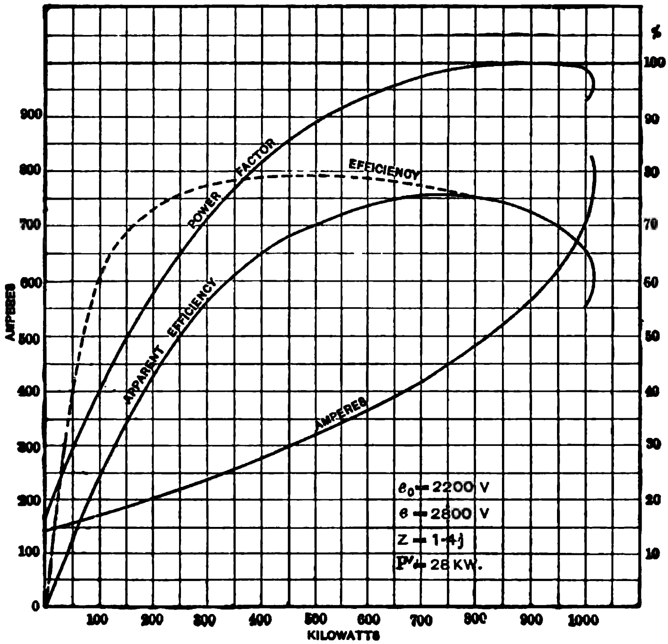


FIG. 179.

The resistance, $r = 1$ ohm, is assumed so as to represent a synchronous motor inclusive of transmission line, with about 9 per cent loss of energy in the line at 400 kw. output.

The friction and core loss are assumed as 20 kw., the excitation as 4 kw. at $e = 2{,}000$.

Considering the intersections of a horizontal line with the constant power curves of Fig. 174, gives the characteristic curves of the synchronous motor when operating on constant current. Such curves are shown for $i = 300$ in Fig. 180. They illustrate that at the same impressed voltage, with the same current input the power output of the synchronous motor can vary over a wide

range, and also that for each value of power output two points exist, one with lagging, the other with leading current.

As regards phase characteristics and load characteristics,

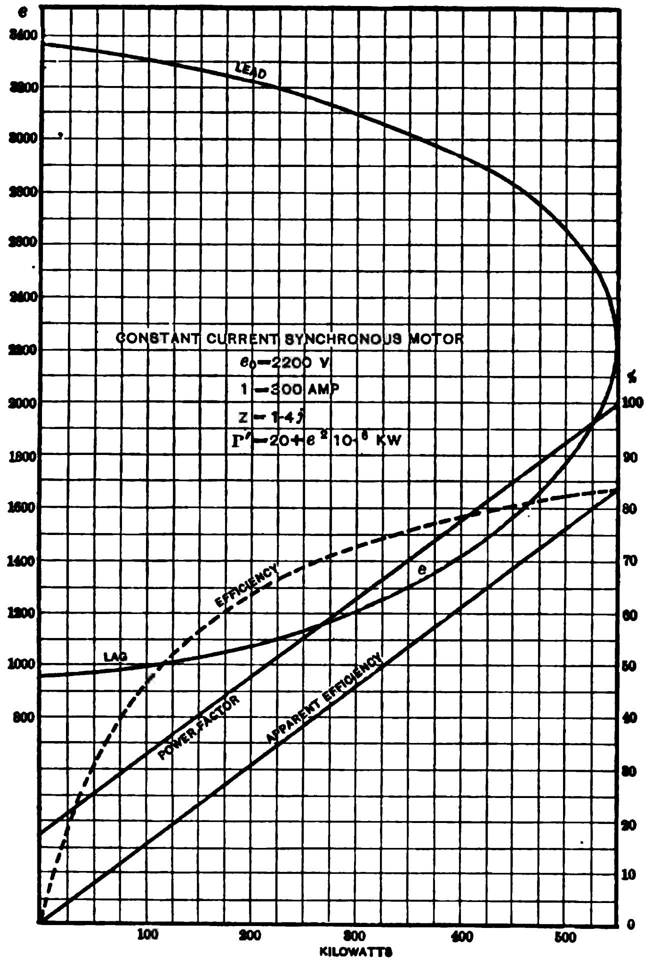


FIG. 180.

the same applies to the synchronous converter as to the synchronous motor, except that in the former the continuous current output affords a means of automatically varying the excitation with the load.

269. The investigation of a variation of the armature reaction and the self-induction, that is, of the synchronous reactance, with the position of the armature in the magnetic field, and so the intensity and phase of the current in its effect on the characteristic curves of the synchronous motor, can be carried out in the same manner as done for the alternating-current generator in Chapter XXII.

In the graphical and the symbolic investigations in Chapter XXII, the current, $\dot{I} = i_1 + ji_2$, has been considered as the output current, and chosen of such phase as to differ less than 90° from the terminal voltage, $\dot{E} = e_1 - je_2$, so representing power output.

Choosing then the current vector $\overline{OI}$ in opposite direction from that chosen in Figs. 148 and 149, and then constructing the diagram in the same manner as done in Chapter XXII, brings the output current, $\overline{OI}$, more than 90° displaced from the terminal voltage, $\overline{OE}$. Then the current consumes power, that is, the machine is a synchronous motor. The graphical representation in Chapter XXII so applies equally well to alternating-current generator as to synchronous motor, and the former corresponds to the case $\measuredangle EOI < 90°$, the latter to the case: $\measuredangle EOI > 90°$.

In the same manner, in the symbolic representation of Chapter XXII, choosing the current as $\dot{I} = -i_1 - ji_2$, or, in the final equation, where the current has been assumed as zero vector, $\dot{I} = -i$, that is, reversing all the signs of the current, gives the equations of the synchronous motor.

Choosing the same denotations as in Chapter XXII, and substituting $-i$ for $+i$ in equation (64) so gives the general equation of the synchronous motor,

$$e_0 = \frac{(e_1 - ri)^2 + (e_2 - x_0'i)(e_2 - x_0''i)}{\sqrt{(e_1 - ri)^2 + (e_2 - x_0''i)^2}},$$

and for non-inductive load,

$$e_0 = \frac{(e - ri)^2 + x_0' x_0'' i^2}{\sqrt{(e - ri)^2 + x_0''^2 i^2}}.$$

Or, by choosing $\overline{OI}$ in the graphic, and $\dot{I} = \dot{I}' + \dot{I}''$ in the symbolic method, as the input current, the diagram can be

constructed by combining the vectors in their proper directions, that is, where they are added in Chapter XXII, they are now subtracted, and inversely. For instance,

$$\underset{.}{E}_1 = \underset{.}{E}_2 + \underset{.}{E}_3, \qquad \underset{.}{E} = \underset{.}{E}_1 + \underset{.}{E}_4, \text{ etc.}$$

The reversal of the sign of the current in the above equations, compared with the equations of Chapter XXII, shows that in the synchronous motor, the effect of lag and of lead of the input current are the opposite of the effect of lag and lead of the output current in the generator, as discussed before.

It also follows herefrom, that the representation of the internal reactions of the synchronous motor by an effective reactance, the "synchronous reactance," is theoretically justified; but that, like in the alternating-current generator, this reactance may have to be resolved in two components, x_0' and x'', parallel and at right angles respectively to the field-poles.

CHAPTER XXV.

SURGING OF SYNCHRONOUS MOTORS.

270. In the preceding theory of the synchronous motor the assumption has been made that the mechanical output of the motor equals the power developed by it. This is the case only if the motor runs at constant speed. If, however, it accelerates, the power input is greater, if it decelerates, less than the power output, by the power stored in and returned by the momentum. Obviously, the motor can neither constantly accelerate nor decelerate, without breaking out of synchronism.

If, for instance, at a certain moment the power produced by the motor exceeds the mechanical load (as in the moment of throwing off a part of the load), the excess power is consumed by the momentum as acceleration, causing an increase of speed. The result thereof is that the phase of the counter e.m.f., e, is not constant, but the vector, e, (in Fig. 173) moves backward to earlier time, or counter-clockwise, at a rate depending upon the momentum. Thereby the current changes and the power developed changes and decreases. As soon as the power produced equals the load, the acceleration ceases, but the vector e still being in motion, due to the increased speed, further reduces the power, causing a retardation and thereby a decrease of speed, at a rate depending upon the mechanical momentum. In this manner a periodic variation of the phase relation between e and e_0, and corresponding variation of speed and current occurs, of an amplitude and period depending upon the circuit conditions and the mechanical momentum.

If the amplitude of this pulsation has a positive decrement, that is, is decreasing, the motor assumes after a while a constant position of e regarding e_0, that is, its speed becomes uniform. If, however, the decrement of the pulsation is negative, an infinitely small pulsation will continuously increase in amplitude, until the motor is thrown out of step, or the decrement becomes zero, by the power consumed by forces opposing the pulsation,

as anti-surging devices, or by the periodic pulsation of the synchronous reactance, § 269. If the decrement is zero, a pulsation started once will continue indefinitely at constant amplitude. This phenomenon, a surging by what may be called electromechanical resonance, must be taken into consideration in a complete theory of the synchronous motor.

Let

$E_0 = e_0 =$ impressed e.m.f. assumed as zero vector.

$E = e(\cos\beta + j\sin\beta) =$ e.m.f. consumed by counter e.m.f. of motor, where,

$$\beta = \text{phase angle between } E_0 \text{ and } E.$$

Let

$$Z = r - jx,$$

$$\text{and } z = \sqrt{r^2 + x^2}$$

$$= \text{impedance of circuit between}$$

E_0 and E, and

$$\tan\alpha = \frac{x}{r}.$$

The current in the system is

$$I_0 = \frac{e_0 - E}{z} = \frac{e_0 - e\cos\beta - je\sin\beta}{r - jx}$$

$$= \frac{1}{z}\{[e_0\cos\alpha - e\cos(\alpha+\beta)] + j[e_0\sin\alpha - e\sin(\alpha+\beta)]\}. \quad (1)$$

The power developed by the synchronous motor is

$$P_0 = [EI]^1 = \frac{e}{z}\{[\cos\beta[e_0\cos\alpha - e\cos(\alpha+\beta)] + \sin\beta$$

$$[e_0\sin\alpha - e\sin(\alpha+\beta)]\} = \frac{e}{z}\{[e_0\cos(\alpha-\beta) - e\cos\alpha]\}. \quad (2)$$

If, now, a pulsation of the synchronous motor occurs, resulting in a change of the phase relation, β, between the counter e.m.f., e, and the impressed e.m.f., e_0 (the latter being of constant frequency, thus constant phase), by an angle, δ, where δ is a periodic

function of time, of a frequency very low compared with the impressed frequency, then the phase angle of the counter e.m.f., e, is $\beta + \delta$; and the counter e.m.f. is

$$E = e \left\{\cos(\beta + \delta) + j \sin(\beta + \delta)\right\},$$

hence the current,

$$I = \frac{1}{z}\{[\dot{e}_0 \cos \alpha - e \cos(\alpha + \beta + \delta)] + j[e_0 \sin \alpha - e \sin$$

$$(\alpha + \beta + \delta)]\} = I_0 + \frac{2e}{z} \sin \frac{\delta}{2} \left\{ \sin\left(\alpha + \beta + \frac{\delta}{2}\right) - j \cos\left(\alpha + \beta + \frac{\delta}{2}\right) \right\} \tag{3}$$

the power,

$$P = \frac{e}{z}\{e_0 \cos(\alpha - \beta - \delta) - e \cos \alpha\}$$

$$= P_0 + \frac{2\,ee_0}{z} \sin \frac{\delta}{2} \sin\left(\alpha - \beta - \frac{\delta}{2}\right). \tag{4}$$

Let now

v_0 = mean velocity (linear, at radius of gyration) of synchronous machine;

s = slip, or decrease of velocity, as fraction of v_0, where s is a (periodic) function of time; hence

$v = v_0 (1 - s)$ = actual velocity, at time t.

During the time element, dt, the position of the synchronous motor armature regarding the impressed e.m.f., e_0, and thereby the phase angle $\beta + \delta$ of e, changes by

$$d\delta = 2\pi f s dt$$
$$= s d\theta, \tag{5}$$

where $\theta = 2\pi f t$,

and f = frequency of impressed e.m.f., e_0.

Let

m = mass of revolving machine elements, and

$M_0 = \frac{1}{2} m v_0^2$ = mean mechanical momentum, reduced to

joules or watt-seconds; then the momentum at time t and velocity $v = v_0 (1 - s)$ is

$$M = \frac{1}{2} m v_0^2 (1 - s)^2,$$

and the change of momentum during the time element, dt, is

$$\frac{dM}{dt} = -m v_0^2 (1 - s) \frac{ds}{dt};$$

hence, for small values of s,

$$\frac{dM}{dt} = -m v_0^2 \frac{ds}{d\theta} \frac{d\theta}{dt}$$

$$= -2 M_0 \frac{ds}{d\theta} \frac{d\theta}{dt}; \tag{6}$$

since

$$\frac{d\theta}{dt} = 2 \pi f$$

and from (5)

$$s = \frac{d\delta}{d\theta},$$

$$\frac{ds}{d\theta} = \frac{d^2\delta}{d\theta^2},$$

it is

$$\frac{dM}{dt} = -4\pi f M_0 \frac{d^2\delta}{d\theta}. \tag{7}$$

Since, as discussed, the change of momentum equals the difference between produced and consumed power, the excess of power being converted into momentum, it is

$$P - P_0 = \frac{dM}{dt}, \tag{8}$$

and, substituting (4) and (7)

$$\frac{ee_0}{z} \sin \frac{\delta}{2} \sin \left(\alpha - \beta - \frac{\delta}{2}\right) + 2 \pi f M_0 \frac{d^2\delta}{d\theta^2} = 0. \tag{9}$$

Assuming δ as a small angle, that is, considering only small oscillations, it is

$$\sin\frac{\delta}{2} = \frac{\delta}{2},$$

$$\sin\left[\alpha - \beta - \frac{\delta}{2}\right] = \sin(\alpha - \beta);$$

hence, substituted in (18),

$$\frac{ee_0}{z}\delta\sin(\alpha - \beta) + 4\pi f M_0 \frac{d^2\delta}{d\theta^2} = 0, \tag{10}$$

and, substituting,

$$a = \frac{ee_0\sin(\alpha - \beta)}{4\pi f z M_0} \tag{11}$$

it is

$$a\delta + \frac{d^2\delta}{d\theta^2} = 0. \tag{12}$$

This differential equation is integrated by

$$\delta = A\epsilon^{C\theta}, \tag{13}$$

which, substituted in (22) gives

$$aA\epsilon^{C\theta} + AC^2\epsilon^{C\theta} = 0,$$

$$a + C^2 = 0,$$

$$C = \pm\sqrt{-a}. \tag{14}$$

271. (1) If $a < 0$, it is

$$\delta = A_1\epsilon^{+m\theta} + A_2\epsilon^{-m\theta},$$

where

$$m = \sqrt{-a} = \sqrt{-\frac{ee_0\sin(\beta - \alpha)}{4\pi f z M_0}}.$$

Since in this case, $\epsilon^{+m\theta}$ is continually increasing, the synchronous motor is unstable. That is, without oscillation it drops out of step, if $\beta > \alpha$.

(2) If $a > 0$, it is, denoting

$$n = + \sqrt{a} = + \sqrt{\frac{ee_0 \sin (\alpha - \beta)}{4 \pi f z M_0}},$$

$$\delta = A_1 \varepsilon^{+jn\theta} + A_2 \varepsilon^{-jn\theta},$$

or, substituting for $\varepsilon^{+jn\theta}$ and $\varepsilon^{+jn\theta}$ the trigonometric functions

$$\delta = (A_1 + A_2) \cos n\theta + j (A_1 - A_2) \sin n\theta,$$

or,
$$\delta = B \cos (n\theta + \gamma). \tag{15}$$

That is, the synchronous motor is in stable equilibrium, when oscillating with a constant amplitude B, depending upon the initial conditions of oscillation, and a period, which for small oscillations gives the frequency of oscillation,

$$f_0 = nf = \sqrt{\frac{fee_0 \sin (\alpha - \beta)}{4 \pi z M_0}}. \tag{16}$$

In the example given in § 267 it is

$e_0 = 2{,}200$ volts. $Z = 1 - 4j$ ohms, or, $z = 4.12$; $\alpha = 76°$.

Let the machine, a 16-polar, 60-cycle, 400-kw., revolving-field, synchronous motor, have the radius of gyration of 20 in., a weight of the revolving part of 6,000 lb.

The momentum then is $M_0 = 850{,}000$ joules.

Deriving the angles, β, corresponding to given values of output P and excitation e, from the polar diagram, or from the symbolic representation, and substituting in (16), gives the frequency of oscillation,

$P = 0$:

$e = 1600$ volts;	$\beta = -2°$;	$f_0 = 2.17$ cycles,	or 130 periods per min.					
2180	"	+ 3°	2.50	"	150	"	"	"
2800	"	+ 5°	2.85	"	169	"	"	"

$P = 400$ kw.

$e = 1600$ volts;	$\beta = 33°$;	$f_0 = 1.90$ cycles,	or 114 periods per min.					
2180	"	21°	2.31	"	139	"	"	"
2800	"	22°	2.61	"	154	"	"	"

As seen, the frequency of oscillation does not vary much with the load and with the excitation. It slightly decreases with increase of load, and it increases with increase of excitation.

In this instance, only the momentum of the motor has been considered, as would be the case for instance in a synchronous converter.

In a direct-connected motor-generator set, assuming the momentum of the direct-current-generator armature equal to 60 per cent of the momentum of the synchronous motor, the total momentum is $M_0 = 1{,}360{,}000$ joules, hence, at no load,

$$P = 0,$$

$e = 1{,}600$ volts; $f_0 = 1.72$ cycles, or 103 periods per min.

1.98	"	119	"	"	"
1.23	"	134	"	"	"

272. In the preceding discussion of the surging of synchronous machines, the assumption has been made that the mechanical power consumed by the load is constant, and that no damping or anti-surging devices were used.

The mechanical power consumed by the load varies, however, more or less with the speed, approximately porportional to the speed if the motor directly drives mechanical apparatus, as pumps, etc., and at a higher power of speed if driving direct-current generators, or as synchronous converter, especially when in parallel with other direct-current generators. Assuming, then, in the general case the mechanical power consumed by the load to vary, within the narrow range of speed variation considered during the oscillation, as the pth power of the speed, in the preceding equation instead of P_0 is to be substituted, $P_0 (1-s)^p = P_0 (1-ps)$.

If anti-surging devices are used, and even without these in machines in which eddy currents can be produced by the oscillation of slip, in solid field-poles, etc., a torque is produced more or less proportional to the deviation of speed from synchronism. This power assumes the form, $P_1 = c^2 s$, where c is a function of the conductivity of the eddy-current circuit and the intensity of the magnetic field of the machine. c^2 is the power which would be required to drive the magnetic field of the motor through the

circuits of the anti-surging device at full frequency, if the same relative proportions could be retained at full frequency as at the frequency of slip s. That is, P_1 is the power produced by the motor as induction machine at slip s. Instead of P, the power generated by the motor, in the preceding equations the value, $P + P_1$ has to be substituted, then

The equation (8) assumes the form,

$$P + P_1 - P_0 (1 - ps) = \frac{dM}{dt},$$

or

$$(P - P_0) + (P_1 + pP_0 s) = \frac{dM}{dt}, \tag{17}$$

or, substituting (7) and (4),

$$2 e \frac{e_0}{z} \sin\frac{\delta}{2} \sin\left[\alpha - \beta - \frac{\delta}{2}\right] + (c^2 + pP_0) \frac{d\delta}{d\theta} + 4 \pi f M_0 \frac{d^2\delta}{d\theta^2} = 0; \tag{18}$$

and, for small values of δ,

$$a\delta + 2 b \frac{d\delta}{d\theta} + \frac{d^2\delta}{d\theta^2} = 0,$$

$$a = \frac{e e_0 \sin (\alpha - \beta)}{4 \pi f z M_0}, \tag{19}$$

$$b = \frac{c^2 + pP_0}{8 \pi f M_0}. \tag{20}$$

Of these two terms b represents the consumption, a the oscillation of energy by the pulsation of phase angle β. b and a thus have a similar relation as resistance and reactance in alternating current circuits, or in the discharge of condensers. a is the same term as in § 270.

Differential equation (19) is integrated by

$$\delta = A \epsilon^{C\theta}, \tag{21}$$

which, substituted in (19), gives

$$a A \epsilon^{C\theta} + 2 bCA \epsilon^{C\theta} + C^2 A \epsilon^{C\theta} = 0,$$

$$a + 2 bC + C^2 = 0,$$

which equation has the two roots

$$C_1 = -b + \sqrt{b^2 - a},$$
$$C_2 = -b - \sqrt{b^2 - a}. \qquad (22)$$

(1) If $a < 0$, or negative, that is, $\beta > \alpha$, C_1 is positive and C_2 negative, and the term with C_1 is continuously increasing, that is, the synchronous motor is unstable, and, without oscillation, drifts out of step.

(2) If $0 < a < b^2$, or a positive, and b^2 larger than a (that is, the energy consuming term very large), C_1 and C_2 are both negative, and, by substituting, $+\sqrt{b^2 - a} = g$, it is,

$$C_1 = -(b - g), \qquad C_2 = -(b + g);$$

hence

$$\delta = A_1 \epsilon^{-(b-g)\theta} + A_2 \epsilon^{-(b+g)\theta}. \qquad (23)$$

That is, the motor steadies down to its mean position logarithmically, or without any oscillation.

$$b^2 < a,$$

hence

$$\frac{(c^2 + pP_0)^2}{16\,\pi f M_0} < \frac{e e_0 \sin(\alpha - \beta)}{2} \qquad (24)$$

is the condition under which no oscillation can occur.

As seen, the left side of (24) contains only mechanical, the right side only electrical terms.

(3) $$a > b^2.$$

In this case, $\sqrt{b^2 - a}$ is imaginary, and, substituting,

$$g = \sqrt{a - b^2},$$

it is

$$C_1 = -b + jg,$$
$$C_2 = -b - jg,$$

hence

$$\delta = \epsilon^{-b\theta} \left[A_1 \epsilon^{+jg\theta} + A_2\, \epsilon^{-jg\theta}\right],$$

and, substituting the trigonometric for the exponential functions, gives ultimately

$$\delta = B\epsilon^{-b\theta} \cos(g\theta + \gamma). \qquad (25)$$

That is, the motor steadies down with an oscillation of period,

$$f_0 = gf$$

$$= \sqrt{\frac{fee_0 \sin(\alpha - \beta)}{4\pi z M_0} - \frac{(c^2 + pP_0)^2}{64\pi^2 M_0^2}}, \tag{26}$$

and decrement or attenuation constant

$$b = \frac{c^2 + pP_0}{8\pi f M_0}. \tag{27}$$

273. It follows, however, that under the conditions considered, a cumulative surging, or an oscillation with continuously increasing amplitude, cannot occur, but that a synchronous motor, when displaced in phase from its mean position, returns thereto either aperiodically, if $b^2 > \alpha$, or with an oscillation of vanishing amplitude, if $b^2 < \alpha$. At the worst, it may oscillate with constant amplitude, if $b = 0$.

Cumulative surging can, therefore, occur only if in the differential equation (19)

$$a\delta + 2b\frac{d\delta}{d\theta} + \frac{d^2\delta}{d\theta^2} = 0, \tag{28}$$

the coefficient b is negative.

Since c^2, representing the induction motor torque of the damping device, etc., is positive, and pP_0 is also positive (p being the exponent of power variation with speed), this presupposes the existence of a third and negative term, $\frac{-h^2}{8\pi f M_0}$, in b,

$$b = \frac{c^2 + pP_0 - h^2}{8\pi f M_0}. \tag{29}$$

This negative term represents a power,

$$P_2 = -h^2 s; \tag{30}$$

that is, a retarding torque during slow speed, or increasing β, and accelerating torque during high speed, or decreasing β.

The source of this torque may be found external to the motor, or internal, in its magnetic circuit.

External sources of negative P_2 may be, for instance, the magnetic field of a self-exciting, direct-current generator, driven by the synchronous motor. With decrease of speed, this field decreases, due to the decrease of generated voltage, and increases with increase of speed. This change of field strength, however, lags behind the exciting voltage and thus speed, that is, during decrease of speed the output is greater than during increase of speed. If this direct-current generator is the exciter of the synchronous motor, the effect may be intensified.

The change of power input into the synchronous motor, with change of speed, may cause the governor to act on the prime mover driving the generator, which supplies power to the motor, and the lag of the governor behind the change of output gives a pulsation of the generator frequency, of e_0, which acts like a negative power, P_2. The pulsation of impressed voltage, caused by the pulsation of β, may give rise to a negative, P_2, also.

An internal cause of a negative term, P_2, is found in the lag of the synchronous motor field behind the resultant m.m.f. In the preceding discussion, e is the "nominal generated e.m.f." of the synchronous machine, corresponding to the field excitation. The actual magnetic flux of the machine, however, does not correspond to e, and thus to the field excitation, but corresponds to the resultant m.m.f. of field excitation and armature reaction, which latter varies in intensity and in phase during the oscillation of β. Hence, while e is constant, the magnetic flux is not constant, but pulsates with the oscillations of the machine. This pulsation of the magnetic flux lags behind the pulsation of m.m.f., and thereby gives rise to a term in b in equation (28). If P_0, β, e, e_0, Z are such that a retardation of the motor increases the magnetizing, or decreases the demagnetizing force of the armature reaction, a negative term, P_2, appears, otherwise a positive term.

P_2 in this case is the energy consumed by the magnetic cycle of the machine at full frequency, assuming the cycle at full frequency as the same as at frequency of slip, s.

Or inversely, e may be said to pulsate, due to the pulsation of armature reaction, with the same frequency as β, but with a phase, which may either be lagging or leading. Lagging of the pulsation of e causes a negative, leading a positive P_2.

P_2, therefore, represents the power due to the pulsation of e

caused by the pulsation of the armature reaction, as discussed in § 269.

Any appliance increasing the area of the magnetic cycle of pulsation, as short-circuits around the field poles, therefore, increases the steadiness of a steady and increases the unsteadiness of an unsteady synchronous motor.

In self-exciting synchronous converters, the pulsation of e is intensified by the pulsation of direct-current voltage caused thereby, and hence of excitation.

Introducing now the term, $P_2 = -h^2 s$, into the differential equations of § 272, gives the additional cases

$b < 0$, or negative, that is,

$$\frac{c^2 + pP_0 - h^2}{8 \pi f M_0} < 0. \tag{31}$$

Hence, denoting

$$b_1 = -b = \frac{h^2 - c^2 - pP_0}{8 \pi f M_0}, \tag{32}$$

gives

(4) If $\qquad b_1^2 > a, \qquad g = +\sqrt{b_1^2 - a},$

$$\delta = A_1 \epsilon^{+(b_1 + f)\theta} + A_2 \epsilon^{+(b_1 - f)\theta}. \tag{33}$$

That is, without oscillation, the motor drifts out of step, in unstable equilibrium.

(5) If $\qquad a > b^2, \qquad g = \sqrt{a - b_1^2},$

$$\delta = B\epsilon^{+b_1\theta} \cos(g\theta + \delta). \tag{34}$$

That is, the motor oscillates, with constantly increasing amplitude, until it drops out of step. This is the typical case of cumulative surging by electromechanical resonance.

The problem of surging of synchronous machines, and its elimination, thus resolves into the investigation of the coefficient

$$b = \frac{c^2 + pP_0 - h^2}{8 \pi f M_0}, \tag{35}$$

while the frequency of surging, where such exists, is given by

$$f_0 = \sqrt{\frac{fee_0 \sin(\alpha - \beta)}{4\pi z_0 M_0} - \frac{(c^2 + pP_0 - h^2)^2}{64\pi^2 M_0^2}}. \tag{36}$$

Case (4), steady drifting out of step, has only rarely been observed.

The avoidance of surging thus requires

(1) An elimination of the term h^2, or reduction as far as possible.

(2) A sufficiently large term, c^2, or

(3) A sufficiently large term, pP_0.

(1) refers to the design of the synchronous machine and the system on which it operates. (2) leads to the use of electromagnetic antisurging devices, as an induction motor winding in the field poles, short circuits between the poles, or around the poles, and (3) leads to flexible connection to a load or a momentum, as flexible connection with a flywheel, or belt drive of the load.

The conditions of steadiness are

$$\beta > \alpha,$$

$$c^2 + pP_0 - h^2 > 0,$$

and if

$$\frac{(c^2 + pP_0 - h^2)^2}{16\pi f M_0} > \frac{ee_0 \sin(\alpha - \beta)}{z},$$

no oscillation at all occurs, otherwise an oscillation with decreasing amplitude.

CHAPTER XXVI.

ALTERNATING-CURRENT MOTORS IN GENERAL.

274. The starting point of the theory of the polyphase and single-phase induction motor usually is the general alternating-current transformer and from the equations of the general alternating-current transformer the induction motor equations have been developed in the preceding chapter. Coming, however, to the commutator motors, this method becomes less suitable, and the following more general method preferable.

In its general form the alternating-current motor consists of one or more stationary electric circuits magnetically related to one or more rotating electric circuits. These circuits can be excited by alternating currents, or some by alternating, others by direct current, or closed upon themselves, etc., and connection can be made to the rotating member either by collector rings — that is, to fixed points of the windings — or by commutator — that is, to fixed points in space.

The alternating-current motors can be subdivided into two classes — those in which the electric and magnetic relations between stationary and moving members do not vary with their relative positions, and those in which they vary with the relative positions of stator and rotor. In the latter a cycle of rotation exists, and therefrom the tendency of the motor results to lock at a speed giving a definite ratio between the frequency of rotation and the frequency of impressed e.m.f. Such motors, therefore, are synchronous motors.

The main types of synchronous motors are as follows:

(1) One member supplied with alternating and the other with direct current — polyphase or single-phase synchronous motors.

(2) One member excited by alternating current, the other containing a single circuit closed upon itself — synchronous-induction motors, as discussed in Chapter XIX.

(3) One member excited by alternating current, the other of

different magnetic reluctance in different directions (as polar construction) — reaction motors (see Chapter XXVII).

(4) One member excited by alternating current, the other by alternating current of different frequency or different direction of rotation — general alternating-current transformer or frequency converter and synchronous induction generator (see Chapters XVIII and XX).

(1) is the synchronous motor of the electrical industry. (2) and (3) are used occasionally to produce synchronous rotation without direct-current excitation, and of very great steadiness of the rate of rotation, where weight efficiency and power-factor are of secondary importance. (4) is used to some extent as frequency converter or alternating current generator.

(2) and (3) are occasionally observed in induction machines, and in the starting of synchronous motors, as a tendency to lock at some intermediate, occasionally low, speed. That is, in starting, the motor does not accelerate up to full speed, but the acceleration stops at some intermediate speed, frequently half-speed, and to carry the motor beyond this speed, the impressed voltage may have to be raised or even external power applied. The appearance of such "dead points" in the speed curve is due to a mechanical defect — as eccentricity of the rotor — or faulty electrical design: an improper distribution of primary and secondary windings causes a periodic variation of the mutual inductive reactance and so of the effective primary inductive reactance, (2) or the use of sharply defined and improperly arranged teeth in both elements causes a periodic magnetic lock (opening and closing of the magnetic circuit, (3) and so a tendency to synchronize at the speed corresponding to this cycle.

Synchronous machines have been discussed in the preceding chapters. Here shall be considered only that type of motor in which the electric and magnetic relations between the stator and rotor do not vary with their relative positions, and the torque is, therefore, not limited to a definite synchronous speed. This requires that the rotor when connected to the outside circuit be connected through a commutator, and when closed upon itself, several closed circuits exist, displaced in position from each other so as to offer a resultant closed circuit in any direction.

The main types of these motors are:

(1) One member supplied with polyphase or single-phase alternating voltage, the other containing several circuits closed upon themselves — polyphase and single-phase induction machines, as discussed in Chapter XIX.

(2) One member supplied with polyphase or single-phase alternating voltage, the other connected by a commutator to an alternating voltage — compensated induction motors, commutator motors with shunt-motor characteristic.

(3) Both members connected, through a commutator, directly or inductively, in series with each other, to an alternating voltage — alternating-current motors with series-motor characteristic.

Herefrom then follow three main classes of alternating-current motors:

Synchronous Motors.

Induction Motors.

Commutator Motors.

There are, however, numerous intermediate forms, which belong in several classes, as the synchronous induction motor, the compensated induction motor, etc.

275. An alternating current, I, in an electric circuit produces a magnetic flux, Φ, interlinked with this circuit. Considering equivalent sine waves of I and Φ, Φ lags behind I by the angle of hysteretic lag, α. This magnetic flux, Φ, generates an e.m.f., $E = 2\,\pi f n \Phi$, where f = frequency, n = number of turns of electric circuit. This generated e.m.f., E, lags 90 time degrees behind the magnetic flux Φ, hence consumes an e.m.f. 90 time degrees ahead of Φ, or $90 - \alpha$ degrees ahead of I. This may be resolved in a wattless component: $E = 2\,\pi f n \Phi \cos\alpha = 2\,\pi f L I = xI$, the e.m.f. consumed by self-induction, and power component: $E'' = 2\,\pi f n \Phi \sin\alpha = 2\,\pi f H I = r'' \dot{I}$ = e.m.f. consumed by hysteresis (eddy currents, etc.), and is, therefore, in vector representation denoted by

$$\dot{E}' = -jx\dot{I} \text{ and } \dot{E}'' = r''\dot{I},$$

where $x = 2\,\pi f L$ = reactance,

and L = inductance,

r'' = effective hysteretic resistance.

The ohmic resistance of the circuit, r', consumes an e.m.f. $r'\dot{I}$, in phase with the current, and the total or effective resistance of the circuit is, therefore, $r = r' + r''$, and the total e.m.f. consumed by the circuit, or the impressed e.m.f., is

$$\dot{E} = (r - jx)\,\dot{I} = Z\dot{I},$$

where

$$Z = r - jx = \text{impedance, in vector denotation,}$$

$$z = \sqrt{r^2 + x^2} = \text{impedance, in absolute terms.}$$

If an electric circuit is in inductive relation to another electric circuit, it is advisable to separate the inductance, L, of the circuit in two parts — the self-inductance, L_s, which refers to that part of the magnetic flux produced by the current in one circuit which is interlinked only with this circuit but not with the other circuit, and the mutual inductance, L_m, which refers to that part of the magnetic flux interlinked also with the second circuit. The desirability of this separation results from the different character of the two components: The self-inductive reactance generates a wattless e.m.f. and thereby causes a lag of the current, while the mutual inductive reactance transfers power into the second circuit, hence generally does the useful work of the apparatus. This leads to the distinction between the self-inductive impedance, $Z_0 = r_0 - jx_0$, and the mutual inductive impedance, $Z = r - jx$.

The same separation of the total inductive reactance into self-inductive reactance and mutual inductive reactance, represented respectively by the self-inductive or "leakage" impedance, and the mutual inductive or "exciting" impedance has been made in the theory of the transformer and the induction machine. In those, the mutual inductive reactance has been represented, not by the mutual inductive impedance, Z, but by its reciprocal value, the exciting admittance: $Y = \frac{1}{Z}\cdot$ It is then:

r_0 is the coefficient of power consumption by ohmic resistance, hysteresis and eddy currents of the self-inductive flux — effective resistance.

x_0 is the coefficient of e.m.f. consumed by the self-inductive or leakage flux — self-inductive reactance.

r is the coefficient of power consumption by hysteresis and eddy currents due to the mutual magnetic flux (hence contains no ohmic resistance component).

x is the coefficient of e.m.f. consumed by the mutual magnetic flux.

The e.m.f. consumed by the circuit is then

$$\underset{\cdot}{E} = Z_0 \underset{\cdot}{I} + Z \underset{\cdot}{I}. \tag{1}$$

If one of the circuits rotates relatively to the other, then in addition to the e.m.f. of self-inductive impedance: $Z_0 \underset{\cdot}{I}$ and the e.m.f. of mutual-inductive impedance or e.m.f. of alternation: $Z \underset{\cdot}{I}$ an e.m.f. is consumed by rotation. This e.m.f. is in phase with the flux through which the coil rotates — that is, the flux parallel to the plane of the coil — and proportional to the speed — that is, the frequency of rotation — while the e.m.f. of alternation is 90 time degrees ahead of the flux alternating through the coil — that is, the flux parallel to the axis of the coil — and proportional to the frequency. If, therefore, Z' is the impedance corresponding to the former flux, the e.m.f. of rotation is $jSZ'\underset{\cdot}{I}$, where S is the ratio of frequency of rotation to frequency of alternation, or the speed expressed in fractions of synchronous speed. The total e.m.f. consumed in the circuit is thus:

$$\underset{\cdot}{E} = Z_0 \underset{\cdot}{I} + Z \underset{\cdot}{I} + jSZ' \underset{\cdot}{I}. \tag{2}$$

Applying now these considerations to the alternating-current motor, we assume all circuits reduced to the same number of turns — that is, selecting one circuit, of n effective turns, as starting point, if n_i = number of effective turns of any other circuit, all the e.m.fs. of the latter circuit are divided, the currents multiplied with the ratio, $\frac{n_i}{n}$, the impedances divided, the admittances multiplied with $\frac{n_i^2}{n}$. This reduction of the constants of all circuits to the same number of effective turns is convenient by eliminating constant factors from the equations, and so permitting a direct comparison. When speaking, therefore, in the following of the impedance, etc., of the different circuits, we always refer to their reduced values, as it is customary in induction-motor designing practice, and has been done in preceding theoretical investigations.

276. Let, then, in Fig. 181:

E_0, I_0, Z_0 = impressed voltage, current and self-inductive impedance respectively of a stationary circuit,

E_1, I_1, Z_1 = impressed voltage, current and self-inductive impedance respectively of a rotating circuit,

τ = space-angle between the axes of the two circuits,

Z = mutual inductive, or exciting impedance in the direction of the axis of the stationary coil,

Z' = mutual inductive, or exciting impedance in the direction of the axis of the rotating coil,

Z'' = mutual inductive or exciting impedance in the direction at right angles to the axis of the rotating coil,

S = speed, as fraction of synchronism.

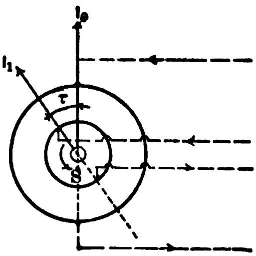


Fig. 181.

It is then:

E.m.f. consumed by self-inductive impedance, $Z_0 I_0$.

E.m.f. consumed by mutual-inductive impedance, $Z(I_0 + I_1 \cos \tau)$ since the m.m.f. acting in the direction of the axis of the stationary coil is the resultant of both currents. Hence,

$$E_0 = Z_0 I_0 + Z(I_0 + I_1 \cos \tau). \tag{3}$$

In the rotating circuit, it is,

E.m.f. consumed by self-inductive impedance, $Z_1 I_1$.

E.m.f. consumed by mutual-inductive impedance or "e.m.f. of alternation": $Z'(I_1 + I_0 \cos \tau)$. (4)

E.m.f. of rotation, $jSZ'' I_0 \sin \tau$. (5)

Hence the impressed e.m.f.,

$$E_1 = Z_1 I_1 + Z'(I_1 + I_0 \cos \tau) + jaZ'' I_0 \sin \tau. \tag{6}$$

In a structure with uniformly distributed winding, as used in induction motors, etc., $Z' = Z'' = Z$, that is, the exciting impedance is the same in all directions.

Z is the reciprocal of the "exciting admittance," Y of the induction-motor theory.

In the most general case, of a motor containing n circuits, of which some are revolving, some stationary, if:

$\dot{E}_k$, $\dot{I}_k$, Z_k = impressed e.m.f., current and self-inductive impedance respectively of any circuit, k.

Z^i, and Z^{ii} = exciting impedance parallel and at right angles respectively to the axis of a circuit, i,

τ_k^i = space-angle between the axes of coils k and i, and

S = speed, as fraction of synchronism, or "frequency of rotation."

It is then, in a coil, i,

$$\dot{E}_i = Z_i \dot{I}_i + Z^i \sum_1^n{}^k \dot{I}_k \cos \tau_k^i + jSZ^{ii} \sum_1^n{}^k \dot{I}_k \sin \tau_k^i, \tag{7}$$

where

$$Z_i \dot{I}_i = \text{e.m.f. of self-inductive impedance}; \tag{8}$$

$$Z^i \sum_1^n{}^k \dot{I}_k \cos \tau_k^i = \text{e.m.f. of alternation}; \tag{9}$$

$$\dot{E}_i' = jSZ^{ii} \sum_1^n{}^k \dot{I}_k \sin \tau_k^i = \text{e.m.f. of rotation}; \tag{10}$$

which latter = 0 in a stationary coil, in which $S = 0$.

The power output of the motor is the sum of the powers of all the e.m.fs. of rotation, hence, in vector denotation,[1]

$$\begin{aligned} \dot{P} &= \sum_1^n{}^i [\dot{E}_i', \dot{I}_i]^1 \\ &= S \sum_1^n{}^i [jZ^{ii} \sum_1^n{}^k \dot{I}_k \sin \tau_k^i, \dot{I}_i]^1, \end{aligned} \tag{11}$$

and herefrom the torque, in synchronous watts,

$$\dot{D} = \frac{\dot{P}}{S} = \sum_1^n{}^i [jZ^{ii} \sum_1^n{}^k \dot{I}_k \sin \tau_k^i, \dot{I}_i]^1. \tag{12}$$

[1] See Chapter XV.

The power input, in vector denotation, is

$$\left.\begin{aligned} P_0 &= \sum_1^n i [E_i, I_i] \\ &= \sum_1^n i [E_i, I_i]^1 + \sum_1^n i [E_i, I_i]^j \\ &= P_0^1 + jP_{0j}; \end{aligned}\right\} \tag{13}$$

and therefore,

$P_0^1 =$ true power input;

$P_0^j =$ wattless volt-ampere input;

$P_{a_0} = \sqrt{{P_0^1}^2 + {P_0^j}^2} =$ apparent, or volt-ampere input.

$\frac{P}{P_0^1} =$ efficiency;

$\frac{P}{P_{a_0}} =$ apparent efficiency;

$\frac{D}{P_0^1} =$ torque efficiency;

$\frac{D}{P_{a_0}} =$ apparent torque efficiency;

$\frac{P_0^1}{P_{a_0}} =$ power-factor.

From the n circuits, $i = 1, 2 \ldots n$, thus result n linear equations, with $2n$ complex variables, $\dot{I}_i$ and $\dot{E}_i$.

Hence n further conditions must be given to determine the variables. These obviously are the conditions of operation of the n circuits.

Impressed e.m.fs. $\dot{E}_i$ may be given.

Or circuits closed upon themselves $\dot{E}_i = 0$.

Or circuits connected in parallel $c_i\dot{E}_i = c_k\dot{E}_k$, where c_i and c_k are the reduction factors of the circuits to equal number of effective turns, as discussed before.

Or circuits connected in series: $\frac{\dot{I}_i}{c_i} = \frac{\dot{I}_k}{c_k}$, etc.

When a rotating circuit is connected through a commutator, the frequency of the current in this circuit obviously is the same as the impressed frequency. Where, however, a rotating circuit is permanently closed upon itself, its frequency may differ from the impressed frequency, as, for instance, in the polyphase induction motor it is the frequency of slip, $s = 1 - S$, and the self-inductive reactance of the circuit, therefore, is sx; though in its reaction upon the stationary system the rotating system necessarily is always of full frequency.

As an illustration of this method, its application to the theory of some motor types shall be considered, especially such motors as have either found an extended industrial application, or have at least been seriously considered.

(1) *Polyphase Induction Motor.*

277. In the polyphase induction motor a number of primary circuits, displaced in position from each other, are excited by polyphase e.m.fs. displaced in time-phase from each other by a phase angle equal to the position angle of the coils. A number of secondary circuits are closed upon themselves. The primary usually is the stator, the secondary the rotor.

In this case the secondary system always offers a resultant closed-circuit in the direction of the axis of each primary coil, irrespective of its position.

Let us assume two primary circuits in quadrature as simplest form, and the secondary system reduced to the same number of phases and the same nmuber of turns per phase as the primary system. With three or more primary phases the method of procedure and the resultant equations are essentially the same.

Let, in the motor shown diagrammatically in Fig. 182,

$\dot{E}_0$ and $j\dot{E}_0$, $\dot{I}_0$ and $j\dot{I}_0$, Z_0 = impressed e.m.f. currents and self-inductive impedance respectively of the primary system.

0, $\dot{I}_1$ and $j\dot{I}_1$, Z_1 = impressed e.m.f., currents and self-inductive impedance respectively of the secondary system, reduced to the primary. Z = mutual-inductive impedance between primary and secondary, constant in all directions.

S = speed; $s = 1 - S$ = slip, as fraction of synchronism.

The equation of the primary circuit is then, by (7),

$$E_0 = Z_0 I_0 + Z (I_0 - I_1). \tag{14}$$

The equation of the secondary circuit,

$$0 = Z_1 I_1 + Z (I_1 - I_0) + jSZ (jI_1 - jI_0), \tag{15}$$

from (15) follows,

$$I_1 = I_0 \frac{Z_0 (1 - S)}{Z (1 - S) + Z_1} = I_0 \frac{Zs}{Zs + Z_1}; \tag{16}$$

and, substituted in (14):

Primary current,

$$I_0 = E_0 \frac{Zs + Z_1}{ZZ_0 s + ZZ_1 + Z_0 Z_1}. \tag{17}$$

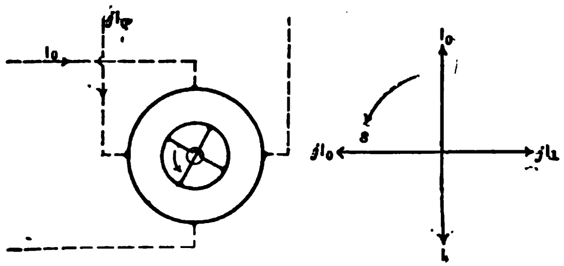


FIG. 182. Polyphase Induction Motor.

Secondary current,

$$I_1 = E_0 \frac{Zs}{ZZ_0 s + ZZ_1 + Z_0 Z_1}. \tag{18}$$

Exciting current,

$$I_{00} = I_0 - I_1 = E_0 \frac{Z_1}{ZZ_0 s + ZZ_1 + Z_0 Z_1}. \tag{19}$$

E.m.f. of rotation,

$$\begin{aligned} E' &= jSZ (jI_1 - jI_0) = SZ \ (I_0 - I_1). \\ &= SE_0 \frac{ZZ_1}{ZZ_0 s + ZZ_1 + Z_0 Z_1} \\ &= (1 - s) E_0 \frac{ZZ_1}{ZZ_0 s + ZZ_1 + Z_0 Z_1} \end{aligned} \tag{20}$$

It is, at synchronism; $s = 0$;

$$\underset{\cdot}{I}_0 = \frac{\dot{E}_0}{Z + Z_0};$$

$$\underset{\cdot}{I}_1 = 0;$$

$$\underset{\cdot}{I}_{00} = \underset{\cdot}{I}_0;$$

$$\underset{\cdot}{E}' = \frac{\dot{E}_0 Z}{Z + Z_0} = \frac{\dot{E}_0}{1 + \frac{Z_0}{Z}}.$$

At standstill $s = 1$;

$$\underset{\cdot}{I}_0 = \frac{\dot{E}_0 (Z + Z_1)}{ZZ_0 + ZZ_1 + Z_0 Z_1};$$

$$\underset{\cdot}{I}_1 = \frac{\dot{E}_0 Z}{ZZ_0 + ZZ_1 + Z_0 Z_1};$$

$$\underset{\cdot}{I}_{00} = \frac{E_0 Z_1}{ZZ_0 + ZZ_1 + Z_0 Z_1};$$

$$\underset{\cdot}{E}' = 0.$$

Introducing as parameter the counter e.m.f., or e.m.f. of mutual induction,

$$\underset{\cdot}{E} = \underset{\cdot}{E}_0 - Z_0 \underset{\cdot}{I}_0, \tag{21}$$

or,

$$\underset{\cdot}{E}_0 = \underset{\cdot}{E} + Z_0 \underset{\cdot}{I}_0, \tag{22}$$

it is, substituted,
Counter e.m.f.,

$$\underset{\cdot}{E} = \underset{\cdot}{E}_0 \frac{ZZ_1}{ZZ_0 s + ZZ_1 + Z_0 Z_1}; \tag{23}$$

hence,
Primary impressed e.m.f.,

$$\underset{\cdot}{E}_0 = \underset{\cdot}{E} \frac{ZZ_0 s + Z_1 + ZZ_0 Z_1}{ZZ_1}, \tag{24}$$

E.m.f. of rotation,

$$\underset{\cdot}{E}' = \underset{\cdot}{E}S = \underset{\cdot}{E}(1 - s). \tag{25}$$

Secondary current,

$$\dot{I}_1 = \frac{\dot{E}s}{Z_1}. \tag{26}$$

Primary current,

$$\dot{I}_0 = \dot{E}\,\frac{Zs + Z_1}{ZZ_1} = \frac{\dot{E}s}{Z_1} + \frac{\dot{E}}{Z}. \tag{27}$$

Exciting current,

$$\dot{I}_{00} = \frac{\dot{E}}{Z} = \dot{E}Y. \tag{28}$$

These are the equations from which the transformer theory of the polyphase induction motor starts.

278. Since the frequency of the secondary currents is the frequency of slip, hence varies with the speed, $S = 1 - s$, the secondary self-inductive reactance also vairies wth the speed, and so the impedance,

$$Z_1 = r_1 - jsx_1. \tag{29}$$

The power output of the motor, per circuit, is,

$$\dot{P} = [\dot{E}', \dot{I}_1]$$

$$= \frac{e_0^2 z^2 s\,(1 - s)}{[ZZ_0 s + ZZ_1 + Z_0 Z_1]^2}\,(r_1 + jsx_1), \tag{30}$$

where the brackets [ ] denote the absolute value of the term included by it, and the small letters, e_0, z, etc., the absolute values of the vectors, $\dot{E}_0$, Z, etc.

Since the imaginary term of power seems to have no physical meaning, it is,

Mechanical power output,

$$P = \frac{e_0^2 z^2 s\,(1 - s) r_1}{[ZZ_0 s + ZZ_1 + Z_0 Z_1]^2}. \tag{31}$$

This is the power output at the armature conductors, hence includes friction and windage.

The torque of the motor is,

$$\dot{D} = \frac{\dot{P}}{1 - s}$$

$$= \frac{e_0^2 z^2 r_1 s}{[ZZ_0 s + ZZ_1 + Z_0 Z_1]^2} + j\,\frac{e_0^2 z^2 x_1 s^2}{[ZZ_0 s + ZZ_1 + Z_0 Z_1]^2}. \tag{32}$$

The imaginary component of torque seems to represent the radial force or thrust acting between stator and rotor. Omitting this we have,

$$D = \frac{e_0^2 z^2 r_1 s}{[ZZ_0 s + ZZ_1 + Z_0 Z_1]^2}. \tag{33}$$

The power input of the motor per circuit is:

$$\begin{aligned} P_0 &= [E_0, I_0] \\ &= e_0^2 \left[1, \frac{Zs + Z_1}{ZZ_0 s + ZZ_1 + Z_0 Z_1}\right] \\ &= P'_0 + j P_0^j \end{aligned} \tag{34}$$

where: $P_0' =$ true power,
$P_0^j =$ reactive or "wattless power,"
$P_{a_0} = \sqrt{P_0'^2 + P_0^{j2}} =$ volt-ampere input.

Herefrom follows power-factor, efficiency, etc.

Introducing the parameter: E, or absolute e, we have:

Power output,

$$\begin{aligned} P &= [E', I_1] \\ &= \left[eS, \frac{es}{Z_1}\right] \\ &= e^2 sS \left[1, \frac{1}{Z_1}\right] \\ &= \frac{e^2 sSr_1}{z_1^2} + j\frac{e^2 s^2 Sx_1}{z_1^2} \\ &= \frac{i_1^2 Sr_1}{s} + ji_1^2 Sx_1. \end{aligned} \tag{35}$$

Power input,

$$\begin{aligned} P_0 &= [E_0, I_0] \\ &= e^2\left[\frac{ZZ_0 s + ZZ_1 + Z_0 Z_1}{ZZ_1}, \quad \frac{Zs + Z_1}{ZZ_1}\right] \\ &= e_2\left[\frac{Z_0 (Zs + Z_1)}{ZZ_1} + 1, \quad \frac{Zs + Z_1}{ZZ_1}\right] \\ &= e^2\left[\frac{Zs + Z_1}{ZZ^1}\right]^2 \left\{\left[Z_0\, 1\, [+ e^2]\, 1, \frac{s}{Z_1} + \frac{1}{Z}\right]\right\} \\ &= e^2\left[\frac{Zs + Z_1}{ZZ_1}\right]^2 \left\{[Z_0, 1] + \frac{e^2 s}{z_1^2}[Z_1, 1] + \frac{e^2}{z^2}[Z, 1]\right\} \\ &= e^2\left[\frac{Zs + Z_1}{ZZ_1}\right]^2 \left\{(r_0 + jx_0) + \frac{e^2 s}{z_1^2}(r_1 + jsx_1) + \frac{e^2}{z^2}(r + jx)\right\} \\ &= i_0^2 (r_0 + jx_0) + i_1^2\left(\frac{r_1}{s} + jx_1\right) + i_{00}^2 (r + jx). \end{aligned} \tag{36}$$

And since,

$$\frac{r_1}{s} = \frac{S + s}{s} r_1 = \frac{S r_1}{s} + r_1,$$

and

$$\frac{i_1^2 S r_1}{s} = P,$$

it is,

$$\dot{P}_0 = (i_0^2 r_0 + i_1^2 r_1 + i_{00}^2 r + P) + j\,(i_0^2 x_0 + i_1^2 x_1 + i_{00}^2 x). \quad (37)$$

Where:

$i_0^2 r_0$ = primary resistance loss,
$i_1^2 r_1$ = secondary resistance loss,
$i_{00}^2 r$ = core loss (and eddy-current loss),
P = output,
$i_0^2 x_0$ = primary reactive volt-amperes,
$i_1^2 x_1$ = secondary reactive volt-amperes,
$i_{00}^2 x$ = magnetizing volt-amperes.

279. Introducing into the equations, (16), (17), (18), (19), (23) the terms,

$$\left.\begin{aligned} \frac{Z_0}{Z} &= \lambda_0, \\ \frac{Z_1}{Z} &= \lambda_1. \end{aligned}\right\} \quad (38)$$

Where λ_0 and λ_1 are small quantities, and: $\lambda = \lambda_0 + \lambda_1$ is the "characteristic constant" of the induction motor theory, it is, Primary current,

$$\dot{I}_0 = \frac{\dot{E}_0}{Z} \frac{s + \lambda_1}{s\lambda_0 + \lambda_1 + \lambda_0\lambda_1} = \frac{\dot{E}_0}{Z} \frac{s + \lambda_1}{s\lambda_0 + \lambda}. \quad (39)$$

Secondary current,

$$\dot{I}_1 = \frac{\dot{E}_0}{Z} \frac{s}{s\lambda_0 + \lambda_1 + \lambda_0\lambda_1} = \frac{\dot{E}_0}{Z} \frac{s}{s\lambda_0 + \lambda_1}. \quad (40)$$

Exciting current,

$$\underset{\cdot}{I}_{00} = \frac{\dot{E}_0}{Z} \frac{\lambda_1}{s\lambda_0 + \lambda_1 + \lambda_0\lambda_1} = \frac{\dot{E}_0}{Z} \frac{\lambda_1}{s\lambda_0 + \lambda_1}. \tag{41}$$

E.m.f. of rotation,

$$\underset{\cdot}{E}' = \underset{\cdot}{E}_0 S \frac{\lambda_1}{s\lambda_0 + \lambda_1 + \lambda_0\lambda_1} = \underset{\cdot}{E}_0 S \frac{\lambda_1}{s\lambda_0 + \lambda_1}. \tag{42}$$

Counter e.m.f.,

$$\underset{\cdot}{E} = \underset{\cdot}{E}_0 \frac{\lambda_1}{s\lambda_0 + \lambda_1 + \lambda_0\lambda_1} = \underset{\cdot}{E}_0 \frac{\lambda_1}{s\lambda_0 + \lambda_1}. \tag{43}$$

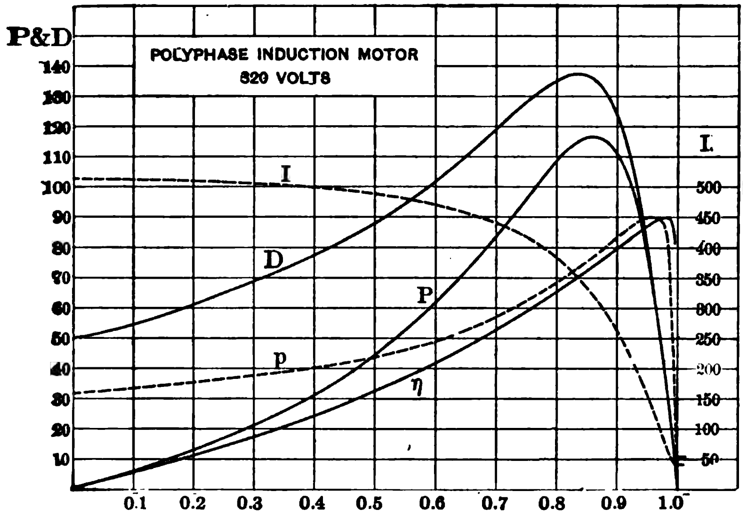


Fig. 183.

280. As an example are shown, in Fig. 183, with the speed as abscissas, the curves of a polyphase induction motor of the constants:

$$e_0 = 320 \text{ volts},$$
$$Z = 1 - 10\,j \text{ ohms},$$
$$Z_0 = Z_1 = 0.1 - 0.3\,j \text{ ohms};$$

hence, $\lambda_0 = \lambda_1 = 0.0307 + 0.0069\,j.$

It is,

$$I_0 = \frac{320 \{10.30\, s + (s + 0.1)\, j\}}{(1.03 + 1.63\, s) + j\,(.11 - 5.99\, s)} \text{ amp.}$$

$$D = \frac{2048\,(1 - s)}{(1.03 + 1.63\, s)^2 + (0.11 - 5.99\, s)^2} \text{ synchronous kw.}$$

$$P = (1 - s)\, D$$

$$\tan \theta'' = \frac{0.11 - 5.99 s}{1.03 + 1.63 s}$$

$$\tan \theta' = \frac{s + 0.1}{10.3\, s};$$

$$\cos (\theta' - \theta'') = \text{power-factor.}$$

Fig. 183 gives, with the speed S as abscissas: the current, I; the power output, P; the torque, D; the power factor, p; the efficiency, η.

The curves show the well-known characteristics of the polyphase induction motor: approximate constancy of speed at all loads, and good efficiency and power-factor within this narrow-speed range, but poor constants at all other speeds.

(2) *Single-phase Induction Motor.*

281. In the single-phase induction motor one primary circuit acts upon a system of closed secondary circuits which are displaced from each other in position on the secondary member.

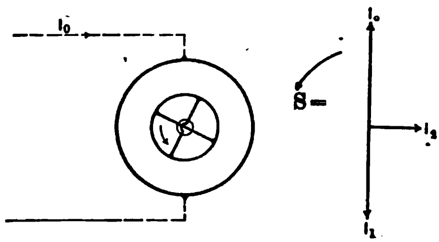


FIG. 184. Single-phase Induction Motor.

Let the secondary be assumed as two-phase, that is, containing or reduced to two circuits closed upon themselves at right angles to each other. While it then offers a resultant closed secondary

circuit to the primary circuit in any position, the electrical disposition of the secondary is not symmetrical, but the directions parallel with the primary circuit and at right angles thereto are to be distinguished. The former may be called the secondary energy circuit, the latter the secondary magnetizing circuit, since in the former direction power is transferred from the primary to the secondary circuit, while in the latter direction the secondary circuit can act magnetizing only.

Let, in the diagram Fig. 184,

$\dot{E}_0$, $\dot{I}_0$, Z_0 = impressed e.m.f., current and self-inductive impedance, respectively, of the primary circuit,

$\dot{I}_1$, Z_1 = current and self-inductive impedance, respectively, of the secondary energy circuit,

$\dot{I}_2$, Z_1 = current and self-inductive impedance, respectively, of the secondary magnetizing circuit,

Z = mutual-inductive impedance,

S = speed,

and let $s_0 = 1 - S^2$ (where s_0 is not the slip).

It is then, by equation (7)

Primary circuit,

$$\dot{E}_0 = Z_0\dot{I}_0 + Z(\dot{I}_0 - \dot{I}_1). \tag{44}$$

Secondary energy circuit,

$$0 = Z_1\dot{I}_1 + Z(\dot{I}_1 - \dot{I}_0) + jSZ\dot{I}_2. \tag{45}$$

Secondary magnetizing circuit,

$$0 = Z_1\dot{I}_2 + Z\dot{I}_2 + jSZ(\dot{I}_0 - \dot{I}_1); \tag{46}$$

hence, from (45) and (46)

$$\dot{I}_1 = \dot{I}_0 \frac{Z(Zs_0 + Z_1)}{Z^2 s_0 + 2ZZ_1 + Z_1^2}, \tag{47}$$

$$\dot{I}_2 = -jS\dot{I}_0 \frac{ZZ_1}{Z^2 s_0 + 2ZZ_1 + Z_1^2}; \tag{48}$$

and, substituted in (44):

Primary current,

$$\dot{I}_0 = \dot{E}_0 \frac{Z^2 s_0 + 2ZZ_1 + Z_1^2}{K}. \tag{49}$$

Secondary energy current,

$$\dot{I}_1 = \dot{E}_0 \frac{Z(Zs_0 + Z_1)}{K}. \tag{50}$$

Secondary magnetizing current,

$$\dot{I}_2 = -jS\dot{E}_0 \frac{ZZ_1}{K}. \tag{51}$$

E.m.f. of rotation of secondary energy circuit,

$$\dot{E}_1 = jSZ\dot{I}_2 = S^2\dot{E}_0 \frac{ZZ_1}{K}. \tag{52}$$

E.m.f. of rotation of secondary magnetizing circuit,

$$\dot{E}_2' = jSZ(\dot{I}_0 - \dot{I}_1) = jS\dot{E}_0 \frac{ZZ_1(Z + Z_1)}{K}; \tag{53}$$

where,

$$K = Z_0(Z^2 s_0 + 2ZZ_1 + Z_1^2) + ZZ_1(Z + Z_1). \tag{54}$$

It is, at synchronism, $S = 1$, $s_0 = 0$:

$$\dot{I}_0 = \dot{E}_0 \frac{2Z + Z_1}{Z_0(2Z + Z_1) + Z(Z + Z_1)};$$

$$\dot{I}_1 = \dot{E}_0 \frac{Z}{Z_0(2Z + Z_1) + Z(Z + Z_1)};$$

$$\dot{I}_2 = -j\dot{E}_0 \frac{Z}{Z_0(2Z + Z_1) + Z(Z + Z_1)}.$$

Hence, at synchronism, the secondary current of the single-phase induction motor does not become zero, as in the polyphase motor, but both components of secondary current become equal.

At standstill, $S = 0$, $s_0 = 1$, it is:

$$\dot{I}_0 = \dot{E}_0 \frac{Z + Z_1}{ZZ_0 + ZZ_1 + Z_0 Z_1};$$

$$\dot{I}_1 = \dot{E}_0 \frac{Z}{ZZ_0 + ZZ_1 + Z_0 Z_1};$$

$$\dot{I}_2 = 0.$$

That is, primary and secondary current corresponding thereto have the same values as in the polyphase induction motor, as was to be expected.

282. Introducing as parameter the counter e.m.f., or e.m.f. of mutual induction,

$$\dot{E} = \dot{E}_0 - Z_0 \dot{I}_0,$$

and substituting for $\dot{I}_0$ from (49), it is,
Primary impressed e.m.f.,

$$\dot{E}_0 = \dot{E}\,\frac{Z_0\,(Z^2 s_0 + 2\,ZZ_1 + Z_1^2) + ZZ_1\,(Z + Z_1)}{ZZ_1\,(Z + Z_1)}\,. \tag{55}$$

Primary current,

$$\dot{I}_0 = \dot{E}\,\frac{Z^2 s_0 + 2\,ZZ_1 + Z_1^2}{ZZ_1\,(Z + Z_1)}\,. \tag{56}$$

Secondary energy circuit,

$$\dot{I}_1 = \dot{E}\,\frac{Zs_0 + Z_1}{Z_1\,(Z + Z_1)} = \frac{s_0 \dot{E}}{Z_1} + \frac{S^2 \dot{E}}{Z + Z_1}\,. \tag{57}$$

$$\dot{E}_1' = S^2 \dot{E}\,\frac{Z}{Z + Z_1}\,. \tag{58}$$

Secondary magnetizing circuit,

$$\dot{I}_2 = -j\,\frac{S\dot{E}}{Z + Z_1}\,. \tag{59}$$

$$\dot{E}_2' = jS\dot{E}. \tag{60}$$

And

$$\dot{I}_0 - \dot{I}_1 = \frac{\dot{E}}{Z}\,. \tag{61}$$

These equations differ from the equations of the polyphase induction motor by containing the term $s_0 = (1 - S^2)$, instead of $s = (1 - S)$, and by the appearance of the terms, $\frac{S\dot{E}}{Z + Z_1}$ and $\frac{S^2\dot{E}}{Z + Z_1}$, of frequency $(1 + S)$, in the secondary circuit.

The power output of the motor is,

$$
\begin{aligned}
P &= [\dot{E}_1, \dot{I}_1] + [\dot{E}_2, \dot{I}_2] \\
&= \frac{S^2 e_0^2 z^2}{[K]^2} \left\{ [ZZ_1, Zs_0 + Z_1] - [Z_1 (Z + Z_1), Z_1] \right\} \\
&= \frac{S^2 e_0^2 z^2 r_1 (s_0 z^2 - z_1^2)}{[K]},
\end{aligned}
\tag{62}
$$

and the torque, in synchronous watts,

$$
D = \frac{P}{S} = \frac{S e_0^2 z^2 r_1 (s_0 z^2 - z_1^2)}{[K]^2}. \tag{63}
$$

From these equations it follows that at synchronism torque and power of the single-phase induction motor are already negative.

Torque and power become zero for,

$$
s_0 z^2 - z_1^2 = o,
$$

hence

$$
S = \sqrt{1 - \left(\frac{z_1}{z}\right)} \tag{64}
$$

that is, very slightly below synchronism.

Let $z = 10$, $z_1 = 0.316$, it is, $S = 0.9995$.

In the single-phase induction motor, the torque contains the speed S as factor, and thus becomes zero at standstill.

Neglecting quantities of secondary order, it is, approximately:

$$
\dot{I}_0 = \dot{E}_0 \frac{Zs_0 + 2 Z_1}{Z (Z_0 s_0 + Z_1) + 2 Z_0 Z_1}, \tag{65}
$$

$$
\dot{I}_1 = \dot{E}_0 \frac{Zs_0 + Z_1}{Z (Z_0 s_0 + Z_1) + 2 Z_0 Z_1}, \tag{66}
$$

$$
\dot{I}_2 = - jS\dot{E}_0 \frac{Z_1}{Z (Z_0 s_0 + Z_1) + 2 Z_0 Z_1}, \tag{67}
$$

$$
\dot{E}_1 = S^2 \dot{E}_0 \frac{ZZ_1}{Z (Z_0 s_0 + Z_1) + 2 Z_0 Z_1}, \tag{68}
$$

$$
\dot{E}_2 = jS\dot{E}_0 \frac{ZZ_1}{Z (Z_0 s_0 + Z_1) + 2 Z_0 Z_1}, \tag{69}
$$

$$
P = \frac{S^2 e_0^2 z^2 r_1 s_0}{[Z (Z_0 s_0 + Z_1) + 2 Z_0 Z_1]^2}, \tag{70}
$$

$$
D = \frac{S e_0^2 z^2 r_1 s_0}{[Z (Z_0 s_0 + Z_1) + 2 Z_0 Z_1]^2}. \tag{71}
$$

This theory of the single-phase induction motor differs from that based on the transformer feature of the motor, Chapter XVI, in that it represents more exactly the phenomena taking place at intermediate speeds, which are only approximated by the transformer theory of the single-phase induction motor.

For studying the action of the motor at intermediate and at low speed, as for instance, when investigating the performance of a starting device, in bringing the motor up to speed, that is, during acceleration, this method so is more suited. An application to the "condenser motor," that is, a single-phase induction motor using a condenser in a stationary tertiary circuit (under an angle, usually 60 degrees, with the primary circuit) is given in the paper on Alternating Current Motors, A. I. E. E. Transactions, 1904.

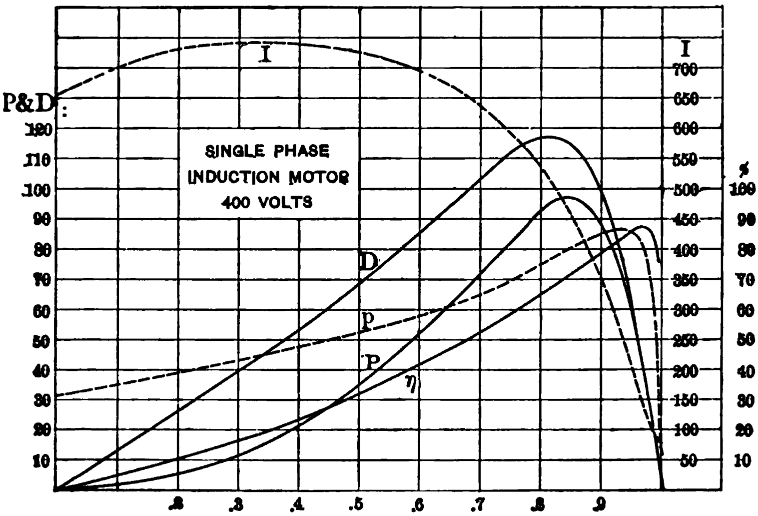


Fig. 185.

283. As example are shown, in Fig. 185, with the speed as abscissas, the curves of a single-phase induction motor, having the constants,

$$e_0 = 400 \text{ volts},$$

$$Z = 1 - 10j \text{ ohms},$$

and

$$Z_0 = Z_1 = 0.1 - 0.3j \text{ ohms};$$

hence,

$$I_0 = 400 \frac{N}{K} \text{ amp.};$$

$$N = (s_0 + 0.2) - j\,(10\,s_0 + 0.6 - 0.6\,S);$$

$$K = (0.1 - 0.3\,j)\,N + (1 - 10\,j)\,(0.1 - j\,(0.3 - 0.3\,S));$$

$$D = \frac{1616\,S s_0}{[K]^2} \text{ synchronous kw.}$$

Fig. 185 gives, with the speed, S, as abscissas: the current, I_0, the power output, P, the torque, D, the power-factor, p, the efficiency, η.

(3) *Polyphase Shunt Motor.*

284. Since the characteristics of the polyphase motor do not depend upon the number of phases, here, as in the preceding, a two-phase system may be assumed: a two-phase stator winding acting upon a two-phase rotor winding, that is, a closed-coil rotor winding connected to the commutator in the same manner as in direct-current machines, but with two sets of brushes in quadrature position excited by a two-phase system of the same frequency. Mechanically the three-phase system here has the advantage of requiring only three sets of brushes instead of four as with the two-phase system, but otherwise the general form of the equations and conclusions are not different.

Let $\dot{E}_0$ and $j\dot{E}_0$ = e.m.fs. impressed upon the stator, $\dot{E}_1$ and $j\dot{E}_1$ = e.m.fs. impressed upon the rotor, θ_0 = phase angle between e.m.f. $\dot{E}_0$ and $\dot{E}_1$, and θ_1 = position angle between the stator and rotor circuits. The e.m.fs. $\dot{E}_0$ and $j\dot{E}_0$ produce the same rotating m.m.f. as two e.m.fs. of equal intensity, but displaced in phase and in position by angle θ_0 from $\dot{E}_0$ and $j\dot{E}_0$, and instead of considering a displacement of phase θ_0 and a displacement of position θ_1 between stator and rotor circuits, we can, therefore, assume zero-phase displacement and displacement in position by angle $\theta_0 + \theta_1 = \theta$. Phase displacement between stator and rotor e.m.fs. is, therefore, equivalent to a shift of brushes, hence gives no additional feature beyond those produced by a shift of the commutator brushes.

Without losing in generality of the problem, we can, therefore, assume the stator e.m.fs. in phase with the rotor e.m.fs., and the polyphase shunt motor can thus be represented diagrammatically by Fig. 186.

285. Let, in the polyphase shunt motor, shown two-phase in diagram Fig. 186,

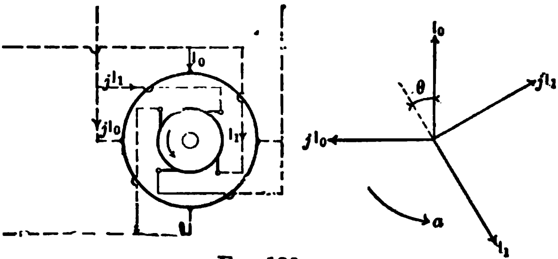


FIG. 186.

$\dot{E}_0$ and $j\dot{E}_0$, $\dot{I}_0$ and $j\dot{I}_0$, Z_0 = impressed e.m.fs., currents and self-inductive impedance respectively of the stator circuits,

$c\dot{E}_0$ and $jc\dot{E}_0$, $\dot{I}_1$ and $j\dot{I}_1$, Z_1 = impressed e.m.fs., currents and self-inductive impedance respectively of the rotor circuits, reduced to the stator circuits by the ratio of effective turns, c,

Z = mutual-inductive impedance,

S = speed; hence $s = 1 - S$ = slip,

θ = position angle between stator and rotor circuits, or "brush angle."

It is then,

Stator,

$$\dot{E}_0 = Z_0\dot{I}_0 + Z(\dot{I}_0 - \dot{I}_1 \cos\theta + j\dot{I}_1 \sin\theta). \tag{72}$$

Rotor,

$$c\dot{E}_0 = Z_1\dot{I}_1 + Z(\dot{I}_1 - \dot{I}_0 \cos\theta - j\dot{I}_0 \sin\theta) + jSZ(j\dot{I}_1 + \dot{I}_0 \sin\theta - j\dot{I}_0 \cos\theta). \tag{73}$$

Substituting,

$$\left.\begin{aligned} \sigma &= \cos\theta + j\sin\theta, \\ \delta &= \cos\theta - j\sin\theta, \end{aligned}\right\} \tag{74}$$

it is,

$$\sigma\delta = 1, \tag{75}$$

and,

$$\dot{E}_0 = Z_0\dot{I}_0 + Z(\dot{I}_0 - \delta\dot{I}_1), \tag{76}$$

$$\begin{aligned} c\dot{E}_0 &= Z_1\dot{I}_1 + Z(\dot{I}_1 - \sigma\dot{I}_0) + jSZ(j\dot{I}_1 - j\sigma\dot{I}_0) \\ &= Z_1\dot{I}_1 + sZ(\dot{I}_1 - \sigma\dot{I}_0). \end{aligned} \tag{77}$$

Herefrom follows,

$$I_0 = E_0 \frac{(s + \delta c)\, Z + Z_1}{sZZ_0 + ZZ_1 + Z_0Z_1}, \tag{78}$$

$$I_1 = E_0 \frac{(\sigma s + c)\, Z + cZ_1}{sZZ_0 + ZZ_1 + Z_0Z_1}, \tag{79}$$

for $c = o$, this gives,

$$I_0 = E_0 \frac{sZ + Z_1}{sZZ_0 + ZZ_1 + Z_0Z_1},$$

$$I_1 = \sigma E_0 \frac{sZ}{sZZ_0 + ZZ_1 + Z_0Z_1},$$

that is, the polyphase induction-motor equations of page 474 *et seq.*, $\sigma = \cos\theta + j\sin\theta = 1^{\frac{\theta}{\pi}}$ representing the displacement of position between stator and rotor currents.

This shows the polyphase induction motor as a special case of the polyphase shunt motor, for $c = o$.

The e.m.fs. of rotation are,

$$\begin{aligned} E_1' &= jSZ\,(jI_1 + I_0 \sin\theta - jI_0\cos\theta) \\ &= SZ\,(\sigma I_0 - I_1); \end{aligned}$$

hence,

$$E_1' \quad SE_0 \frac{Z\,(\sigma Z_1 - cZ_0)}{sZZ_0 + ZZ_1 + Z_0Z_1}. \tag{80}$$

The power output of the motor is

$$\begin{aligned} P &= [E_1, I_1] \\ &= \frac{Se_0^2}{[sZZ_0 + ZZ_1 + Z_0Z_1]^2}\left[(\sigma Z_1 - cZ_0)\, Z,\ (\sigma s + c)\, Z + cZ_0\right], \end{aligned} \tag{81}$$

which, suppressing terms of secondary order, gives,

$$P = \frac{Se_0^2 z^2 \{s(r_1 + c(x_0\sin\theta - r_0\cos\theta)) + c\,(r_1\cos\theta + x_1\sin\theta - cr_0)\}}{[sZZ_0 + ZZ_1 + Z_0Z_1]^2}, \tag{82}$$

for $Sc = o$, this gives,

$$P = \frac{Se_0^2\, z^2 s r_1}{[sZZ_0 + ZZ_1 + Z_0Z_1]^2},$$

the same value as for the polyphase induction motor.

In general, the power output, as given by equation (82), becomes zero,

$$P = o,$$

for the slip,

$$s_0 = -c\frac{r_1 \cos\theta + x_1 \sin\theta - cr_0}{r_1 + c\,(x_0 \sin\theta - r_0 \cos\theta)}. \tag{83}$$

286. It follows herefrom, that the speed of the polyphase shunt motor is limited to a definite value, just as that of a direct-current shunt motor, or alternating-current induction motor. In other words, the polyphase shunt motor is a constant-speed motor, approaching with decreasing load, and reaching at no-load a definite speed,

$$S_0 = 1 - s_0. \tag{84}$$

The no-load speed, S_0, of the polyphase shunt motor is, however, in general not synchronous speed, as that of the induction motor, but depends upon the brush angle, θ, and the ratio, c, of rotor ÷ stator impressed voltage.

At this no-load speed, S_0, the armature current, I_1, of the polyphase shunt motor is in general not equal to zero, as it is in the polyphase induction motor.

Two cases are therefore of special interest:

(1) Armature current $I_1 = o$ at no-load, that is, at slip s_0.

(2) No-load speed equals synchronism, $s_0 = o$.

(1) The armature or rotor current (79),

$$\dot{I}_1 = \dot{E}_0\frac{\sigma s Z + c(Z + Z_1)}{sZZ_0 + ZZ_1 + Z_0Z_1},$$

becomes zero, if,

$$c = -\sigma s\frac{Z}{Z + Z_1},$$

or, since Z_1 is small compared with Z, approximately,

$$c = -\sigma s = -s\,(\cos\theta + j \sin\theta);$$

hence, resolved,

$$c = -s \cos\theta,$$

$$o = s \sin\theta;$$

hence,

$$\left.\begin{aligned} \theta &= o, \\ c &= -s. \end{aligned}\right\} \tag{85}$$

That is, the rotor current can become zero only if the brushes are set in line with the stator circuit or without shift, and in this case the rotor current, and therewith the output of the motor, becomes zero at the slip $s = -c$.

Hence such a motor gives a characteristic curve very similar to that of the polyphase induction motor, except that the stator tends not toward synchronism but toward a definite speed equal to $(1 + c)$ times synchronism.

The speed of such a polyphase motor with commutator can, therefore, be varied from synchronism by the insertion of an e.m.f. in the rotor circuit, and the percentage of variation is the same as the ratio of the impressed rotor e.m.f. to the impressed stator e.m.f. A rotor e.m.f., in opposition to the stator e.m.f. reduces, in phase with the stator e.m.f., increases the free running speed of the motor. In the former case the rotor impressed e.m.f. is in opposition to the rotor current, that is, the rotor returns power to the system in the proportion in which the speed is reduced, and the speed variation, therefore, occurs without loss of efficiency, and is similar in its character to the speed-control of a direct-current shunt motor by varying the ratio between the e.m.f. impressed upon the armature and that impressed upon the field.

Substituting in the equations,

$$\left.\begin{aligned} \theta &= 0, \\ s + c &= s_1 \end{aligned}\right\} \tag{86}$$

it is

$$I_0 = E_0 \frac{s_1 Z + Z_1}{sZZ_0 + ZZ_1 + Z_0 Z_1}, \tag{87}$$

$$I_1 = E_0 \frac{s_1 Z}{sZZ_0 + ZZ_1 + Z_0 Z_1}, \tag{88}$$

$$P = \frac{S e_0^2 z^2 s_1 (r_1 - c r_0)}{[sZZ_0 + ZZ_1 + Z_0 Z_1]^2}. \tag{89}$$

These equations of I_0 and I_1 are the same as the polyphase induction-motor equations, except that the slip from synchronism, s, of the induction motor, is, in the numerator, replaced by the slip from the no-load speed, s_1.

Insertion of voltages into the armature of an induction motor

in time-phase with the primary impressed voltages, and by a commutator, so gives a speed-control of the induction motor without sacrifice of efficiency, with a serious sacrifice, however, of the power-factor, as can be shown from equation (87).

287. (2) The no-load speed of the polyphase shunt motor is in synchronism, that is, the no-load slip, $s_0 = o$, or the motor output becomes zero at synchronism, just as the ordinary induction motor, if, in equation (83),

$$r_1 \cos \theta + x_1 \sin \theta - cr_0 = o;$$

hence,

$$c = \frac{r_1 \cos \theta + x_1 \sin \theta}{r_0}; \tag{90}$$

or, substituting,

$$\frac{x_1}{r_1} = \tan \alpha_1, \tag{91}$$

where α_1 is the phase angle of the rotor impedance, it is,

$$c = \frac{z_1}{r_0} \cos (\alpha_1 - \theta),$$

or,

$$\cos (\alpha_1 - \theta) = \frac{r_0}{z_1} c, \tag{92}$$

or,

$$c = \frac{z_1 \cos (\alpha_1 - \theta)}{r_0}. \tag{93}$$

Since r_0 is usually very much smaller than z_1, if c is not very large, it is,

$$\cos (\alpha_1 - \theta) = o;$$

hence,

$$\theta = 90° - \alpha_1. \tag{94}$$

That is, if the brush angle, θ, is complementary to the phase angle of the self-inductive rotor impedance, α_1, the motor tends toward approximate synchronism at no-load.

Hence:

At given brush angle θ, a value of secondary impressed e.m.f.,

$c\dot{E}_0$, exists, which makes the motor tend to synchronize at no-load (93), and,

At given rotor-impressed e.m.f., $c\dot{E}_0$, a brush angle, θ, exists, which makes the motor synchronize at no-load (92).

288. (3) In the general equations of the polyphase shunt motor, the stator current, equation (78):

$$\dot{I}_0 = \dot{E}_0 \frac{sZ + Z_1 + \delta c Z}{sZZ_0 + ZZ_1 + Z_0 Z_1},$$

can be resolved into a component,

$$\dot{I}_0'' = \dot{E}_0 \frac{sZ + Z_1}{sZZ_0 + ZZ_1 + Z_0 Z_1}, \tag{95}$$

which does not contain c, and is the same value as the primary current of the polyphase induction motor, and a component,

$$\dot{I}_0'' = \dot{E}_0 \frac{\delta c Z}{sZZ_0 + ZZ_1 + Z_0 Z_1}. \tag{96}$$

Resolving $\dot{I}_0''$, it assumes the form,

$$\begin{aligned} \dot{I}_0'' &= \dot{E}_0 \delta c (A_1 + jA_2) \\ &= c\{A_1 \cos\theta + A_2 \sin\theta) - j\,(A_1 \sin\theta - A_2 \cos\theta)\}. \end{aligned} \tag{97}$$

This second component of primary current, $\dot{I}_0''$, which is produced by the insertion of the voltage, $c\dot{E}$, into the secondary circuit, so contains a power component,

$$i_0' = c\,(A_1 \cos\theta + A_2 \sin\theta), \tag{98}$$

and a wattless or reactive component,

$$i_0'' = -jc\,(A_1 \sin\theta - A_2 \cos\theta); \tag{99}$$

where,

$$\dot{I}_0'' = i_0' + ji_0''. \tag{100}$$

The reactive component, i_0'', is zero, if,

$$A_1 \sin\theta - A_2 \cos\theta = o; \tag{101}$$

hence,

$$\tan\theta_1 = +\frac{A_2}{A_1}. \tag{102}$$

In this case, that is, with brush angle θ_1, the secondary impressed voltage, $c\dot{E}$, does not change the reactive current, but

adds or subtracts, depending on the sign of c, energy, and so raises or lowers the speed of the motor: case (1).

The power component, i_0', is zero, if

$$A_1 \cos \theta + A_2 \sin \theta = o, \tag{103}$$

hence,

$$\tan \theta_2 = -\frac{A_1}{A_2}. \tag{104}$$

In this case, that is, with brush angle θ_2, the secondary impressed voltage, $c\dot{E}$, does not change power or speed, but produces wattless lagging or leading current. That is, with the brush position, θ_2, the polyphase shunt motor can be made to produce lagging or leading currents, by varying the voltage impressed upon the secondary, $c\dot{E}$, just as a synchronous motor can be made to produce lagging or leading currents by varying its field excitation, and plotting the stator current, $\dot{I}_0$, of such a polyphase shunt motor, gives the same V-shaped phase characteristics as shown for the synchronous motor in Chapter XXIV.

These two phase angles or brush positions, θ_1 and θ_2, are in quadrature with each other.

There result then two distinct phenomena from the insertion of a voltage by commutator, into an induction motor armature: a change of speed, in the brush position, θ_1, and a change of phase angle, in the brush position, θ_2, at right angles to θ_1.

For any intermediate brush position, θ, a change of speed so results corresponding to a voltage,

$$c\dot{E} \cos (\theta_1 - \theta);$$

and a change of phase angle corresponding to a voltage,

$$\begin{aligned} &c\dot{E} \cos (\theta_2 - \theta), \\ = \; &c\dot{E} \sin (\theta_1 - \theta), \end{aligned}$$

and by choosing then such a position, θ, that the wattless current produced by the component in phase with θ_2, is equal and opposite to the wattless lagging current of the motor proper, $\dot{I}_0'$, the polyphase shunt motor can be made to operate at unity power-factor at all speeds (except very low speeds) and loads. This, however, requires shifting the brushes with every change of load or speed.

When using the polyphase shunt motor as generator of wattless current, that is, at no-load and with brush position, θ_2, it is,

$$s = 0;$$

hence, from (78),

$$\dot{I}_0 = \dot{E}_0 \frac{\delta c Z + Z_1}{Z_1 (Z + Z_0)}, \qquad (105)$$

$$\dot{I}_0' = \frac{\dot{E}_0}{Z + Z_0} \qquad (106)$$

or, approximately,

$$I_0' = \frac{\dot{E}_0}{Z},$$

that is, primary exciting current,

$$\dot{I}_0'' = \dot{E}_0 \frac{\delta c Z}{Z_1 (Z + Z_0)}, \qquad (107)$$

or, approximately, neglecting Z_0 against Z,

$$\left.\begin{aligned} \dot{I}_0'' &= \frac{\dot{E}_0 \delta c}{Z_1} \\ &= \frac{\dot{E}_0 c (\cos\theta - j \sin\theta)}{r_1 - j x_1}, \\ &= \frac{\dot{E}_0 c}{z_1^2} \left\{ (r_1 \cos\theta + x_1 \sin\theta) + j (x_1 \cos\theta - r_1 \sin\theta) \right\}, \end{aligned}\right\} \qquad (108)$$

and, since the power component vanishes,

$$r_1 \cos\theta + x_1 \sin\theta = 0,$$

or

$$\tan\theta_2 = -\frac{r_1}{x_1}. \qquad (109)$$

Substituting (109) in (108) gives,

$$\dot{I}_0'' = \frac{\dot{E} j_0 c}{z_1^2} (x_1 \cos\theta_2 - r_1 \sin\theta_2)$$

$$= j \frac{\dot{E}_0 c}{z_1}; \qquad (110)$$

and,

$$\dot{I}_0 = \frac{\dot{E}_0}{Z} + j \frac{E_0 c}{z_1}$$

$$= \dot{E}_0 \left\{ \frac{r}{z^2} + j \left(\frac{c}{z_1} - \frac{x}{z^2} \right) \right\}. \qquad (111)$$

289. In the exact predetermination of the characteristics of such a motor, the effect of the short-circuit current under the brushes has to be taken into consideration, however. When a commutator is used, by the passage of the brushes from segment to segment coils are short-circuited. Therefore, in addition to the circuits considered above, a closed circuit on the rotor has to be introduced in the equations for every set of brushes. Reduced to the stator circuit by the ratio of turns, the self-inductive impedance of the short-circuit under the brushes is very high, the current, therefore, small, but still sufficient to noticeably affect the motor characteristics, at least at certain speeds. Since, however, this phenomenon will be considered in the chapters on the single-phase motors, it may be omitted here.

(4) *Polyphase Series Motor.*

290. If in a polyphase commutator motor the rotor circuits are connected in series to the stator circuits, entirely different characteristics result, and the motor no more tends to synchronize

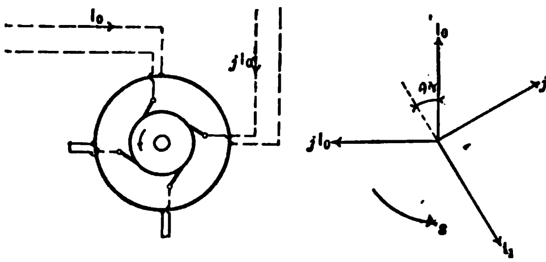


FIG. 187.

as the induction motor with short-circuited secondary, nor approaches a definite speed at no-load, as a shunt motor, but with decreasing load the speed increases indefinitely. In short, the motor has similar characteristics as the direct-current series motor.

In this case we may assume the stator reduced to the rotor by the ratio of effective turns.

Let then, in the motor shown diagrammatically in Fig. 187:

$\dot{E}_0$ and $j\dot{E}_0$, $\dot{I}_0$ and $j\dot{I}_0$, Z_0 = impressed e.m.fs., currents and self-inductive impedance of stator circuits, assumed as two-

phase, and reduced to the rotor circuits by the ratio of effective turns, c,

E_1 and jE_1, I_1, and jI_1, Z_1 = impressed e.m.fs. currents and self-inductive impedance of rotor circuits,

Z = mutual-inductance impedance,

S = speed; and, $s = 1 - S$ = slip,

θ = brush angle,

c = ratio of effective stator turns to rotor turns.

If, then,

E and jE = impressed e.m.fs., I and jI = currents of motor, it is,

$$I_1 = I, \tag{112}$$

$$I_0 = cI, \tag{113}$$

$$cE_0 + E_1 = E; \tag{114}$$

and, stator, by equation (7),

$$E_0 = Z_0 I_0 + Z\,(I_0 - I_1 \cos\theta + jI_1 \sin\theta); \tag{115}$$

rotor,

$$E_1 = {}_1 I_1 + Z\,(I_1 - I_0 \cos\theta - jI_0 \sin\theta) + jSZ\,(jI_1 + I_0 \sin\theta - jI_0 \cos\theta); \tag{116}$$

and, e.m.f. of rotation,

$$E_1' = jSZ\,(jI_1 + I_0 \sin\theta - jI_1 \cos\theta). \tag{117}$$

Substituting (112), (113) in (115), (116), (117), and (115), (116) in (114) gives,

$$I = \frac{E}{(c^2 Z_0 + Z_1) + Z\,(1 + c^2 - 2\,c \cos\theta) + SZ\,(c\sigma - 1)}; \tag{118}$$

where,

$$\sigma = \cos\theta + j \sin\theta, \tag{119}$$

and

$$E_1' = \frac{SZ\dot{E}\,(c\sigma - 1)}{(c^2 Z_0 + Z_1) = Z\,(1 + c^2 - 2\,c\cos\theta) + SZ\,(c\sigma - 1)}; \tag{120}$$

and the power output,

$$P = [E_1', I_1]'$$

$$= \frac{Se^2\{c\,(r\cos\theta + x\sin\theta) - r\}}{[(c^2 Z_0 + Z_1) + Z\,(1 + c^2 - 2\,c\cos\theta) + SZ\,(c\sigma - 1)]^2}. \tag{121}$$

The characteristics of this motor entirely vary with a change of the brush angle, θ. It is, for $\theta = 0$: $P = \frac{Se^2r(c-1)}{[K]^2}$, hence very small, while for $\theta = 90°$: $P = \frac{Se^2(xc-r)}{[K]^2}$, hence considerable. Some brush angles give positive P: motor, others negative, P, generator.

In such a motor, by choosing θ and c appropriately, unity power-factor or leading current as well as lagging current can be produced.

That is, by varying c and θ, the power output and therefore the speed, as well as the phase angle of the supply current or

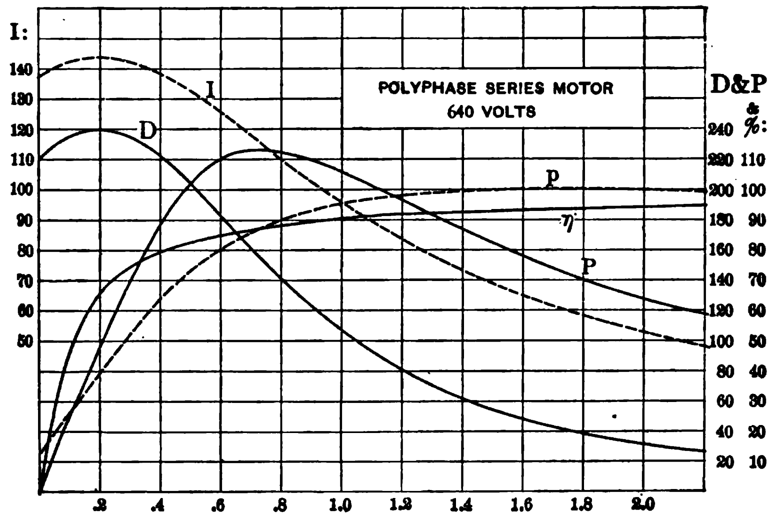


FIG. 188.

the power-factor can be varied, and the machine used to produce lagging as well as leading current, similarly as the polyphase shunt motor or the synchronous motor. Or, the motor can be operated at constant unity power factor at all loads and speeds (except very low speeds), but in this case requires changing the brush angle, θ, and the ratio, c, with the change of load and speed. Such a change of the ratio, c, of rotor $\div$ stator turns can be produced by feeding the rotor (or stator) through

a transformer of variable ratio of transformation, connected with its primary circuit in series to the stator (or rotor).

291. As example is shown in Fig. 188, with the speed as abscissas, and values from standstill to over double synchronous speed, the characteristic curves of a polyphase series motor of the constants,

$$e = 640 \text{ volts},$$

$$Z = 1 - 10\,j \text{ ohms},$$

$$Z_0 = Z_1 = 0.1 - 0.3\,j \text{ ohms},$$

$$c = 1,$$

$$\theta = 37°;\ (\sin\theta = 0.6;\ \cos\theta = 0.8);$$

hence,

$$\dot{I} = \frac{640}{(0.6 + 5.8\,S) - j\,(4.6 - 2.6\,S)} \text{ amp.},$$

$$P = \frac{4673\,S}{(0.6 + 5.8S)^2 + (4.6 - 2.6S)^2} \text{ kw.}$$

As seen, the motor characteristics are similar to those of the direct-current series motor: very high torque in starting and at low speed, and a speed which increases indefinitely with the decrease of load. That is, the curves are entirely different from those of the induction motors shown in the preceding. The power-factor is very high, much higher than in induction motors, and becomes unity at the speed $S = 1.77$, or about one and three-quarter synchronous speed.

CHAPTER XXVII.

SINGLE-PHASE COMMUTATOR MOTORS.

292. The three main types of alternating-current motors in extended use are:

(1) The synchronous motor. This is the alternating-current generator with direct-current field excitation, operating as motor. It is characterized by its absolute rigidity of speed, that is, the motor runs in step with the generator. There is nothing corresponding to the synchronous motor in the field of direct-current motors.

(2) The polyphase induction motor, characterized by approximately constant speed, and corresponding to the direct-current shunt motor.

(3) The single-phase commutator motor, usually built so that its speed varies over a wide range with a variation of load, that is, increases with decrease of load, and so corresponding to the direct-current series motor.

The single-phase commutator motor then consists of an armature connected to a multisegmental commutator and brushes, in the same manner as in direct-current machines, and a field, which in this case must be well laminated, to carry an alternating magnetic flux. The field structure or stator contains a field exciting winding, corresponding to that of a direct-current machine, and usually a second winding at right angles thereto, which transfers power between stator and rotor by alternating magnetic induction, and compensates for armature reaction and self-induction. Both stator windings sometimes are combined as components of one single winding. Occasionally also, a second set of brushes, at right angles to the main brushes, is used on the armature for excitation.

To investigate the action and performance of single-phase commutator motors, the method described in the preceding chapter is best suited.

In polyphase induction motors a distributed rotor and stator

winding is used, that is, a structure having uniform effective reluctance and thus exciting impedance in all directions; a polar construction of the stator winding results in lower power-factor, and thus is permissible only in very small motors — as fan motors, etc. In direct-current motors a polar construction of the stator is almost exclusively used, that is, a construction in which the reluctance in the direction of the magnetic field, which produces the e.m.f. of rotation, is very much smaller than in the direction at right angles thereto. In single-phase alternating commutator motors (as series motors, repulsion motors, etc.) both stator constructions may be used, and in the most general case we must, therefore, assume the mutual induction and therewith the exciting impedance in the direction of the axis of the rotor circuit, Z', as different from the exciting impedance, Z, at right angles to this axis. When different, the latter Z is frequently larger than the former, Z', since Z is in the direction of the magnetic flux which produces the e.m.f. of rotation, that is, corresponds to the field excitation, while in the direction of Z' energy transfer between stator and rotor, or compensation of rotor reaction takes place, but magnetic flux in the direction, Z', does not produce e.m.f. and thereby power by the rotation of the motor.

The stator winding can, therefore, be considered as consisting of two components, or may be constructed of two separate circuits, in the directions in line and at right angles to the rotor winding, which circuits may be connected in series or energized in any other manner, as, for instance, by exciting one by the impressed e.m.f., short-circuiting the other upon itself, etc.

The stator winding at right angles to the rotor winding, which corresponds to the field winding of the direct-current machine, is called *field coil* or-*winding*, *exciting winding*, *main winding*, etc., the stator winding parallel with, or in line with the armature winding, is called the *cross-coil* or-*winding*, *transformer winding primary winding*, *power-transferring winding*, or *compensating, winding*. With a completely distributed winding and an angle θ between the axes of the stator and the rotor circuits (the angle of brush position), the exciting or magnetizing component of the stator winding is $I_0 \sin \theta$, the compensating or power transferring component $I_0 \cos \theta$, if I_0 = stator current, as shown in diagram Fig. 189. When using separate circuits for the two stator com-

ponents, they can even magnetically be arranged differently, and usually are so, and a unitooth or polar arrangement chosen for the field-exciting circuit, a distributed winding for the compensating circuit. In this case obviously, when reducing all circuits to each other by the ratio of effective turns, the resultant vector of the distributed winding has to be used.

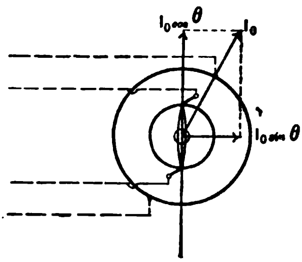


FIG. 189.

As limit case, with zero compensating winding, appears the plain, uncompensated series motor, consisting of a polar-field exciting circuit and an armature with brushes at the neutral or at right angles to the field, as shown in Fig. 190; as a further limit case may be considered a motor with zero field exciting winding on the stator and excitation of

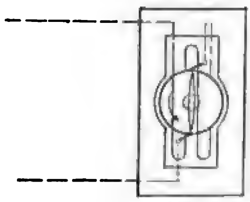


FIG. 190.

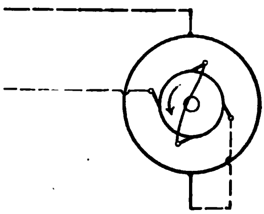


FIG. 191.

the rotor by a second system of brushes at right angles to the main or power brushes, as shown diagrammatically in Fig. 191, and developed by Winter and Eichberg and Latour.

293. In alternating-current commutator motors, the short-circuit current in the coils under the brushes during commutation has to be taken into consideration. While with numerous commutator segments, carbon brushes and possibly an additional resistance in the commutator leads, as occasionally used in such motors, these short-circuit currents may be moderate, they still are sufficient noticeably to affect the constants of the motor, especially at high speeds, where the main current is small, and at standstill, where the main magnetic flux is very large. Furthermore, the character of the commutation of the motor,

and therefore its operativeness, essentially depends upon the value and the phase of the short-circuit currents under the commutator brushes. An excessive short-circuit current gives destructive sparking by high-current density under the brushes and arcing at the edge of the brushes due to the great and sudden change of current in the armature coil when leaving the brush. But even with a moderate short-circuit current, the sparking at the commutator may be destructive and the motor therefore inoperative, if the phase of the short-circuit current greatly differs from that of the current in the armature coil after it leaves the brush, and so a considerable and sudden change of current must take place at the moment when the armature coil leaves the brush. That is, perfect commutation occurs, if the short-circuit current in the armature coil under the commutator brush at the moment when the coil leaves the brush has the same value and the same phase as the main-armature current in the coil after leaving the brush. The commutation of such a motor therefore is essentially characterized by the difference between the main-armature current after, and the short-circuit current before leaving the brush. The investigation of the short-circuit current under the commutator brushes therefore is of fundamental importance in the study of the alternating-current commutator motor, and the control of this short-circuit current the main problem of alternating-current commutator motor design.

294. In its most general form, the single-phase commutator motor, as represented by Fig. 192, comprises two armature or rotor circuits in quadrature with each other, the *main*, or *energy*, and the *exciting* circuit of the armature, which by a multisegmental commutator are connected to two sets of brushes in quadrature position with each other, and gives rise to two short circuits, also in quadrature position with each other and caused respectively by the main and by the exciting brushes, and two stator circuits, the field, or exciting, and the cross, or compensating circuit, also in quadrature with each other, and in line respectively with the exciting and the main armature circuit.

These circuits may be separate, or may be parts or components of the same circuit. They may be massed together in a single

slot of the magnetic structure, as frequently the case with the field circuit, and shown so in Fig. 190, or may be distributed over the whole periphery, as frequently done with the armature windings, and then as their effective number of turns must be considered their vector resultant, that is,

$$n = \frac{2}{\pi} n';$$

where n' = actual number of turns in series between the armature brushes, and distributed over the whole periphery, that is, an arc of 180 degrees electrical. Or the windings of the circuit may be distributed only over an arc of the periphery of angle ω,

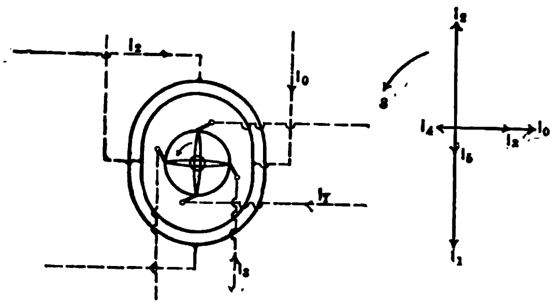


FIG. 192.

as frequently the case with the compensating winding distributed in the pole-face of pole-arc, ω; or with fractional pitch armature windings of pitch ω. In this case, the effective number of turns is,

$$n = \frac{\sin \omega}{\omega} n',$$

where n' with a fractional pitch armature winding is the number of series turns in the pitch angle, ω, that is,

$$n' = \frac{\omega}{\pi} n'',$$

n'' being the number of turns in series between the brushes, since in the space $(\pi - \omega)$ outside of the pitch angle the armature conductors neutralize each other, that is, conductors

carrying current in opposite direction are superposed upon each other. See fractional pitch windings, chapter "Commutating Machine," Theoretical Elements of Electrical Engineering.

These four main circuits of the single-phase commutator motor may be energized in many different ways: connected in series, or in parallel with each other, or connected to different e.m.fs., or some short-circuited and so energized by induction from others, etc., or some of the circuits may be absent, as frequently the armature exciting circuit is, to avoid the use of two sets of brushes.

295. Let:

E_0, I_0, Z_0 = impressed e.m.f., current and self-inductive impedance of the magnetizing or exciter circuit of stator (field coils), reduced to the rotor energy circuit by the ratio of effective turns, c_0,

E_1, I_1, Z_1 = impressed e.m.f., current and self-inductive impedance of the rotor energy circuit (or circuit at right angles to I_0),

E_2, I_2, Z_2 = impressed e.m.f., current and self-inductive impedance of the stator compensating circuit (or circuit parallel to I_1; the "cross-coil" of the Eickemeyer motor), reduced to the rotor circuit by the ratio of effective turns, c_2.

E_3, I_3, Z_1 = impressed e.m.f., current and self-inductive impedance of the exciting circuit of the rotor, or circuit parallel to I_0,

I_4, Z_4 = current and self-inductive impedance of the short-circuit under the brushes, I_1, reduced to the rotor circuit,

I_5, Z_5 = current and self-inductive impedance of the short-circuit under the brushes, I_3, reduced to the rotor circuit,

Z = mutual impedance of field excitation, that is, in the direction of I_0, I_3, I_4,

Z' = mutual impedance of armature reaction, that is, in the direction of I_1, I_2, I_5.

Z' usually either equals Z, or is smaller than Z.

I_4 and I_5 are very small, Z_4 and Z_5 very large quantities.

Let S = speed, as fraction of synchronism.

Using then the general equations (7) Chapter XXVI, which apply to any alternating-current circuit revolving with speed S

through a magnetic field energized by alternating-current circuits, gives for the six circuits of the general single-phase commutator motor the six equations:

$$\dot{E}_0 = Z_0\dot{I}_0 + Z\ (\dot{I}_0 + \dot{I}_3 - \dot{I}_4), \tag{1}$$
$$\dot{E}_1 = Z_1\dot{I}_1 + Z'\ (\dot{I}_1 + \dot{I}_5 - \dot{I}_2) + jSZ\ (\dot{I}_0 + \dot{I}_3 - \dot{I}_4), \tag{2}$$
$$\dot{E}_2 = Z_2\dot{I}_2 + Z'\ (\dot{I}_2 - \dot{I}_1 - \dot{I}_5), \tag{3}$$
$$\dot{E}_3 = Z_1\dot{I}_3 + Z\ (\dot{I}_3 + \dot{I}_0 - \dot{I}_4) + jSZ\ (\dot{I}_2 - \dot{I}_1 - \dot{I}_5), \tag{4}$$
$$o = Z_4\dot{I}_4 + Z\ (\dot{I}_4 - \dot{I}_0 - \dot{I}_3) + jSZ\ (\dot{I}_1 + \dot{I}_5 - \dot{I}_2), \tag{5}$$
$$o = Z_5\dot{I}_5 + Z'\ (\dot{I}_5 + \dot{I}_1 - \dot{I}_2) + jSZ\ (\dot{I}_0 + \dot{I}_3 - \dot{I}_4). \tag{6}$$

These 6 equations contain 10 variables,

$$\dot{I}_0, \dot{I}_1, \dot{I}_2, \dot{I}_3, \dot{I}_4, \dot{I}_5, \dot{E}_0, \dot{E}_1, \dot{E}_2, \dot{E}_3,$$

and so leave 4 independent variables, that is, 4 conditions, which may be chosen.

Properly choosing these 4 conditions, and substituting them into the 6 equations (1) to (6), so determines all 10 variables. That is, the equations of practically all single-phase commutator motors are contained as special cases in above equations, and derived therefrom, by substituting the 4 conditions, which characterize the motor.

Let then, in the following, the reduction factors to the armature circuit, or the ratio of effective turns of a circuit i to the effective turns of the armature circuit, be represented by c_i. That is,

$$c_i = \frac{\text{number of effective turns of circuit } i}{\text{number of effective turns of armature circuit}};$$

and if $\dot{E}_i$, $\dot{I}_i$, Z_i are voltage, current and impedance of circuit i, reduced to the armature circuit, then the actual voltage, current and impedance of circuit i are,

$$c_i\dot{E}_i, \frac{\dot{I}_i}{c_i}, c_i^2 Z_i,$$

as discussed in preceding chapters.

296. The different forms of single-phase commutator motors, which have been considered for railway work, are, as shown diagrammatically in Fig. 193:

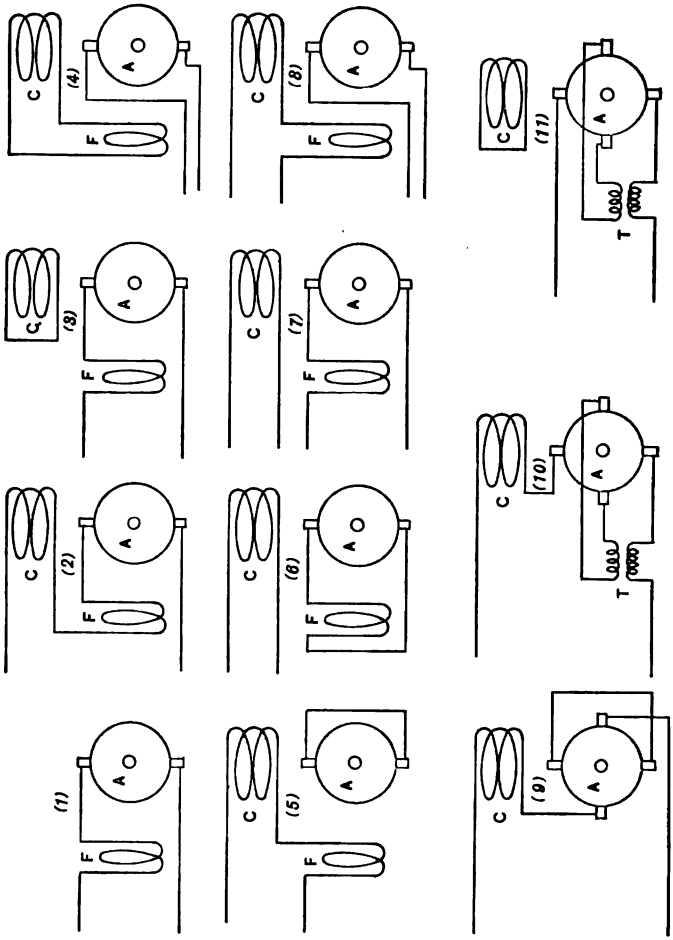


FIG. 193.

(1) Series Motor:

$$e = c_0 E_0 + E_1; \; I_0 = c_0 I_1; \; I_2 = 0; \; I_3 = 0.$$

(2) Conductively-compensated series motor (Eickemeyer motor):

$$e = c_0 E_0 + E_1 + c_2 E_2; \; I_0 = c_0 I_1; \; I_2 = c_2 I_1; \; I_3 = 0.$$

(3) Inductively-compensated series motor (Eickemeyer motor):

$$e = c_0 E_0 + E_1;\ E_2 = 0;\ \underset{\cdot}{I}_0 = c_0 \underset{\cdot}{I}_1;\ \underset{\cdot}{I}_3 = 0.$$

(4) Inverted repulsion motor, or series motor with secondary excitation:

$$e = \underset{\cdot}{E}_1;\ c_0 \underset{\cdot}{E}_0 + c_2 \underset{\cdot}{E}_2 = 0;\ c_2 \underset{\cdot}{I}_0 = c_0 \underset{\cdot}{I}_2;\ \underset{\cdot}{I}_3 = 0.$$

(5) Repulsion motor (Thomson motor):

$$e = c_0 \underset{\cdot}{E}_0 + c_2 \underset{\cdot}{E}_2;\ \underset{\cdot}{E}_1 = 0;\ c_2 \underset{\cdot}{I}_0 = c_0 \underset{\cdot}{I}_2;\ I_3 = 0.$$

(6) Repulsion motor with secondary excitation:

$$e = c_2 \underset{\cdot}{E}_2;\ c_0 \underset{\cdot}{E}_0 + \underset{\cdot}{E}_1 = 0;\ \underset{\cdot}{I}_0 = c_0 \underset{\cdot}{I}_1;\ I_3 = 0.$$

(7) Series repulsion motor with secondary excitation:

$$e_1 = c_0 \underset{\cdot}{E}_0 + E_1:\ e_2 = \underset{\cdot}{E}_2;\ \underset{\cdot}{I}_0 = c_0 \underset{\cdot}{I}_1;\ \underset{\cdot}{I}_3 = 0.$$

(8) Series repulsion motor with primary excitation:

$$e_1 = \underset{\cdot}{E}_1;\ e_2 = c_0 \underset{\cdot}{E}_0 + c_2 \underset{\cdot}{E}_2;\ c_2 \underset{\cdot}{I}_0 = c_0 \underset{\cdot}{I}_2;\ I_3 = 0.$$

(9) Compensated repulsion motor (Winter and Eichberg motor):

$$e = c_2 \underset{\cdot}{E}_2 + c_3 \underset{\cdot}{E}_3;\ \underset{\cdot}{E}_1 = 0;\ \underset{\cdot}{I}_0 = 0;\ c_3 \underset{\cdot}{I}_2 = c_2 \underset{\cdot}{I}_3.$$

(10) Rotor-excited series motor with conductive compensation:

$$e = \underset{\cdot}{E}_1 + c_2 \underset{\cdot}{E}_2 + c_3 \underset{\cdot}{E}_3;\ \underset{\cdot}{I}_2 = c_2 \underset{\cdot}{I}_1;\ \underset{\cdot}{I}_3 = c_3 \underset{\cdot}{I}_1;\ \underset{\cdot}{I}_0 = 0.$$

(11) Rotor-excited series motor with inductive compensation:

$$e = \underset{\cdot}{E}_1 + c_3 \underset{\cdot}{E}_3;\ \underset{\cdot}{E}_2 = 0;\ \underset{\cdot}{I}_0 = 0;\ \underset{\cdot}{I}_3 = c_3 \underset{\cdot}{I}_1.$$

Numerous other combinations can be made and have been proposed.

All of these motors have series characteristics, that is, a speed increasing with decrease of load.

(1) to (8) contain only one set of brushes on the armature; (9) to (11) two sets of brushes in quadrature.

Motors with shunt characteristic, that is, a speed which does not vary greatly with the load, and reaches such a definite limiting value at no load that the motor can be considered a

constant-speed motor, can also be derived from above equations. For instance:

Compensated Shunt Motor (Fig. 194.):

$$\underset{\cdot}{E}_1 = 0; \; c_2\underset{\cdot}{E}_2 = c_3\underset{\cdot}{E}_3 = e; \; \underset{\cdot}{I}_0 = 0.$$

In general, a series characteristic results, if the field-exciting circuit and the armature energy circuit are connected in series with each other directly or inductively, or related to each other so that the currents in the two circuits are more or less proportional to each other. Shunt characteristic results, if the voltage impressed upon the armature energy circuit, and the field excitation, or rather the magnetic field-flux, whether produced or induced by the internal reactions of the motor, are constant, or, more generally, proportional to each other.

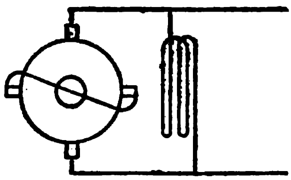


Fig. 194.

It is not possible here to discuss the different types of motors in detail; this must be left to the student. However, as illustrative of the method of treatment the repulsion motor, and the series repulsion motor, which latter includes as limit cases the inductively-compensated series motor, and the repulsion motor with secondary excitation, may be investigated.

A. Repulsion Motor.

297. The repulsion motor in some respects is an induction motor or transformer motor; that is, a motor in which the main current enters the primary member or field only, while in the secondary member, or armature, a current is produced, and the action may be said to be due to the repulsive thrust between this current and the magnetism which causes it, and this feature gave the motor its name.

As stated under the heading of induction motors, a multiple-circuit armature is required for the purpose of having always secondary circuits in inductive relation to the primary during the rotation. If with a single-coil field, these secondary circuits are constantly closed upon themselves as in an induction motor, the primary circuit does not exert a rotary effect on the armature while at rest, since in half of the armature coils the

current is in such a direction as to give a rotary effort in the one direction, and in the other half the current is so directed as to give a rotary effort in the opposite direction, as shown by the arrows in Fig. 195.

In the induction motor a second magnetic field is used to act upon the currents produced by the first, or inducing magnetic field, and thereby cause a rotation. That means, the motor

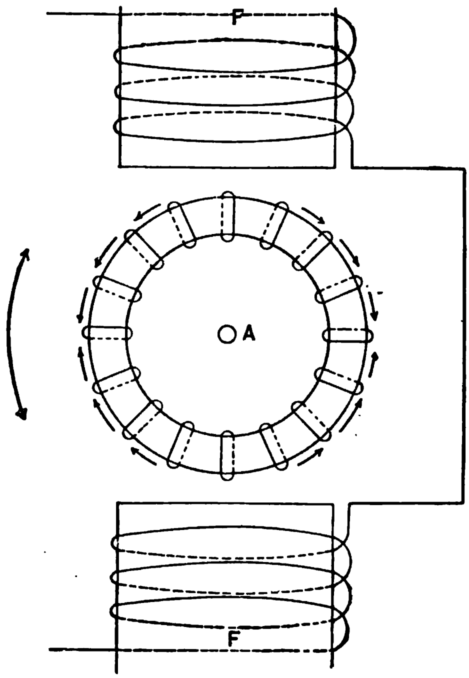


FIG. 195.

consists of a primary energy circuit, producing in the armature the secondary currents, and a primary magnetizing circuit producing the magnetism to act upon these secondary currents.

In the polyphase induction motor both functions of the primary circuit are usually combined in the same coils; that is, each primary coil produces secondary currents, and produces magnetic flux acting upon secondary currents due to another primary coil.

In the single-phase induction motor the primary coil produces currents in the armature winding, and the magnetic flux, which acts upon these currents to produce the torque of the motor, is set up by the armature reactions at speed, and has to be established by some starting device at standstill.

In the repulsion motor the difficulty due to the equal and opposite rotary efforts, caused by the secondary currents when

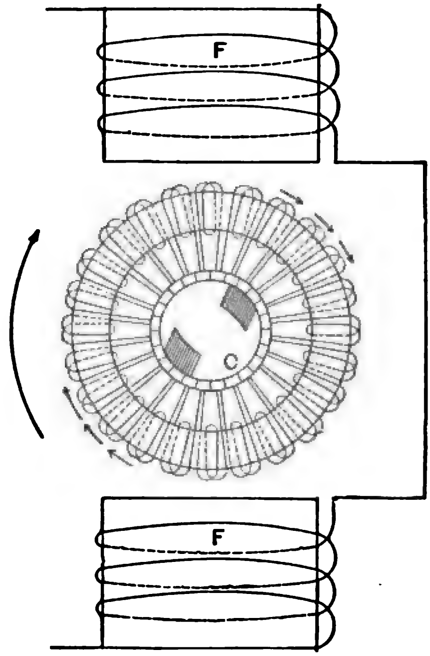


FIG. 196.

acted upon by the inducing magnetic field, is overcome by having the armature coils closed upon themselves, either short-circuited or through resistance, in that position where the secondary currents give a rotary effort in the right direction, while the armature coils are open-circuited in the position where the rotary effort of the secondary currents would be in opposition to the desired rotation. This requires means to open or close the circuit of the armature coils and thereby introduces the commutator.

Thus the general construction of the repulsion motor is as shown in Figs. 196 and 197 diagrammatically as bipolar motor. The stator construction is similar to that of a single-phase induction motor, *F*, the rotor, *A*, shown diagrammatically as ring-wound, consists of a number of coils connected to a segmental commutator, *C*, in general in the same way as in continuous-current machines. Brushes standing under an angle with the

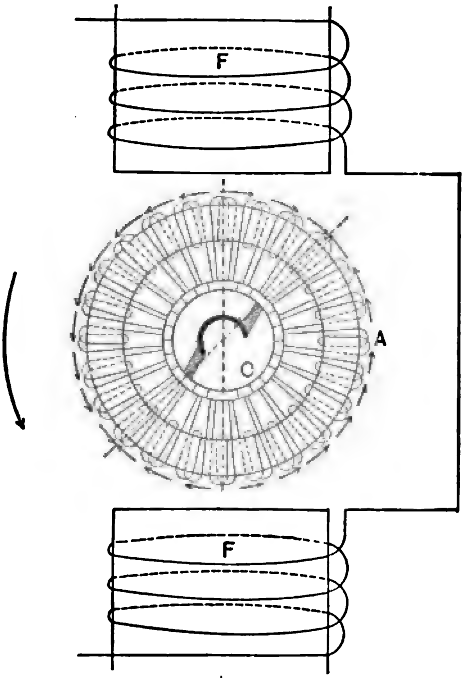


FIG. 197.

direction of the magnetic field short-circuit either a part of the armature coils as shown in Fig. 196, or the whole armature by a connection from brush to brush as shown in Fig. 197. The latter arrangement is usually employed, since it utilizes the whole armature winding and with a distributed field-winding, as mostly used, gives a better mutual induction.

As primary winding in the repulsion motor usually a more or less completely distributed winding is employed, to get good mutual induction between primary and secondary.

Since primary and secondary current are approximately opposite to each other and of approximately equal m.m.f., it follows that these currents are in the same direction, and so add in some parts of the periphery of the motor, an arc, F, equal to twice the angle between the axes of stator and rotor circuit, as shown in Fig. 198 for a motor with full pitch, and in Fig. 199 for a motor with fractional pitch armature winding, while in the supplementary arc, C, of the periphery, the two currents are in opposition to each other, and so neutralize each other.

The primary turns in the arc, F, thus act as magnetizing or field-exciting winding, and represent the circuit, I_0, in the preceding equations, while the turns in the arc, C, act as com-

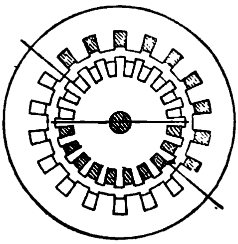


Fig. 198.

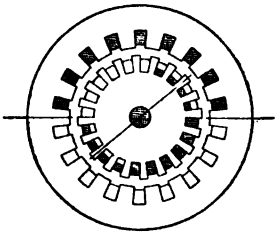


Fig. 199.

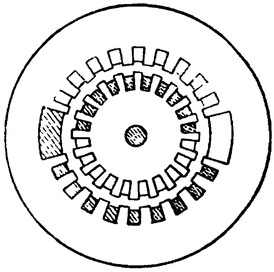


Fig. 200.

pensating and power transferring winding, the circuit, I_2, of the preceding general equations.

Frequently separate windings or circuits are used for F and C, and the winding, F, massed in a single slot or interpolar space, while the winding, C, is distributed in the pole-face, as shown diagrammatically in Fig. 200.

298. Amongst the single-phase commutator motors, the repulsion motor takes a separate and distinctive position by the characteristics of its magnetic field, which exert an important influence on the commutation.

In the uncompensated series motor, the magnetic field at right angles with the main field, F, or the crossfield in the direction, I_1, is in phase with the main field, and the magnetic field of the armature reaction is approximately proportional to the armature current. In the perfectly compensated series motor

this crossfield disappears, and practically disappears in the inductively compensated series motor (in which the compensation is practically complete). In an undercompensated motor, a part of this crossfield of armature reaction remains, while in an over-compensated motor it is reversed, but in either case it is still in phase with the main field.

In the repulsion motor, however, in the direction of the armature axis, $\dot{I}_1$, power is transferred from stator to rotor, and so a transformer field exists, due to the resultant m.m.f. of primary and secondary current, hence, as in any transformer, approximately in quadrature with the current, while the main field is in phase with the current.

The two components of the magnetic field of the repulsion motor, the main field in direction F, and the crossfield C, in quadrature position thereto, in space, so are in quadrature with each other in time phase, hence produce by their combination a rotating field, which revolves with uniform intensity and velocity at that speed, where the two components are equals.

The main field, Φ_f, is approximately proportional and in phase with the current, hence decreases with increasing speed, while the crossfield, Φ'_c, is approximately proportional to the impressed voltage — since it generates the e.m.f. which consumes this voltage — and in quadrature thereto.

299. In the short-circuited armature of the repulsion motor, an e.m.f., $\dot{E}$, is generated by the alternation of the crossfield through it, which e.m.f. is proportional to the frequency, f, and to the intensity of the crossfield, Φ'_c, and in quadrature therewith, hence,

$$\dot{E} = -jbf\Phi'_c,$$

where b is a constant.

A second e.m.f., $\dot{E}'$, is generated by the rotation of the armature through the main field, and is proportional to the speed of the motor, or frequency of rotation, f_0, and to the intensity of the main field, Φ_f, and in phase therewith, hence,

$$\dot{E}' = bf_0\Phi_f.$$

Neglecting then the voltage consumed by the self-inductive impedance of the armature circuit, these two voltages, $\dot{E}$ and

$\dot{E}'$, must be equal and opposite, since the armature is short-circuited. Thus it is,

$$f_0 \Phi_f = + j f \Phi'_c,$$

or,

$$\frac{j \Phi'_c}{\Phi_f} = \frac{f_0}{f}.$$

That is, the two quadrature components, in space, of the repulsion motor, in the direction of the main field, and in the direction of the armature circuit, are in quadrature with each other in time-phase, and have the ratio $\frac{\text{speed}}{\text{synchronism}}$, that is, are equal, and the motor field a uniformly rotating field at synchronism, while below synchronism the main field, above synchronism the crossfield or transformer field is stronger.

If now into the armature circuit an e.m.f., $\dot{E}''$, is inserted, in opposition to $\dot{E}'$, by impressing it upon the armature terminals, it is,

$$\dot{E} + \dot{E}' - \dot{E}'' = 0;$$

hence,

$$- j b f \Phi'_c + b f_0 \Phi_f - E'' = 0,$$

or,

$$\Phi'_c = \Phi_f;$$

for,

$$\frac{f_0}{f} = \frac{j \Phi'_c}{\Phi_f} + \frac{E''}{b f \Phi_f}.$$

That is, impressing an e.m.f., $\dot{E}''$, upon the armature circuit of the repulsion motor, in opposition to that generated by the rotation of the armature — that is, approximately of the phase of the impressed e.m.f. of a series motor — raises the speed at which the motor field becomes a uniformly rotating, or polyphase field, while an opposite e.m.f. lowers this speed.

The inverse result takes place by inserting an e.m.f. in phase with $\dot{E}$, that is, an e.m.f. approximately in quadrature with the impressed e.m.f. of the motor. This, however, requires a polyphase power supply, and so leads to the polyphase commutator motor.

300. Assuming in the following the armature of the repulsion motor as short-circuited upon itself, and applying to the motor the equations (1) to (6), the four conditions characteristic of the repulsion motor are:

(1) Armature short-circuited upon itself. Hence,

$$E_1 = 0.$$

(2) Field circuit and cross circuit in series with each other connected to a source of impressed voltage, e. Hence, assuming the compensating circuit or cross circuit of the same number of effective turns as the rotor circuit, or, $c_2 = 1$,

$$c_0 E_0 + E_2 = e.$$

Herefrom follows,

(3) $I_0 = c_0 I_2$.

(4) No armature excitation used, but only one set of commutator brushes; hence,

$$I_3 = 0,$$

and therefore,

$$I_5 = 0.$$

Substituting these four conditions in the six equations (1) to (6), gives the three repulsion motor equations:

Primary circuit,

$$Z_2 I_2 + Z' (I_2 - I_1) + c_0^2 Z_0 I_2 + c_0 Z (c_0 I_2 - I_4) = e; \quad (7)$$

Secondary circuit,

$$Z_1 I_1 + Z' (I_1 - I_2) + jSZ (c_0 I_2 - I_4) = 0; \quad (8)$$

Brush short-circuit,

$$Z_4 I_4 + Z (I_4 - c_0 I_2) + jSZ' (I_1 - I_2) = 0; \quad (9)$$

Substituting now the abbreviations,

$$Z_2 + c_0^2 (Z_0 + Z) = Z_3, \quad (10)$$

$$\frac{Z'}{Z} = A, \quad (11)$$

$$\frac{Z_1}{Z'} = \lambda_1, \quad (12)$$

$$\frac{Z}{Z_4 + Z} = \lambda_4; \quad (13)$$

where λ_1 and λ_4, especially the former, are small quantities.

From (9) then follows,

$$\underset{\cdot}{I}_4 = \lambda_4 \left\{ I_2 (c_0 + jSA) - jS\underset{\cdot}{I}_1 A \right\}; \tag{14}$$

from (8) follows, by substituting (14) and rearranging,

$$\underset{\cdot}{I}_1 = \underset{\cdot}{I}_2 \frac{1 - \frac{jSc_0}{A} - \lambda_4 S \left(S - \frac{jc_0}{A} \right)}{1 + \lambda_1 - \lambda_4 S^2}, \tag{15}$$

and, substituting (15) in (14), gives,

$$\underset{\cdot}{I}_4 = \lambda_4 \underset{\cdot}{I}_2 \frac{(c_0 + jSA)\ (1 + \lambda_1 - \lambda_4 S^2) - jSA - S^2 c_0 + \lambda_4 jS\ (SA - jc_0)}{1 + \lambda_1 - \lambda_4 S^2},$$

or, canceling terms of secondary order in the numerator,

$$\underset{\cdot}{I}_4 = \lambda_4 \underset{\cdot}{I}_2 \frac{c_0 (1 - S^2)}{1 + \lambda_1 - \lambda_4 S^2}. \tag{16}$$

Equation (7) gives, substituting (10) and rearranging,

$$\underset{\cdot}{I}_2 (Z_3 + Z') - I_1 Z' - I_4 c_0 Z = e. \tag{17}$$

Substituting (15) and (16) herein, and rearranging, gives:

Primary Current,

$$\underset{\cdot}{I}_2 = \frac{e\,(1 + \lambda_1 - S^2 \lambda_4)}{ZK}, \tag{18}$$

where,

$$K = (A_3 + jSc_0) + \lambda_1 (A_3 + A) - \lambda_4 (S^2 A_3 - S^2 c_0 + c_0^2 + jSc_0), \tag{19}$$

and,

$$A_3 = \frac{Z_3}{Z}; \tag{20}$$

or, since approximately,

$$A_3 = c_0^2, \tag{21}$$

it is,

$$K = (A_3 + jSc_0) + \lambda_1 (c_0^2 + A) - \lambda_4 c_0 (c_0 + jS). \tag{22}$$

Substituting (18), (19), (20) in (15) and (16), gives:

Secondary Current,

$$\dot{I}_1 = \frac{e\left\{1 - \frac{jSc_0}{A} - \lambda_4 S\left(S - \frac{jc_0}{A}\right)\right\}}{ZK}. \tag{23}$$

Brush Short-Circuit Current,

$$\dot{I}_4 = \frac{\lambda_4 e c_0 (1 - S^2)}{ZK}. \tag{24}$$

As seen, for $S = 1$, or at synchronism, $\dot{I}_4 = 0$, that is, the short-circuit current under the commutator brushes of the repulsion motor disappears at synchronism, as was to be expected, since the armature coils revolve synchronously in a rotating field.

301. The *e.m.f. of rotation*, that is, the e.m.f. generated in the rotor by its rotation through the magnetic field, which e.m.f., with the current in the respective circuit, produces the torque and so gives the power developed by the motor, is,

Main circuit,

$$\dot{E}_1' = jSZ(c_0 \dot{I}_2 - \dot{I}_4). \tag{25}$$

Brush short circuit,

$$\dot{E}_4' = jSZ'(\dot{I}_1 - \dot{I}_2). \tag{26}$$

Substituting (18), (23), (24) into (25) and (26), and rearranging, gives:

Main Circuit e.m.f. of Rotation,

$$\dot{E}_1' = \frac{jSc_0 e}{K}\{1 + \lambda_1 - \lambda_4\}. \tag{27}$$

Brush Short-circuit e.m.f. of Rotation,

$$\dot{E}_4' = \frac{Se}{K}\{Sc_0 - j\lambda_1 A - c_0\lambda_4\}; \tag{28}$$

or, neglecting smaller terms,

$$\dot{E}_4' = \frac{S^2 c_0 e}{K}. \tag{29}$$

The *Power* produced by the main armature circuit is,

$$P_1 = [\dot{E}_1', \dot{I}_1]^1,$$

hence, substituting (22) and (27),

$$P_1 = \left[\frac{jSc_0e}{K}\{1 + \lambda_1 - \lambda_4\}, \frac{e\left\{1 - \frac{jSc_0}{A} - \lambda_4 S\left(S - \frac{jc_0}{A}\right)\right\}}{ZK}\right]^1. \tag{30}$$

Let

$$m = [ZK] \tag{31}$$

be the absolute value of the complex product, ZK, and,

$$\left.\begin{aligned} \frac{1}{A} &= \alpha' - j\alpha'' \\ \lambda_1 &= \lambda_1' + j\lambda_1'' \\ \lambda_4 &= \lambda_4' - j\lambda_4'' \end{aligned}\right\}, \tag{32}$$

it is, substituting (31), (32) in (30), and expanding:

$$\begin{aligned} P_1 = \frac{Sc_0e^2}{m^2}\{&[x(1 - Sc_0\alpha'') - rSc_0\alpha'] + (1 - Sc_0r'')[x(\lambda_1' - \lambda_4') \\ &- r(\lambda_1'' + \lambda_4'')] - Sc_0\alpha'[r(\lambda_1' - \lambda_4') + x(\lambda_1'' + \lambda_4'')] \\ &- x(\lambda_4'S^2 - \lambda_4' Sc_0\alpha'' + \lambda_4''Sc_0\alpha') + r(\lambda_4'Sc_0\alpha' \\ &- \lambda_4''S^2 + \lambda_4''Sc_0\alpha'')\}, \end{aligned} \tag{33}$$

after canceling terms of secondary order.

As first approximation follows herefrom:

$$\begin{aligned} P_1 &= \frac{Sc_0e^2x}{m^2}\left(1 - Sc_0\alpha'' - \frac{r}{x}Sc_0\alpha'\right) \\ &= \frac{Sc_0e^2x}{m^2}\left\{1 - Sc_0\left(\alpha'' + \frac{r}{x}\alpha'\right)\right\} \\ &= \frac{Se^2x\left\{1 - Sc_0\left(\alpha'' + \frac{r}{x}\alpha'\right)\right\}}{c_0(1 + S^2)}, \end{aligned} \tag{34}$$

hence a maximum for the speed S, given by

$$\frac{dP_1}{dS} = 0,$$

or,
$$S_0 = \sqrt{1 + c_0^2\left(\alpha'' + \frac{r}{x}\alpha'\right)^2} - c_0\left(\alpha'' + \frac{r}{x}\alpha'\right), \tag{35}$$

and equal to

$$P_1^0 = \frac{e^2 x}{2}\left\{\sqrt{1 + c_0^2\left(\alpha'' + \frac{r}{x}\alpha'\right)^2} - c_0\left(\alpha'' + \frac{r}{x}\alpha'\right)\right\}. \tag{36}$$

The complete expression of the power of the main circuit is, from (33):

$$P_1 = \frac{Sc_0e^2x}{m^2}\left\{\left[1 - Sc_0\left(\alpha'' + \frac{r}{x}\alpha'\right)\right] - b_0 - b_1S - b_2S^2\right\}, \tag{37}$$

where b_0, b_1, b_2 are functions of λ_1', λ_1'', λ_4', λ_4'', as derived by rearranging (33).

The *Power* produced in the brush short-circuit is,

$$P_4 = [\dot{E}_4', \dot{I}_4']^1;$$

hence, substituting (24) and (28),

$$\begin{aligned} P_4 &= \left[\frac{S^2c_0e}{K}, \frac{\lambda_4 ec_0(1 - S^2)}{ZK}\right]^1 \\ &= \frac{S^2c_0^2e^2(1 - S^2)}{m^2}\left[r - jx, \lambda_4' - j\lambda_4''\right] \\ &= \frac{S^2c_0^2e^2(1 - S^2)}{m^2}(r\lambda_4' + x\lambda_4''); \end{aligned} \tag{38}$$

hence positive, or assisting, below synchronism, retarding above synchronism.

The total *Power*, or *Output* of the motor then is,

$$P = P_1 + P_4,$$

or

Power Output:

$$\begin{aligned} P = \frac{Sc_0e^2x}{m^2}\Big\{&\left[1 - Sc_0\left(\alpha'' + \frac{r}{x}\alpha'\right)\right] - b_0 + S\left[c_0\left(\lambda_4'' + \frac{r}{x}\lambda_4'\right) - b_1\right] \\ &- S^2b_2 - S^3c_0\left(\lambda_4'' + \frac{r}{x}\lambda_4'\right)\Big\}; \end{aligned} \tag{39}$$

or, approximately,

$$P = \frac{Sc_0e^2x}{m^2}\left\{1 - Sc_0\left(\alpha'' + \frac{r}{x}\alpha'\right)\right\} \tag{40}$$

$$= \frac{Se^2x\left\{1 - Sc_0\left(\alpha'' + \frac{r}{x}\alpha'\right)\right\}}{c_0\,(1 + S^2)}; \tag{41}$$

hence,

Torque:

$$D = \frac{P}{S},$$

given in synchronous watts.

The power input into the motor, and the volt-ampere input, are, if

$$\left.\begin{aligned} \dot{I}_2 &= i_2' + ji_2'', \\ \text{and,} \qquad & \\ i_2 &= \sqrt{i_2'^2 + i_2''^2}, \end{aligned}\right\} \tag{42}$$

given by,

Power Input,

$$P_0 = ei_2', \tag{43}$$

Volt-ampere Input,

$$P_{a_0} = ei_2, \tag{44}$$

Power-Factor,

$$p = \frac{i_2'}{i_2}, \tag{45}$$

Efficiency,

$$\eta = \frac{P}{P_0}, \tag{46}$$

Apparent Efficiency,

$$p\eta = \frac{P}{P_{a_0}}, \tag{47}$$

etc.

302. While excessive values of the short-circuit current under the commutator brushes, $\underset{\cdot}{I}_4$, give bad commutation, due to excessive current densities under the brushes, the best commutation corresponds not to the minimum value of $\underset{\cdot}{I}_4$ – as the zero value at synchronism in the repulsion motor — but to that value of I_4 for which the sudden change of current in the armature coil is a minimum, at the moment where the coil leaves the commutator brush.

$\underset{\cdot}{I}_4$ is the short-current in the armature coil during commutation, reduced to the armature circuit, $\underset{\cdot}{I}_1$, by the ratio of effective turns,

$$c_4 = \frac{\text{short-circuited turns under brushes}}{\text{total effective armature turns}}. \tag{48}$$

The actual current in the short-circuited coils during commutation then, is

$$\underset{\cdot}{I}_4' = \frac{\dot{I}_4}{c_4}, \tag{49}$$

or, if we denote,

$$\frac{\lambda_4}{c_4} = A_4, \tag{50}$$

where A_4 is a fairly large quantity, and substitute (24), it is,

$$\underset{\cdot}{I}_4' = \frac{A_4 e c_0 (1 - S^2)}{ZK}. \tag{51}$$

Before an armature coil passes under the commutator brushes, it carries the current, $-\underset{\cdot}{I}_1$; while under the brushes, it carries the current, $\underset{\cdot}{I}_4'$; and after leaving the brushes, it carries the current, $+\underset{\cdot}{I}_1$.

While passing under the commutator brushes, the current in the armature coils must change from, $-\underset{\cdot}{I}_1$, to $\underset{\cdot}{I}_4'$, or by,

$$\underset{\cdot}{I}_g' = \underset{\cdot}{I}_4' + \underset{\cdot}{I}_1. \tag{52}$$

In the moment of leaving the commutator brushes, the current in the armature coils must change from, $\underset{\cdot}{I}_4'$ to $+\underset{\cdot}{I}_1$, or by,

$$\underset{\cdot}{I}_g = \underset{\cdot}{I}_1 - \underset{\cdot}{I}_4'. \tag{53}$$

The value, I_g', or the current change in the armature coils while entering commutation, is of less importance, since during this change the armature coils are short-circuited by the brushes.

Of fundamental importance for the commutation is the value, I_g, of the current change in the armature coils while leaving the commutator brushes, since this change has to be brought about by the resistance of the brush contact while the coil approaches the edge of the brush, and if considerable, cannot be completed thereby, but the current, I_g, passes as arc beyond the edge of the brushes.

Essential for good commutation, therefore, is that the current, $\dot{I}_g$, should be zero or a minimum, and the study of the commutation of the single-phase commutator thus resolves itself largely into an investigation of the *Commutation Current*, I_g, or its absolute value, i_g.

The ratio of the commutation current, i_g, to the main armature current, i_1, can be called the *Commutation Constant*,

$$k = \frac{i_g}{i_1}. \tag{54}$$

For good commutation, this ratio should be small or zero.

The product of the commutation current, i_g, and the speed, S, is proportional to the voltage induced by the break of this current, or the voltage which maintains the arc at the edge of the commutator brushes, if sufficiently high, and may be called the *commutation voltage*,

$$e_c = Si_g. \tag{55}$$

In the repulsion motor, it is, substituting (23) and (51) in (53), and dropping the term with λ_4, as of secondary order,

Commutation Current,

$$\dot{I}_g = \frac{e\left\{1 - \frac{jSc_0}{A} - A_4 c_0 (1 - S^2)\right\}}{ZK}. \tag{56}$$

Commutation Constant,

$$\left.\begin{aligned} \frac{\dot{I}_g}{\dot{I}_1} &= \frac{1 - \frac{jSc_0}{A} - A_4 c_0 (1 - S^2)}{1 - \frac{jSc_0}{A}} \\ &= 1 - \frac{A_4 c_0 (1 - S^2)}{1 - \frac{jSc_0}{A}}. \end{aligned}\right\} \tag{57}$$

Or, denoting,

$$A_4 = \alpha_4' - j\alpha_4''; \tag{58}$$

substituting (32) and expanding,

$$\left.\begin{aligned} \dot{I}_g &= \frac{e\{1 - c_0[S\alpha'' + (1-S^2)\alpha_4'] + jc_0[(1-S^2)\alpha_4'' - S\alpha']\}}{ZD} \\ \frac{\dot{I}_g}{\dot{I}_1} &= \frac{e\{1 - c_0[S\alpha'' + (1-S^2)\alpha_4'] + jc_0[(1-S^2)\alpha_4'' - S\alpha']\}}{(1 - Sc_0\alpha'') - jSc_0\alpha'} \end{aligned}\right\} \tag{59}$$

and, absolute,

$$i_g = \frac{e}{m}\sqrt{\{1 - c_0[S\alpha'' + (1-S^2)\alpha_4']\}^2 + c_0^2\{(1-S^2)\alpha_4'' - S\alpha'\}^2} \tag{60}$$

$$k = \sqrt{\frac{\{1 - c_0[S\alpha'' + (1-S^2)\alpha_4']\}^2 + c_0^2\{(1-S^2)\alpha_4'' - S\alpha'\}^2}{(1 - Sc_0\alpha'')^2 + S^2c_0^2\alpha'^2}} \tag{61}$$

Perfect commutation, or $\dot{I}_g = 0$, would require from equation (58),

$$\left.\begin{aligned} 1 - c_0[S\alpha'' + (1-S^2)\alpha_4'] &= 0, \\ (1-S^2)\alpha_4'' - S\alpha' &= 0; \end{aligned}\right\} \tag{62}$$

or,

$$\left.\begin{aligned} \alpha_4' &= \frac{1 - c_0 S\alpha''}{c_0(1-S^2)}, \\ \alpha_4'' &= \frac{S\alpha'}{1-S^2} = 1 - \alpha_4'. \end{aligned}\right\} \tag{63}$$

This condition can usually not be fulfilled.

The commutation is best for that speed, S, when the commutation current, i_g, is a minimum, that is,

$$\frac{di_g}{dS} = 0;$$

hence,

$$\frac{d}{dS}\{(1 - c_0[S\alpha'' + (1-S^2)\alpha_4'])^2 + c_0^2((1-S^2)\alpha_4'' - S\alpha')^2\} = 0 \tag{64}$$

This gives a cubic equation in S, of which one root, $0 < S_1 < 1$, represents a minimum.

The relative commutation, that is, relative to the current

consumed by the motor, is best for the value of speed, S_2, where the commutation factor k is a minimum, that is,

$$\frac{dk}{dS} = 0. \tag{65}$$

303. The power output of the repulsion motor becomes zero at the approximate speed given by substituting $P = 0$ in the approximate equation (40), as,

$$S_0 = \frac{1}{c_0 \left(\alpha'' + \frac{r}{x}\alpha'\right)} \tag{66}$$

and above this speed, the power, P, is negative, that is, the repulsion motor consumes power, acting as brake.

This value, S_0, however, is considerably reduced by using the complete equations (39), that is, considering the effect of the short-circuit current under the brushes, etc.

For $S < 0$, $P < 0$; that is, the power is negative, and the machine a generator, when driven backwards, or, what amounts to the same electrically, when reversing either the field-circuit, $\dot{I}_0$, or the primary energy circuit, $\dot{I}_2$. In this case, the machine then is a *repulsion generator*.

The equations of the *repulsion generator* are derived from those of the *repulsion motor*, given heretofore, by reversing the sign of S.

The power, P_4, of the short-circuit current under the brushes reverses at synchronism, and becomes negative above synchronism. The explanation is: This short-circuit current, $\dot{I}_4$, and a corresponding component of the main current, $\dot{I}_1$, are two currents produced in quadrature in an armature or secondary, short-circuited in two directions at right angles with each other, and so offering a short-circuited secondary to the single-phase primary, in any direction, that is, constituting a single-phase induction motor. The short-circuit current under the brushes so superimposes in the repulsion motor, upon the repulsion-motor torque, a single-phase induction-motor torque, which is positive below synchronism, zero at synchronism, and negative above synchronism, as induction generator torque. It thereby lowers the speed, S_0, at which the total torque vanishes, and reduces the power-factor and efficiency.

304. As an example are shown in Fig. 201 the characteristic curves of a repulsion motor, with the speed, S, as abscissas, for the constants:

$$
\begin{aligned}
\text{Impressed voltage: } e &= 500 \text{ volts.}\\
\text{Exciting impedance, main field: } Z &= 0.25 - 3\,j \text{ ohms.}\\
\text{cross field: } Z' &= 0.25 - 2.5\,j \text{ ohms.}\\
\text{Self-inductive impedance, main field: } Z_0 &= 0.1 - 0.3\,j \text{ ohms.}\\
\text{cross field: } Z_2 &= 0.025 - 0.075\,j \text{ ohms.}\\
\text{armature: } Z_1 &= 0.025 - 0.075\,j \text{ ohms.}\\
\text{brush short-circuit: } Z_4 &= 7.5 - 10\,j \text{ ohms.}\\
\text{Reduction factor, main field: } c_0 &= 0.4\\
\text{brush short-circuit: } c_4 &= 0.04.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
Z_3 &= 0.08 - 0.60\,j \text{ ohms.}\\
A &= 0.835 + 0.014\,j.\\
\frac{1}{A} &= \alpha' - j\alpha'' = 1.20 - 0.02\,j.\\
\lambda_1 &= 0.031 + 0.007\,j.\\
\lambda_4 &= 0.179 - 0.087\,j.\\
A_4 &= 4.475 - 2.175\,j.\\
A_3 &= 0.202 + 0.010\,j.
\end{aligned}
$$

Then, substituting in the preceding equations,

$$
\begin{aligned}
K &= (0.204 - 0.035\,S) + j\,(0.031 + 0.328\,S),\\
ZK &= (0.144 + .975\,S) - j\,(0.604 - .187\,S).
\end{aligned}
$$

Primary or Supply Current,

$$
I_2 = \frac{500\,\{(1.031 - 0.179\,S^2) + j\,(0.007 + 0.087\,S^2)\}}{ZK}.
$$

Secondary or Armature Current,

$$
I_1 = \frac{500\,\{(1 + 0.048\,S - 0.179\,S^2) - j\,0.4\,S - 0.087\,S^2)\}}{ZK}.
$$

Brush Short-Circuit Current,

$$I_4 = \frac{500\,(1 - S^2)\,(0.072 - 0.035\,j)}{ZK},$$

and absolute,

$$i_4 = \frac{40\,(1 - S^2)}{m}.$$

Commutation Factor,

$$k = \sqrt{\frac{(1.508\,S^2 - 0.673)^2 + (0.718 - 0.4\,S - 0.704\,S^2)^2}{(0.697 + 0.4\,S - 0.014)^2}}.$$

Main e.m.f. of Rotation,

$$E_1' = \frac{500\,S\,(4.052 + 0.792\,j)}{ZK}.$$

Commutation e.m.f. of Rotation,

$$E_4' = \frac{500\,S^2\,(0.4 - 4.8\,j)}{ZK}.$$

Power of Main Armature Circuit,

$$P_1 = \frac{250\,S}{m^2}\,(4.052 - .122\,S - .657\,S^2), \text{ in kw.}$$

Power of Brush Short-Circuit;

$$P_4 = \frac{49.2\,S^2\,(1 - S^2)}{m^2}, \text{ in kw.}$$

Total Power Output,

$$P = \frac{250\,S}{m^2}\,(4.052 + 0.075\,S - 0.657\,S^2 - 0.197\,S^3).$$

Torque:

$$D = \frac{250}{m^2}\,(4.052 + 0.075\,S - 0.657\,S^2 - 0.197\,S^3),$$

etc.

These curves are derived by calculating numerical values in tabular form, for $S = 0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, 2.2, 2.4$.

As seen from Fig. 201, the power-factor p rises rapidly, reaching fairly high values at comparatively low speeds, and remains near its maximum of 90 per cent over a wide range of speed. The efficiency η follows a similar curve, with 90 per cent maximum near synchronism. The power P reaches a maximum of 192 kw. at 60 per cent of synchronism — 450 revolutions with a 4-pole 25-cycle motor — is 143 kw. at synchronism, and vanishes, together with the torque D, at double-synchronism. The torque at synchronism corresponds to 143 kw., the starting torque to 657 synchronous kw.

The commutation factor k starts with 1.18 at standstill, the

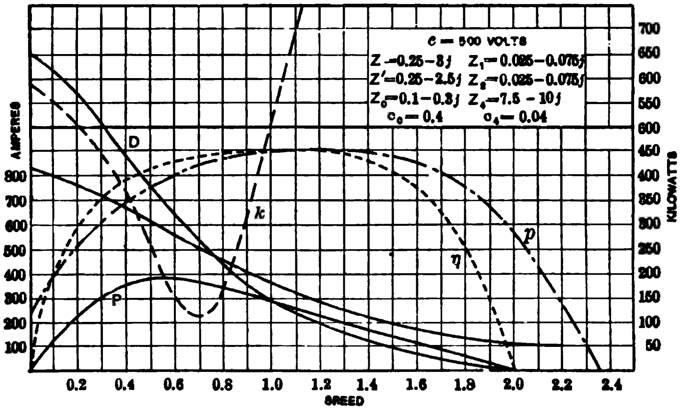


FIG. 201.

same value which the same motor would have as series motor, but rapidly decreases, and reaches a minimum of 0.23 at 70 per cent of synchronism, and then rises again to 1.00 at synchronism, and very high values above synchronism. That is, the commutation of the repulsion is fair already at very low speeds, becomes very good somewhat below synchronism, but poor at speeds considerably above synchronism: this agrees with the experience on such motors.

In the study of the commutation, the short-circuit current under the commutator brushes has been assumed as secondary alternating current. This is completely the case only at standstill, but at speed, due to the limited duration of the short-circuit current in each armature coil — the time of passage of

the coil under the brush — an exponential term superimposes upon the alternating, and so modifies the short-circuit current and thereby the commutation factor, the more, the higher the speed, and greater thereby the exponential term is. The determination of this exponential term is beyond the scope of the present work, but requires the methods of evaluation of transient or momentary electric phenomena, as discussed in "Theory and Calculation of Transient Electric Phenomena and Oscillations."

B. Series Repulsion Motor.

305. As further illustration of the application of these fundamental equations of the single-phase commutator motor, (1) to (6), a motor may be investigated, in which the four independent constants are chosen as follows:

(1) Armature and field connected in series with each other. That is:

$$\dot{E}_1 + c_0\dot{E}_0 = \dot{E} = e_1, \tag{67}$$

where

c_0 = reduction factor of field winding to armature; that is, ratio of effective $\frac{\text{field turns}}{\text{armature turns}}$.

It follows herefrom,

$$\dot{I}_0 = c_0\dot{I}_1. \tag{68}$$

(2) The e.m.f. impressed upon the compensating winding is given, and is in phase with the e.m.f. e_1, which is impressed upon field plus armature,

$$\dot{E}_2 = e_2. \tag{69}$$

That is, $\dot{E}_2$ is supplied by the same transformer or compensator as e_1, in series or in shunt therewith.

(3) No rotor-exciting circuit is used,

$$\dot{I}_3 = 0, \tag{70}$$

and therefore:

(4) No rotor-exciting brushes, or brushes in quadrature position with the main-armature brushes, are used, and so

$$\dot{I}_5 = 0, \tag{71}$$

that is, the armature carries only one set of brushes, which give the short-circuit current, $\dot{I}_4$.

Since the compensating circuit, e_2, is an independent circuit, it can be assumed as of the same number of effective turns as the armature, that is, e_2 is the e.m.f. impressed upon the compensating circuit, reduced to the armature circuit. (The actual e.m.f. impressed upon the compensating circuit thus would be: $c_2 e_2$, where c_2 = ratio effective $\frac{\text{compensating turns}}{\text{armature turns.}}$.)

306. Substituting (68) into (1), (2), (3), and (5), and (1) and (2) into (67), gives the three motor equations:

$$\left.\begin{aligned} e_1 = Z_1 I_1 + Z'(I_1 - I_2) + jSZ\,(c_0 I_1 - I_4) \\ + c_0^2 Z_0 I_1 + c_0 Z\,(c_0 I_0 - I_4), \end{aligned}\right\} \tag{72}$$

$$e_2 = Z_2 I_2 + Z'\,(I_2 - I_1), \tag{73}$$

$$0 = Z_4 I_4 + Z\,(I_4 - c_0 I_1) + jSZ'\,(I_1 - I_2). \tag{74}$$

Substituting now:

$$\left.\begin{aligned} Y' &= \frac{1}{Z'} = \text{quadrature, or transformer exciting admittance,} \\ \frac{Z_2}{Z'} &= \lambda_2 = \lambda_2' + j\lambda_2'', \\ \frac{Z}{Z + Z_4} &= \lambda_4 = \lambda_4' - j\lambda_4'', \\ \frac{Z'}{Z} &= A = \alpha' + j\alpha'' = \text{impedance ratio of the two quadrature fluxes,} \\ Z_1 + c_0^2\,(Z_0 + Z) &= Z_3, \\ \frac{Z_3}{Z} &= A_3 = \alpha_3' + j\alpha_3'', \end{aligned}\right\} \tag{75}$$

and

$$\left.\begin{aligned} e &= e_1 + e_2, \\ t &= \frac{e_2}{e}. \end{aligned}\right\} \tag{76}$$

Adding (72) and (73), and rearranging, gives

$$e = Z_2 \dot{I}_2 + \dot{I}_1 (Z_3 + jSc_0Z) - \dot{I}_4 Z (c_0 + jS);$$

or

$$\frac{e}{Z} = \lambda_2 A \dot{I}_2 + \dot{I}_1 (A_3 + jSc_0) - \dot{I}_4 (c_0 + jS). \tag{77}$$

From (73) follows

$$e_2 Y' = \dot{I}_2 (1 + \lambda_2) - \dot{I}_1,$$

or

$$\left.\begin{aligned} \dot{I}_1 &= \dot{I}_2 (1 + \lambda_2) - etY', \\ \text{and} \qquad \dot{I}_2 &= \dot{I}_1 (1 - \lambda_2) + etY'. \end{aligned}\right\} \tag{78}$$

From (74) follows

$$0 = \dot{I}_4 (Z + Z_4) - \dot{I}_1 (c_0 Z - jSZ') - jSZ' \dot{I}_2. \tag{79}$$

Since $\dot{I}_4$ is a small current, small terms, as λ_2, can be neglected in its evaluation. That is, when substituting (78) in (79), λ_2 can be dropped:

$$\text{or} \quad \left.\begin{aligned} \dot{I}_1 &= \dot{I}_2 - etY', \\ I_2 &= I_1 + etY', \end{aligned}\right\} \text{approximately.} \tag{80}$$

Hence, (80) substituted in (79) gives,

$$0 = \dot{I}_4 (Z + Z_4) - c_0 Z \dot{I}_1 - jSet,$$

or,

$$0 = \frac{\dot{I}_4}{\lambda_4} - c_0 I_1 - \frac{jSet}{Z}.$$

Hence,

$$\dot{I}_4 = \lambda_4 \left\{ c_0 \dot{I}_1 + \frac{jSet}{Z} \right\}. \tag{81}$$

and actual value of short-circuit current,

$$\dot{I}_4' = b\lambda_4 \left\{ \dot{I}_1 + \frac{jSet}{c_0 Z} \right\}, \tag{82}$$

where

$$\left.\begin{aligned} b &= \frac{c_0}{c_4}, \text{ a fairly large quantity, and} \\ c_4 &= \text{reduction factor of brush short-circuit} \\ &\quad \text{to armature circuit.} \end{aligned}\right\} \tag{83}$$

The commutation current then is

$$\begin{aligned} \dot{I}_0 &= \dot{I}_1 - \dot{I}_4' \\ &= \dot{I}_1 (1 - b\lambda_4) - \frac{jSetb\lambda_4}{c_0 Z}. \end{aligned} \tag{84}$$

Substituting (81) and (80) into (77), gives,

$$I_1 = \frac{e}{Z} \frac{1 + jSt\lambda_4 (c_0 + jS) - t\lambda_2}{A_3 + jSc_0 - \lambda_4 c_0 (c_0 + jS) + \lambda_2 A},$$

or, denoting,

$$K = A_3 + jSc_0 - \lambda_4 c_0 (c_0 + jS) + \lambda_2 A,$$

it is,

$$\dot{I}_1 = \frac{e\{1 + jSt\lambda_4 (c_0 + jS) - t\lambda_2\}}{ZK}. \tag{85}$$

It is, approximately,

$$\left.\begin{aligned} A_3 &= \frac{Z_3}{Z} = c_0^2, \\ \lambda_2 &= 0, \end{aligned}\right\} \tag{86}$$

hence,

$$K = c_0 (1 - c_0\lambda_4) (c_0 + jS), \tag{87}$$

$$\begin{aligned} \dot{I}_1 &= \frac{e\{1 + jSt\lambda_4 (c_0 + jS)\}}{c_0 Z (1 - c_0\lambda_4) (c_0 + jS)} \\ &= \frac{e}{c_0 Z (1 - c_0\lambda_4)} \left\{ \frac{1}{c_0 + jS} + jSt\lambda_4 \right\}. \end{aligned} \tag{88}$$

Substituting now (85) respectively (87), (88) into (78), (81) (84), and into

$$\left.\begin{aligned} \dot{E}_1' &= jSZ\,(c_0 \dot{I}_1 - \dot{I}_4), \\ \dot{E}_4' &= jSZ'\,(\dot{I}_1 - \dot{I}_2), \end{aligned}\right\} \quad (89)$$

gives the

Equations of the Series Repulsion Motor:

$$K = A_3 + jSc_0 - \lambda_4 c_0\,(c_0 + jS) + \lambda_2 A,$$

approximately,

$$K = c_0\,(1 - c_0\lambda_4)\,(c_0 + jS). \quad (90)$$

Inducing, or Compensator Current:

$$\dot{I}_2 = \frac{e\{1 + jSt\lambda_4\,(c_0 + jS) - (1 + t)\,\lambda_2\}}{ZK} + \frac{et\,(1 - \lambda_2)}{Z'}$$

approximately,

$$\dot{I}_2 = \frac{e}{c_0 Z\,(1 - c_0\lambda_4)\,(c_0 + jS)} + \frac{jSt\lambda_4 e}{c_0 Z\,(1 - c_0\lambda_4)} + \frac{te}{Z'}. \quad (91)$$

Armature, or Secondary Current:

$$\dot{I}_1 = \frac{e\,\{1 + jSt\lambda_4\,(c_0 + jS) - t\lambda_2\}}{ZK},$$

approximately,

$$\dot{I}_1 = \frac{e}{c_0 Z\,(1 - c_0\lambda_4)} \left\{ \frac{1}{c_0 + jS} + jSt\lambda_4 \right\}. \quad (92)$$

Brush Short-Circuit Current:

$$\dot{I}_4 = \frac{e\lambda_4}{Z\,(1 - c_0\lambda_4)} \left\{ \frac{1}{c_0 + jS} + jSt\,(1 + \lambda_4 - c_0\lambda_4) \right\},$$

approximately,

$$\dot{I}_4 = \frac{e\lambda_4}{Z\,(1 - c_0\lambda_4)} \left\{ \frac{1}{c_0 + jS} + jSt \right\}. \quad (93)$$

Commutation Current:

$$\left.\begin{aligned} \dot{I}_g &= \frac{e}{c_0 Z\,(1 - c_0\lambda_4)} \left\{ \frac{1 - \lambda_4 b}{c_0 + jS} - jSt\lambda_4\,[(b-1) \right. \\ &\qquad \left. + b\lambda_4\,(1 - c_0)] \right\}, \\ &\text{approximately,} \\ \dot{I}_g &= \frac{e}{c_0 Z\,(1 - c_0\lambda_4)} \left\{ \frac{1 - \lambda_4 b}{c_0 + jS} - jSt\lambda_4 b \right\}. \end{aligned}\right\} \tag{94}$$

Main e.m.f. of Rotation:

$$\left.\begin{aligned} \dot{E}_1' &= \frac{jSe}{1 - c_0\lambda_4} \left\{ \frac{1 - \lambda_4}{c_0 + jS} - jSt\lambda_4^2\,(1 - c_0) \right\}, \\ &\text{approximately,} \\ \dot{E}_1' &= \frac{jSe\,(1 - \lambda_4)}{(1 - c_0\lambda_4)\,(c_0 + jS)}. \end{aligned}\right\} \tag{95}$$

Quadrature e.m.f. of Rotation:

$$\dot{E}_4' = -jSte. \tag{96}$$

307. Since in calculating motor curves from the constants, the speed, S, is usually chosen as an independent variable, and also for the investigations of the effects caused by a variation of speed, S, it is convenient to arrange the equations (90) to (96) by orders of S, that is, in the form,

$$F = \frac{B_0 + B_1 S + B_2 S^2 \ldots}{C_0 + C_1 S + C_2 S^2 \ldots},$$

or, after separating the two components of the complex terms,

$$\dot{F} = \frac{(b_0 + b_1 S + b_2 S^2 \ldots) + j\,(b_0' + b_1' S + l_2' S^2 \ldots)}{(c_0 + c_1 S + c_2 S^2 \ldots) + j\,(c_0' + c_1' S + c_2' S^2 \ldots)}.$$

These equations (90) to (96) contain two terms, one with, and one without $t = \frac{e_2}{e}$, and so, for the purpose of investigating

the effect of the distribution of voltage, e, between the circuits, e_1 and e_2, can be arranged in the form

$$F = K_1 + tK_2.$$

In first approximation, neglecting even the effect of the short-circuit current under the brushes, $\dot{I}_4$, that is, assuming

$$\dot{I}_4 = 0,$$

these equations (90) to (96) assume the form

$$\left.\begin{aligned}
K &= A_3 + jSc_0 \\
&= c_0(c_0 + jS); \\
\dot{I}_1 &= \frac{e}{ZK} \\
&= \frac{e}{c_0(c_0 + jS)Z}; \\
\dot{I}_2 &= \frac{e}{ZK} + \frac{e_2}{Z'}, \\
&= \frac{e}{ZK} \\
&= \frac{e}{c_0(c_0 + jS)Z}; \\
\dot{E}_1' &= \frac{jSc_0e}{K} \\
&= \frac{jSe}{c_0 + jS}.
\end{aligned}\right\} \qquad (97)$$

That is, the compensator current, $\dot{I}_2$, the armature current, $\dot{I}_1$, and the e.m.f. of rotation, $\dot{E}_1'$, as first approximation depend only upon the sum of the two impressed e.m.fs.; $e = e_1 + e_2$, and current, torque, power, etc., therefore remain approximately the same, regardless of the distribution of the voltage e between the two circuits, e_1 and e_2.

The character of the magnetic field, however, that is, the relation between its two quadrature components, and therefore the

speed, S, at which the field is a uniformly revolving field, and thus the change of the character of the commutation with the speed, depend upon the ratio of e_1 and e_2, or upon

$$t = \frac{e_2}{e}.$$

From above equations follow:

$$\left.\begin{aligned}
&\textit{Power Output;}\\
&\qquad P = P_1 + P_4\\
&\qquad\quad = [\dot{E}_1', \dot{I}_1]^1 + [\dot{E}_4', \dot{I}_4]^1.\\
&\textit{Power Input;}\\
&\qquad P_0 = [e_1, \dot{I}_1]^1 + [e_2, \dot{I}_2]^1.\\
&\textit{Volt-ampere Input;}\\
&\qquad P = e_1 i_1 + e_2 i_2\\
&\qquad\quad = e\{(1-t)\, i_1 + t i_2\},
\end{aligned}\right\} \qquad (98)$$

where the small letters, i_1 and i_2, denote the absolute values of the currents, $\dot{I}_1$ and $\dot{I}_2$.

When i_1 and i_2 are derived from the same compensator or transformer (or are in shunt with each other, as branches of the same circuit, if $e_1 = e_2$), as usually the case, in the primary circuit the current corresponds not to the sum, $\{(1-t)i_1 + ti_2\}$ of the secondary currents, but to their resultant, $[(1-t)\dot{I}_1 + t\dot{I}_2]^1$, and if the currents, $\dot{I}_1$ and $\dot{I}_2$, are out of phase with each other, as is more or less the case, the absolute value of their resultant is less than the sum of the absolute values of the components. The volt-ampere input, reduced to the primary source of power, then is,

$$P_{a_0} = e[(1-t)\, \dot{I}_1 + t\dot{I}_2]^1, \qquad (99)$$

and

$$P_{a_0} < P_a.$$

From these equations then follows the torque: $D = \frac{P}{S}$, the power-factor, $p = \frac{P_0}{P_{a_0}}$, etc.

For $$t = 0,$$

that is, all the voltage impressed upon the armature circuit, and the compensating circuit short-circuited, these equations are those of the *inductively-compensated series motor*.

For $$t = 1,$$

that is, all the voltage impressed upon the compensating or inducing circuit, and the armature circuit closed in short-circuit, that is, the armature energizing the field, the equations are those of the *repulsion motor with secondary excitation*.

For $$t > 1$$

a reverse voltage is impressed upon the armature circuit.

Study of Commutation.

308. The commutation of the alternating-current commutator motor mainly depends upon:

(*a*) the short-circuit current under the commutator brush, which has the actual value: $\underset{\cdot}{I}_4' = \frac{\dot{I}_4}{c_4}$. High short-circuit current causes arcing under the brushes, and glowing, by high current density;

(*b*) the commutation current, that is, the current change in the armature coil in the moment of leaving the brush short-circuit, $\underset{\cdot}{I}_g = \underset{\cdot}{I}_1 - \underset{\cdot}{I}_4'$. This current, and the e.m.f. produced by it, SI_g, produce sparking at the edge of the commutator brushes, and is destructive, if considerable.

(*a*) *Short-circuit Current under Brushes.*

Using the approximate equation (93), the actual value of the short-circuit current under the brushes is

$$\underset{\cdot}{I}_4' = \frac{e\lambda_4 b}{c_0 Z\,(1 - c_0\lambda_4)} \left\{ \frac{1}{c_0 + jS} + jSt \right\}; \tag{100}$$

where

$b = \frac{c_0}{c_4}$, or $\frac{1}{b}$ = reduction factor of short-circuit under brushes, to field circuit, that is,

$$b = \frac{\text{number of field turns}}{\text{number of effective short-circuit turns}}, \tag{101}$$

hence a large quantity.

The absolute value of the short-circuit current, therefore, is

$$i_4' = \frac{e\lambda_4^0 b \sqrt{c_0^2 + S^2 (1 - t(c_0^2 + S^2))^2}}{c_0 z[1 - c_0\lambda_4](c_0^2 + S^2)}; \tag{102}$$

hence a minimum for that value of t, where

$$f = c_0^2 + S^2 (1 - t(c_0^2 + S^2))^2 = \text{minimum, or}$$
$$= 1 - t(c_0^2 + S^2) = 0, \text{ hence,}$$

$$t = \frac{1}{c_0^2 + S^2},$$

and,

$$S = \sqrt{\frac{1}{t} - c_0^2}.$$

That is, $t = \frac{e_2}{e} = \frac{1}{S^2 + c_0^2}$ gives minimum short-circuit current at speed, S, and inversely, speed, $S = \sqrt{\frac{1}{t} - c_0^2}$, gives minimum short-circuit current at voltage ratio t.

For $t = 1$, or the repulsion motor with secondary excitation, the short-circuit current is minimum at speed $S = \sqrt{1 - c_0^2}$, or somewhat below synchronism, and is $i_4' = \frac{{}_4c_0 e}{z}$, while in the repulsion motor with primary excitation, the short-circuit current is a minimum, and equals zero, at synchronism $S = 1$.

The lower the voltage ratio, $t = \frac{e_2}{e}$, the higher is the speed, S, at which the short-circuit current reaches a minimum.

The short-circuit current, $\dot{I}_4'$, however, is of far less importance than the commutation current, $\dot{I}_g$.

(*b*) *Commutation Current.*

309. While the value, $\dot{I}_g' = \dot{I}_4' + \dot{I}_1$, or the current change in the armature coils while entering commutation, is of minor importance, of foremost importance for good commutation is that the current change in the armature coils, when leaving the short-circuit under the brushes,

$$\dot{I}_g = \dot{I}_1 - \dot{I}_4' \tag{103}$$

is zero or a minimum.

Using the approximate equation of the commutation current (94), it is,

$$\dot{I}_g = \frac{e}{c_0 Z\,(1 - c_0\lambda_4)} \left\{ \frac{1 - \lambda_4 b}{c_0 + jS} - jSt\lambda_4 b \right\}$$

$$= \frac{e}{c_0 Z\,(1 - c_0\lambda_4)\,(c_0 + jS)} \{1 - \lambda_4 b - jS\,(c_0 + jS)\,t\lambda_4 b\}; \tag{104}$$

and, denoting

$$\lambda_4 = \lambda_4' - j\lambda_4'',$$

it is, expanded,

$$\dot{I}_g = \frac{e}{c_0 Z\,(1 - c_0\lambda_4)\,(c_0 + jS)} \{[1 - \lambda_4' b + Stb\,(S\lambda_4' - c_0\lambda_4'')] + j[\lambda_4'' b - Stb\,(c_0\lambda_4' + S\lambda_4'')]\}; \tag{105}$$

hence, absolute,

$$i_g = \frac{e}{c_0 z\,[1 - c_0\lambda_4]\sqrt{c_0^2 + S^2}} \sqrt{[1 - \lambda_4' b + Stb\,(S\lambda_4' - c_0\lambda_4'')]^2 + [\lambda_4'' b - Stb\,(c_0\lambda_4' + S\lambda_4'')]^2}, \tag{106}$$

where $[1 - c_0\lambda_4]$ denotes the absolute value of $(1 - c_0\lambda_4)$.

The commutation current is zero, if either $S = \infty$, that is, infinite speed, which is obvious but of no practical interest, or the parenthesis in (105) vanishes.

Since this parenthesis is complex, it vanishes when both of its terms vanish. This gives the two equations

$$\left.\begin{aligned} 1 - \lambda_4' b + Stb\,(S\lambda_4' - c_0\lambda_4'') &= 0, \\ \lambda_4'' b - Stb\,(c_0\lambda_4' + S\lambda_4'') &= 0. \end{aligned}\right\} \tag{107}$$

From these two equations are calculated the two values, the speed, S, and the voltage ratio, t, as,

$$\left.\begin{aligned} S_0 &= \frac{c_0\,(b\lambda_4^2 - \lambda_4')}{\lambda_4''}, \\ t_0 &= \frac{\lambda_4''^2}{c_0^2 b\lambda_4^2(b\lambda_4^2 - \lambda_4')}; \\ \text{hence,} \\ S_0 t_0 &= \frac{\lambda_4''}{c_0 b\lambda_4^2}. \end{aligned}\right\} \tag{108}$$

For instance, if,

$$Z = 0.25 - 3j,$$

$$Z_4 = 5 \quad - 2.5j:$$

hence,

$$\lambda_4 = \frac{Z}{Z + Z_4} = 0.307 - .248j = \lambda_4' - j\lambda_4'',$$

$$c_0 = 0.4,$$

$$c_4 = 0.04;$$

hence,

$$b = 10;$$

and herefrom,

$$S_0 = 2.02,$$

$$t_0 = .197,$$

that is, at about double synchronism, for $e_2 = te = 0.197e$, or about 20 per cent of e, the commutation current vanishes.

In general, there is thus in the series repulsion motor only one speed, S_0, at which, if the voltage ratio has the proper value, t_0, the commutation current, i_g, vanishes, and the commutation is perfect. At any other speed some commutation current is left, regardless of the value of the voltage ratio, t.

With the two voltages, e_1 and e_2, in phase with each other, the commutation current cannot be made to vanish at any desired speed, S.

310. It remains to be seen, therefore, whether by a phase displacement between e_1 and e_2, that is, if e_2 is chosen out of phase with the total voltage, e, the commutation current can be made to vanish at any speed, S, by properly choosing the value of the voltage ratio, and the phase difference.

Assuming, then, e_2 out of phase with the total voltage, e, hence denoting it by

$$\dot{E}_2 = e_2 (\cos \theta_2 + j \sin \theta_2), \tag{109}$$

the voltage ratio, t, now also is a complex quantity, and expressed by

$$T = \frac{\dot{E}_2}{e} = t (\cos \theta_2 + j \sin \theta_2) = t' + jt''. \tag{110}$$

Substituting (110) in (105), and rearranging, gives

$$\begin{aligned} \dot{I}_g = \frac{e}{c_0 Z (1 - c_0 \lambda_4)(c_0 + jS)} & \{[1 - \lambda_4' b + St'b (S\lambda_4' - c_0 \lambda_4'') \\ & + St''b (c_0 \lambda_4' + S\lambda_4'')] + j[\lambda_4'' b - St'b (c_0 \lambda_4' + S\lambda_4'') \\ & + St''b (S\lambda_4' - c_0 \lambda_4'')]\}; \end{aligned} \tag{111}$$

and this expression vanishes, if

$$\left.\begin{aligned} 1 - \lambda_4' b + St'b (S\lambda_4' - c_0 \lambda_4'') + St''b (c_0 \lambda_4' + S\lambda_4'') &= 0, \\ \lambda_4'' b - St'b (c_0 \lambda_4' + S\lambda_4'') + St''b (S\lambda_4' - c_0 \lambda_4'') &= 0: \end{aligned}\right\}$$

and herefrom follows:

$$\left.\begin{aligned} t' &= \frac{Sb\lambda_4^2 - S\lambda_4' + c_0 \lambda_4''}{Sb\lambda_4^2 (S^2 + c_0^2)} = \frac{1}{c_0^2 + S^2} \left\{ 1 - \frac{S\lambda_4' - c_0 \lambda_4''}{Sb\lambda_4^2} \right\}, \\ t'' &= \frac{c_0 b\lambda_4^2 - c_0 \lambda_4' - S\lambda_4''}{Sb\lambda_4^2 (S^2 + c_0^2)} = \frac{1}{c_0^2 + S^2} \left\{ \frac{c_0}{S} - \frac{c_0 \lambda_4' + S\lambda_4''}{Sb\lambda_4^2} \right\}; \end{aligned}\right\} \tag{112}$$

or approximately,

$$\left.\begin{aligned} t' &= \frac{1}{c_0^2 + S^2}, \\ t'' &= \frac{c_0}{S (c_0^2 + S^2)} \end{aligned}\right\} \tag{113}$$

$t'' = 0$ substituted in equation (112) gives $S = S_0$, the value recorded in equation (108).

It follows herefrom, that with increasing speed, S, t' and still more t'', decrease rapidly. For $S = 0$, t' and t'' become infinite. That is, at standstill, it is not possible by this method to produce zero commutation current.

The phase angle, θ_2, of the voltage ratio, $T = t' + jt''$, is given by

$$\tan \theta_2 = \frac{t''}{t'} = \frac{c_0 b \lambda_4^2 - c_0 \lambda_4' - S \lambda_4''}{S b \lambda_4^2 - S \lambda_4' + c_0 \lambda_4''}; \tag{114}$$

rearranged, this gives,

$$\frac{c_0 \sin \theta_2 + S \cos \theta_2}{c_0 \cos \theta_2 - S \sin \theta_2} = \frac{b \lambda_4^2 - \lambda_4'}{\lambda_4''}; \tag{115}$$

and, denoting,

$$\frac{S}{c_0} = \tan \sigma, \tag{116}$$

where σ may be called the "speed angle," it is, substituted in (115),

$$\tan (\theta_2 + \sigma) = \frac{b \lambda_4^2 - \lambda_4'}{\lambda_4''}, \tag{117}$$

$$= \text{constant};$$

hence,

$$\theta_2 + \sigma = \gamma, \tag{118}$$

and,

$$\theta_2 = \gamma - \sigma. \tag{119}$$

$\frac{b \lambda_4^2 - \lambda_4'}{\lambda_4''}$ is a large quantity, hence γ near 90°.

σ is also near 90° for all speeds S except very slow speeds, since in (116) c_0 is a small quantity.

Hence θ_2 is near zero for all except very low speeds.

For very low speeds, σ is small, and θ_2 thus large and positive.

That is, the voltage, $\dot{E}_2$, impressed upon the compensating circuit to get negligible commutation current, must be approximately in phase with e for all except low speeds. At low speeds, it must lag, the more, the lower the speed. Its absolute value is very large at low speeds, but decreases rapidly with increasing speed, to very low values.

For instance, let, as before,

$$\lambda_4 = 0.304 - 0.248\,j,$$

$$c_0 = 0.4,$$

$$b = 10;$$

it is,

$$\tan(\theta_2 + \sigma) = 5.05,$$

$$\theta_2 + \sigma = 79°;$$

hence,

$$\theta_2 = 79° - \sigma.$$

$\theta_2 = 0$ for $\sigma = 79°$; hence, by (116), $S_0 = 2.02$, or double synchronism. Above this speed, θ_2 is leading, but very small, since the maximum leading value, for infinite speed, $S = \infty$, is given by $\sigma = 90°$, as, $\theta_2 = -11°$. Below the speed, S_0, θ_2 is positive, or lagging;

for $S = 1$, it is $\sigma = 68°$, $\theta_2 = +11°$, hence still approximately in phase;

for $S = 0.4$, it is $\sigma = 45°$, $\theta_2 = 34°$; hence $\dot{E}_2$ is still nearer in phase than in quadrature to e.

The corresponding values of $T = t' + t''$ are, from (112),

$$S = 2.02,\ \theta_2 = 0,\qquad T = 0.197,\qquad t = 0.197,$$

$$S = 1,\ \theta_2 = +11°,\qquad T = 0.747 + 0.140\,j,\qquad t = 0.760,$$

$$S = .4,\ \theta_2 = 34°,\qquad T = 3.00 + 2.00\,j,\qquad t = 3.61.$$

311. The introduction of a phase displacement between the compensating voltage, $\dot{E}_2$, and the total voltage, e, in general is more complicated, and since for all but the lowest speeds the required phase displacement, θ_2, is small, it is usually sufficient to employ a compensating voltage, e_2, in phase with e.

In this case, no value of t exists, which makes the commutation current vanish entirely, except at the speed, S_0.

The problem then is, to determine for any speed, S, that value of the voltage ratio, t, which makes the commutation current, i_g, a minimum. This value is given by

$$\frac{di_g}{dt} = 0, \tag{120}$$

where i_g is given by equation (106).

Since equation (106) contains t only under the square root, the minimum value of i_g is given also by,

$$\frac{dK}{dt} = 0,$$

where

$$K = [1 - b\lambda_4' + Stb\,(S\lambda_4' - c_0\lambda_4'')]^2 + [b\lambda_4'' - Stb\,(c_0\lambda_4' + S\lambda_4'')]^2.$$

Carrying out this differentiation, and expanding, gives,

$$t = \frac{Sb\lambda_4^2 - S\lambda_4' + c_0\lambda_4''}{Sb\lambda_4^2\,(c_0^2 + S^2)} = \frac{1}{c_0^2 + S^2}\left\{1 - \frac{S\lambda_4' - c_0\lambda_4''}{Sb\lambda_4^2}\right\}. \tag{121}$$

This is the same value as the real component t' of the complex voltage ratio, T_1, which caused the commutation current to vanish entirely, and was given by equation (112).

It is, approximately,

$$t = \frac{1}{c_0^2 + S^2}. \tag{122}$$

Substituting (121) into (105) gives the value of the minimum commutation current, i_{g_0}.

Since the expression is somewhat complicated, it is preferable to introduce trigonometric functions, that is, substitute,

$$\tan\delta = \frac{\lambda_4''}{\lambda_4'}, \tag{123}$$

where δ is the phase angle of λ_4, and therefore,

$$\left.\begin{aligned} \lambda_4'' &= \lambda_4 \sin\delta, \\ \lambda_4' &= \lambda_4 \cos\delta, \end{aligned}\right\} \tag{124}$$

and also to introduce, as before, the speed angle (116),

$$\left.\begin{aligned} \tan\sigma &= \frac{S}{c_0}, \\ q &= \sqrt{c_0^2 + S^2}; \end{aligned}\right| \tag{125}$$

hence,

$$\left.\begin{aligned} S &= q\sin\sigma, \\ c_0 &= q\cos\sigma. \end{aligned}\right\} \tag{126}$$

Substituting these trigonometric values into the expression (121) of the voltage ratio for minimum commutation current, it is,

$$t = \frac{1}{q^2} - \frac{\sin(\sigma - \delta)}{Sbq\lambda_4}. \tag{127}$$

Substituting (117) into (106) and expanding gives a relatively simple value, since most terms eliminate:

$$\dot{I}_g = \frac{e\{[\cos^2(\sigma-\delta) + b\lambda\ (\sin\sigma \sin(\sigma-\delta) - \cos\delta)] + j\,[\sin(\sigma-\delta)\cos(\sigma-\delta) - b\lambda_4\ (\sin\sigma\cos(\sigma-\delta) - \sin\delta)]\}}{c_0 Z\,(1 - c_0\lambda_4,\ (c_0 + jS)} \tag{128}$$

and the absolute value

$$\dot{i}_{g_0} = \frac{e\,(\cos(\sigma-\delta) - b\lambda_4\cos\sigma)}{c_0 z\,[1 - c_0\lambda_4]\sqrt{c_0^2 + S^2}}; \tag{129}$$

or, resubstituting for σ and δ,

$$\dot{i}_{g_0} = \frac{e\{S\lambda_4'' - c_0\,(\lambda_4^2 b - \lambda_4')\}}{c_0 z\,[1 - c_0\lambda_4]\,(c_0^2 + S^2)}. \tag{130}$$

From (129) and (130) follows, that $\dot{i}_{g_0} = 0$, or the commutation current vanishes, if

$$\cos(\sigma - \delta) - b\lambda_4\cos\sigma = 0, \tag{131}$$

or,

$$S\lambda_4'' - c_0\,(\lambda_4^2 b - \lambda_4') = 0.$$

This gives, substituting, $\lambda_4'' = \sqrt{\lambda_4^2 - \lambda_4'^2}$, and expanding,

$$\left.\begin{aligned} \lambda_4' &= \frac{\lambda_4}{c_0^2 + S^2}\{b\lambda_4 c_0^2 \pm S\sqrt{S^2 - c_0^2\,(b^2\lambda_4^2 - 1)}\}, \\ \cos(\sigma - \delta) &= \frac{b\lambda_4 c_0}{\sqrt{c_0^2 + S^2}}. \end{aligned}\right\} \tag{132}$$

From (131) follows

$$\cos(\sigma - \delta) = b\lambda_4\cos\sigma.$$

Since $\cos(\sigma - \delta)$ must be less than one, this means

$$b\lambda_4\cos\sigma < 1,$$

or,

$$\lambda_4 < \frac{1}{b \cos \sigma},$$

or,

$$\left.\begin{aligned} \lambda_4 &< \frac{\sqrt{c_0^2 + S^2}}{c_0 b}, \\ \text{or, inversely,} \quad S &> c_0 \sqrt{b^2 \lambda_4^2 - 1}. \end{aligned}\right\} \quad (133)$$

That is:

The commutation current, i_g, can be made to vanish at any speed, S, at given impedance factor, λ_4, by choosing the phase angle of the impedance of the short-circuited coil, δ, or the resistance component, λ', provided that λ_4 is sufficiently small, or the speed, S, sufficiently high, to conform with equations (133).

From (132) follows as the minimum value of speed S, at which the commutation current can be made to vanish, at given λ_4,

$$S_1 = c_0 \sqrt{b^2 \lambda_4^2 - 1},$$

and,

$$\lambda_4' = \frac{1}{b};$$

hence,

$$\lambda_4'' = \sqrt{\lambda_4^2 - \frac{1}{b^2}}.$$

For high values of speed S, it is, approximately,

$$\cos(\sigma - \delta) = 0,$$

$$\sigma - \delta = 90°,$$

$$\tan \sigma = \frac{S}{c_0};$$

hence,

$$\frac{\sigma = 90°}{\delta = 0},$$

$$\lambda_4' = \lambda_4.$$

That is, the short-circuited coil under the brush contains no inductive reactance, hence:

At low and medium speeds, some inductive reactance in the short-circuited coils is advantageous, but for high speeds it is objectionable for good commutation.

312. As an example are shown, in Figs. 202 to 205, the characteristic curves of series-repulsion motors, for the constants:

Impressed voltage:	$e = 500$ volts,
Exciting impedance, main field:	$Z = 0.25 - 3\,j$ ohms,
Exciting impedance, cross field:	$Z' = 0.25 - 2.5\,j$ ohms,
Self-inductive impedance, main field:	$Z_0 = 0.1 - 0.3\,j$ ohms,
Self-inductive impedance, cross-field:	$Z_2 = 0.025 - 0.075\,j$ ohms,
Self-inductive impedance armature:	$Z_1 = 0.025 - 0.075\,j$ ohms,
Self-inductive impedance, brush short-circuit:	$Z_4 = 7.5 - 10\,j$ ohms,
Reduction factor, main field:	$c_0 = 0.4$,
brush short-circuit:	$c_4 = 0.04$;

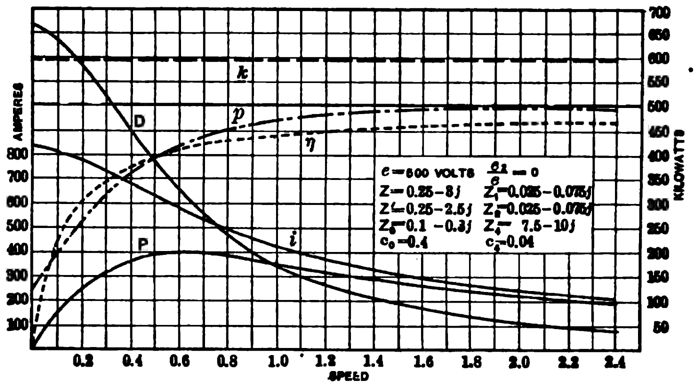


FIG. 202. — Inductively Compensated Series Motor.

that is, the same constants as used in the repulsion motor, § 304, Fig. 201.

Curves are plotted for the voltage ratios:

$t = 0$: inductively compensated series-motor, Fig. 202,
$t = 0.2$: series repulsion motor, high speed, Fig. 203,
$t = 0.5$: series repulsion motor, medium speed, Fig. 204,
$t = 1.0$: repulsion motor with secondary excitation, low speed, Fig. 205.

It is, from above constants:

$$Z_3 = Z_1 + c_0^2 (Z_0 + Z) = 0.08 \quad - 0.60 \ j.$$

$$A_3 = \frac{Z_3}{Z} = 0.202 + 0.010 j.$$

$$A = \frac{Z'}{Z} = 0.835 + 0.014 j.$$

$$\lambda_2 = \frac{Z_2}{Z'} = 0.031 + 0.007 j.$$

$$\lambda_4 = \frac{Z}{Q + Z_4} = 0.179 - 0.087 j.$$

$$b = \frac{c_0}{c_4} = 10$$

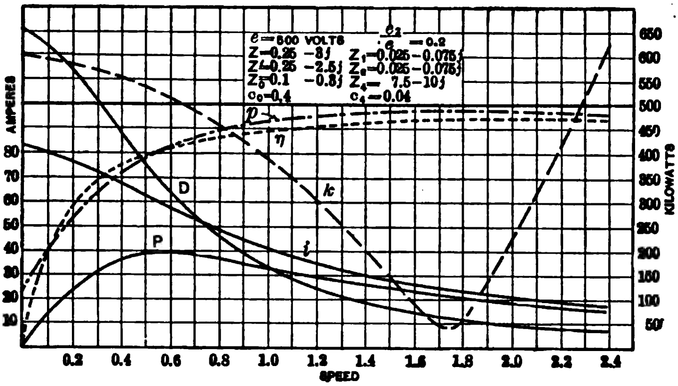


FIG. 203. — Series Repulsion Motor.

Hence, substituting into the preceding equations:

$$(90) \quad ZK = Z_3 + jSc_0Z - \lambda_4 c_0 Z (c_0 + jS) + \lambda_2 Z'$$
$$= (0.160 + 0.975 S) - j (0.590 - 0.187 S),$$

$$(92) \quad I_1 = \frac{e}{ZK} + \frac{et}{ZK} \{jS\lambda_4 (c_0 + jS) - \lambda_2\}$$
$$= \frac{e}{ZK} + \frac{et}{ZK} \{(-0.031 + 0.035 S - 0.179 S^2)$$
$$+ j (-0.007 + 0.072 S + 0.087 S^2)\},$$

$$(91) \quad \dot{I}_2 = \dot{I}_1 (0.969 - 0.007\,j) + et\,(0.010 + 0.096\,j),$$

(93)

$$\dot{I}_4 = \frac{e\,(0.072 - 0.035\,j) + Set\,\{(0.016 - 0.072\,S) + j\,0.045 + 0.035\,S)\}}{ZK},$$

etc.

313. As seen, these four curves are very similar to each other and to those of the repulsion motor, with the exception of the commutation current, i_g, and commutation factor, $k = \frac{\dot{i}_g}{\dot{i}}$.

The commutation factor of the compensated series motor,

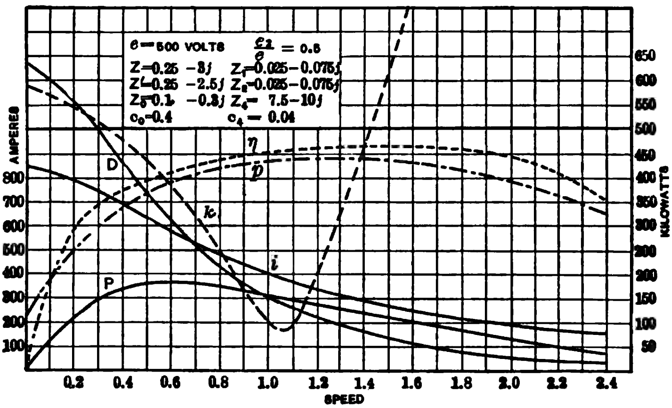


FIG. 204. — Series Repulsion Motor.

that is, the ratio of current change in the armature coil while leaving the brushes, to total armature current, is constant in the series motor, at all speeds. In the series repulsion motors, the commutation factor, k, starts with the same value at standstill, as the series motor, but decreases with increasing speed, thus giving a superior commutation to that of the series motor, reaches a minimum, and then increases again. Beyond the minimum commutation factor, the efficiency, power-factor, torque and output of the motor first slowly, then rapidly increase, due to the rapid increase of the commutation losses. These higher values, however, are of little practical value, since the commutation is bad.

The higher the voltage ratio, t, that is, the more voltage is impressed upon the compensating circuit, and the less upon the armature circuit, the lower is the speed at which the commutation factor is a minimum, and the commutation so good or perfect. That is, with $t = 1$, or the repulsion motor with secondary excitation, the commutation is best at 70 per cent of synchronism, and gets poor above synchronism. With $t = 0.5$, or a series repulsion motor with half the voltage on the compensating, half on the armature circuit, the commutation is best just above synchronism, with the motor constants chosen in this instance, and gets poor at speeds above 150 per cent of synchronism. With $t = 0.2$, or only 20 per cent of the voltage on

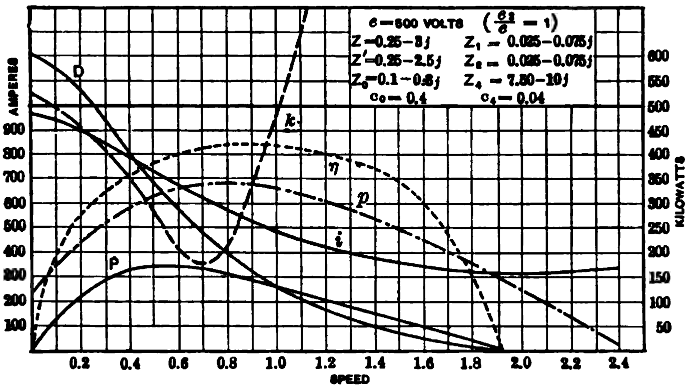


FIG. 205.—Repulsion Motor, Secondary Excitation.

the compensating circuit, the commutation gets perfect at double synchronism.

Best commutation thus is secured by shifting the supply voltage with increasing speed from the compensating to the armature circuit.

$t > 1$, or a reverse voltage $-e_1$ impressed upon the armature circuit, so still further improves the commutation at very low speeds.

For high values of t, however, the power-factor of the motor falls off somewhat.

The impedance of the short-circuited armature coils, chosen in the preceding example,

$$Z_4 = 7.5 - 10j,$$

corresponds to fairly high resistance and inductive reactance in the commutator leads, as frequently used in such motors.

314. As a further example are shown in Fig. 206 and Fig. 207 curves of a motor with low-resistance and low-reactance commutator leads, and high number of armature turns, that is, low reduction factor of field to armature circuit, of the constants

$$Z_4 = 4 - 2j;$$

hence,
$$\lambda_4 = 0.373 - 0.267j,$$

and
$$c_0 = 0.3,$$
$$c_4 = 0.03,$$

the other constants being the same as before.

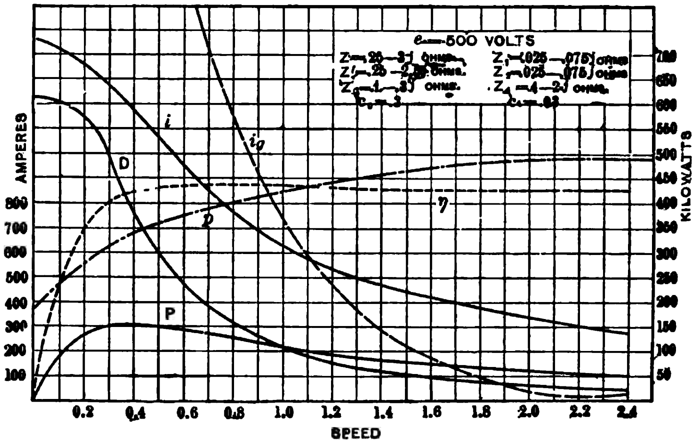


FIG. 206.

Fig. 206 shows, with the speed as abscissas, the current, torque, power output, power-factor, efficiency and commutation current i_g, under such a condition of operation, that at low speeds $t = 1.0$, that is, the motor is a repulsion motor with secondary excitation, and above the speed at which $t = 1.0$ gives best commutation (90 per cent of synchronism in this example), t is gradually decreased, so as to maintain i_g a minimum, that is, to maintain

best commutation. The values of t required heretofore are plotted in dotted lines.

As seen, at 10 per cent above synchronism, i_g drops below i, that is, the commutation of the motor becomes superior to that of a good direct-current motor.

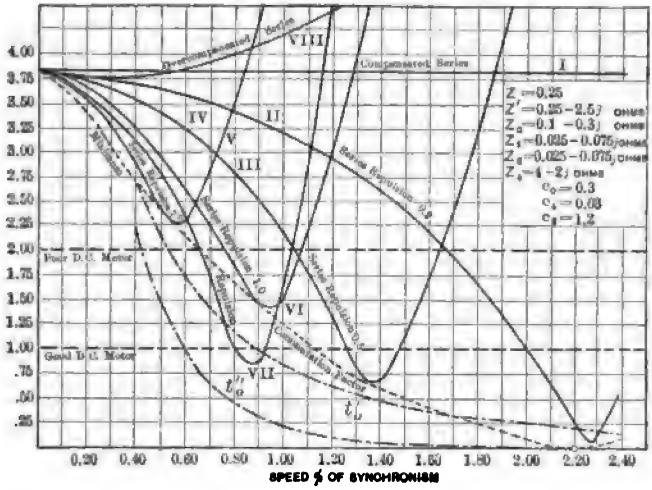


FIG. 207.

Fig. 207 then shows the commutation factors, $k = \frac{i_g}{i}$, of the different motors, all under the assumption of the same constants:

$$
\begin{aligned}
Z &= 0.25 \quad - 3\,j, \\
Z' &= 0.25 \quad - 2.5\,j, \\
Z_0 &= 0.1 \quad\;\; - 0.3\,j, \\
Z_2 &= 0.025 - 0.075\,j, \\
Z_1 &= 0.025 - 0.075\,j, \\
Z_4 &= 4 \qquad\; - 2\,j, \\
c_0 &= 0.3, \\
c_4 &= 0.03.
\end{aligned}
$$

Curve I gives the commutation factor of the motor as inductively compensated series motor ($t = 0$), as constant, $k = 3.82$,

that is, the current change at leaving the brushes is 3.82 times the main current. Such condition, under continued operation, would give destructive sparking.

Curve II shows the series repulsion motor, with 20 per cent of the voltage on the compensating winding, $t = 0.2$; and

Curve III with half the voltage on the compensating winding, $t = 0.5$.

Curve IV corresponds to $t = 1$, or all the voltage on the compensating winding, and the armature circuit closed upon itself: repulsion motor with secondary excitation.

Curve V corresponds to $t = 2$, or full voltage in reverse direction impressed upon the armature, double voltage on the compensating winding.

Curve VI gives the minimum commutation factor, as derived by varying t with the speed, in the manner discussed before.

For further comparison are given, for the same motor constants:

Curve VII, the plain repulsion motor, showing its good commutation below synchronism, and poor commutation above synchronism, and:

Curve VIII, an overcompensated series motor, that is, conductively compensated series motor, in which the compensating winding contains 20 per cent more ampere-turns than the armature, so giving 20 per cent overcompensation.

As seen, overcompensation does not appreciably improve commutation at low speeds, and spoils it at higher speeds.

Fig. 207 also gives the two components of the compensating e.m.f., E_2, which are required to give perfect commutation, or zero commutation current;

$t_0' = \frac{e_2'}{e}$ = component in phase with e, giving quadrature flux;

$t_0'' = \frac{e_2''}{e}$ = component in quadrature with e, giving flux in phase with e.

315. In direct-current motors, overcompensation greatly improves commutation, and so is used in the form of a compensating winding, commutating pole or interpole. In such direct-current motors, the reverse field of the interpole produces

a current in the short-circuited armature coil, by its rotation, in the same direction as the armature current in the coil after leaving the brushes, and by proper proportioning of the commutating field, the commutation current, i_g, thus can be made to vanish, that is, perfect commutation produced.

In alternating-current motors, to make the commutation current vanish and so produce perfect commutation, the current in the short-circuited coil must not only be equal to the armature current in intensity, but also in phase, that is, the commutating field must not only have the proper intensity, but also the proper phase.

In paragraph 310 we have seen that the commutating field has the proper phase to make i_g vanish, if produced by a voltage impressed upon the compensating winding

$$E_2 = Te,$$

which for all except very low speeds is very nearly in phase with e. The magnetic flux produced by this voltage, or the commutating flux, so is nearly in quadrature with e, and therefore approximately in quadrature with the current in the motor, at such speeds where the current, i, is nearly in phase with e. The commutating flux produced by conductive overcompensation, however, is in phase with the current, i, hence is of a wrong phase properly to commutate.

That is, in the alternating-current commutator motor, the commutating flux should be approximately in quadrature with the main flux or main current, and so cannot be produced by the main current by overcompensation, but is produced by the combined magnetizing action of the main current and a secondary current produced thereby, since in a transformer the resultant flux lags approximately 90 degrees behind the primary current.

The same results we can get directly by investigating the commutation current of the overcompensated series motor. This motor is characterized by

(1) $\quad \dot{e} = \dot{E}_1 + c_0\dot{E}_0 + c_2\dot{E}_2;$

where $c_2 = 1 + q$ = reduction factor of compensating circuit to armature.

(2) $\quad \dot{I}_0 = c_0\dot{I}, \; \dot{I}_2 = c_2\dot{I}, \; \dot{I}_1 = \dot{I}.$

Substituting into the fundamental equations of the single-phase commutator motor gives the results,

$$\underset{\cdot}{I} = \frac{e}{ZK},$$

$$\underset{\cdot}{I}_4 = \frac{\lambda_4 (c_0 + jSqA)}{ZK} e,$$

$$\underset{\cdot}{I}_g = \frac{e}{ZK} \left\{ 1 - b\lambda_4 \left(1 + jS\frac{q}{c_0}A \right) \right\};$$

where,

$$ZK = (Z_3 + Z_5 - jSc_0Z) - jS\lambda_4 (c_0Z + jSqZ').$$

To make $\underset{\cdot}{I}_g$ vanish, it must therefore be,

$$q = \frac{c}{SbA} \left\{ \frac{\lambda_4''}{\lambda_4^2} + j\left(b - \frac{\lambda_4'}{\lambda_4^2} \right) \right\},$$

or approximately,

$$q = j\frac{c_0}{S}\frac{x}{x'}$$

or, with the numerical values of the preceding instance,

$$q = \frac{0.046 + 0.295\,j}{S}.$$

That is, the overcompensating component, q, must be approximately in quadrature with the current, $\underset{\cdot}{I}$, hence cannot be produced by this current under the conditions considered here; and overcompensation, while it may under certain conditions improve the commutation, can as a rule not give perfect commutation in a series alternating-current motor.

A discussion and comparison of the different types of asynchronous alternating-current motors is given in International Electrical Congress, St. Louis Transactions, 1904.

316. The preceding study of commutation is based on the assumption of the short-circuit current under the brush as alternating current. This, however, is strictly the case only at standstill, as already discussed in the paragraphs on the repulsion motor. At speed, an exponential term, due to the abrupt

change of current in the armature coil when passing under the brush, superimposes upon the e.m.f. generated in the short-circuited coil, and so on the short-circuit current under the brush, and modifies it the more, the higher the speed, that is, the quicker the current change. This exponential term of e.m.f. generated in the armature coil short-circuited by the commutator brush, is the so-called "e.m.f. of self-induction of commutation." It exists in direct-current motors as well as in alternating-current motors, and is controlled by overcompensation, that is, by a commutating field in phase with the main field, and approximately proportional to the armature current.

The investigation of the exponential term of generated e.m.f. and of short-circuit current, the change of the commutation current and commutation factor brought about thereby and the study of the commutating field required to control this exponential term leads into the theory of transient phenomena, that is, phenomena temporarily occurring during and immediately after a change of circuit condition.*

The general conclusions are:

The control of the e.m.f. of self-induction of commutation of the single-phase commutator motor requires a commutating field, that is, a field in quadrature position in space to the main field, approximately proportional to the armature current and in phase with the armature current, hence approximately in phase with the main field.

Since the commutating field required to control, in the armature coil under the commutator brush, the e.m.f. of alternation of the main field, is approximately in quadrature behind the main field — and usually larger than the field controlling the e.m.f. of self-induction of commutation — it follows that the total commutating field, or the quadrature flux required to give best commutation, must be ahead of the values derived in paragraphs 309 to 311.

As the field required by the e.m.f. of alternation in the short-circuited coil was found to lag for speeds below the speed of best commutation, and to lead above this speed, from the position in quadrature behind the main field, the total commutating

* See "Theory and Calculations of Transient Electric Phenomena and Oscillations," Sections I and II.

field must lead this field controlling the e.m.f. of alternation, and it follows:

Choosing the e.m.f., $\dot{E}_2$, impressed upon the compensating winding in phase with, and its magnetic flux, therefore in quadrature (approximately), behind the main field, gives a commutation in the repulsion and the series repulsion motor which is better than that calculated from paragraphs 309 to 311, for all speeds up to the speed of best commutation, but becomes inferior for speeds above this. Hence the commutation of the repulsion motor and of the series repulsion motor, when considering the self-induction of commutation, is superior to the calculated values below, inferior above the critical speed, that is, the speed of minimum commutation current. The commutation of the overcompensated series motor is superior to the values calculated in the preceding, though not of the same magnitude as in the motors with quadrature commutating flux.

It also follows that an increase of the inductive reactance of the armature coil increases the exponential and decreases the alternating term of e.m.f and therewith the current in the short-circuited coil, and therefore requires a commutating flux earlier in phase than that required by an armature coil of lower reactance, hence improves the commutation of the series repulsion and the repulsion motor at low speeds, and spoils it at high speeds, as seen from the phase angles of the commutating flux calculated in paragraphs 309 to 311.

Causing the armature current to lag, by inserting external inductive reactance into the armature circuit, has the same effect as leading commutating flux: it improves commutation at low, impairs it at high speeds. In consequence hereof the commutation of the repulsion motor with secondary excitation — in which the inductive reactance of the main field circuit is in the armature circuit — is usually superior, at moderate speeds, to that of the repulsion motor with primary excitation, except at very low speeds, where the angle of lag of the armature current is very large.

CHAPTER XXVIII.

REACTION MACHINES.

317. In the chapters on Alternating-Current Generators and on Induction Motors, the assumption has been made that the reactance, x, of the machine is a constant. While this is more or less approximately the case in many alternators, in others, especially in machines of large armature reaction, the reactance, x, is variable, and is different in the different positions of the armature coils in the magnetic circuit. This variation of the reactance causes phenomena which do not find their explanation by the theoretical calculations made under the assumption of constant reactance.

It is known that synchronous motors of large and variable reactance keep in synchronism, and are able to do a considerable amount of work, and even carry under circumstances full load, if the field-exciting circuit is broken, and thereby the counter e.m.f., E_1, reduced to zero, and sometimes even if the field circuit is reversed and the counter e.m.f.. E_1, made negative.

Inversely, under certain conditions of load, the current and the e.m.f. of a generator do not disappear if the generator field circuit is broken, or even reversed to a small negative value, in which latter case the current is against the e.m.f., E_0, of the generator.

Furthermore, a shuttle armature without any winding will in an alternating magnetic field revolve when once brought up to synchronism, and do considerable work as a motor.

These phenomena are not due to remanent magnetism nor to the magnetizing effect of Foucault currents, because they exist also in machines with laminated fields, and exist if the alternator is brought up to synchronism by external means and the remanent magnetism of the field poles destroyed beforehand by application of an alternating current.

318. These phenomena cannot be explained under the assumption of a constant synchronous reactance; because in this case, at no-field excitation, the e.m.f. or counter e.m.f. of the machine is zero, and the only e.m.f. existing in the alternator is the e.m.f. of self-induction; that is, the e.m.f. induced by the alternating current upon itself. If, however, the synchronous reactance is constant, the counter e.m.f. of self-induction is in quadrature with the current and wattless; that is, can neither produce nor consume energy.

In the synchronous motor running without field excitation, always a large lag of the current behind the impressed e.m.f. exists; and an alternating generator will yield an e.m.f. without field excitation only when closed by an external circuit of large negative reactance; that is, a circuit in which the current leads the e.m.f., as a condenser, or an over-excited synchronous motor, etc.

Self-excitation of the alternator by armature reaction can be explained by the fact that the counter e.m.f. of self-induction is not wattless or in quadrature with the current, but contains an energy component; that is, that the reactance is of the form $X = h - jx$, where x is the wattless component of reactance and h the energy component of reactance, and h is positive if the reactance consumes power — in which case the counter e.m.f. of self-induction lags more than 90° behind the current — while h is negative if the reactance produces power — in which case the counter e.m.f. of self-induction lags less than 90° behind the current.

319. A case of this nature has been discussed already in the chapter on Hysteresis, from a different point of view. There the effect of magnetic hysteresis was found to distort the current wave in such a way that the equivalent sine wave, that is, the sine wave of equal effective strength and equal power with the distorted wave, is in advance of the wave of magnetism by what is called the angle of hysteretic advance of phase α. Since the e.m.f. generated by the magnetism, or counter e.m.f. of self-induction lags 90° behind the magnetism, it lags 90° + α behind the current; that is, the self-induction in a circuit containing iron is not in quadrature with the current and thereby wattless, but lags more than 90° and thereby consumes power, so that

the reactance has to be represented by $X = h - jx$, where h is what has been called the "effective hysteretic resistance."

A similar phenomenon takes place in alternators of variable reactance, or, what is the same, variable magnetic reluctance.

320. Obviously, if the reactance or reluctance is variable, it will perform a complete cycle during the time the armature coil moves from one field pole to the next field pole, that is, during one-half wave of the main current. That is, in other words, the reluctance and reactance vary with twice the frequency of the alternating main current. Such a case is shown in Figs. 208 and 209. The impressed e.m.f., and thus at negligible resistance, the counter e.m.f., is represented by the sine wave, E, thus the magnetism produced thereby is a sine wave, Φ, 90° ahead of E. The reactance is represented by the sine wave, x, varying with the double frequency of E, and shown in Fig. 208 to reach the maximum value during the rise of magnetism, in Fig. 209 during the decrease of magnetism. The current, I, required to produce the magnetism, Φ, is found from Φ and x in combination with the cycle of molecular magnetic friction of the material, and the power, P, is the product, IE. As seen in Fig. 208, the positive part of P is larger than the negative part; that is, the machine produces electrical energy as generator. In Fig. 209 the negative part of P is larger than the positive; that is, the machine consumes electrical energy and produces mechanical energy as synchronous motor. In Figs. 210 and 211 are given the two hysteretic cycles or looped curves, Φ, I under the two conditions. They show that, due to the variation of reactance, x, in the first case, the hysteretic cycle has been overturned so as to represent, not consumption, but production of electrical energy, while in the second case the hysteretic cycle has been widened, representing not only the electrical energy consumed by molecular magnetic friction, but also the mechanical output.

321. It is evident that the variation of reluctance must be symmetrical with regard to the field poles; that is, that the two extreme values of reluctance, maximum and minimum, will take place at the moment when the armature coil stands in front of the field pole, and at the moment when it stands midway between the field poles.

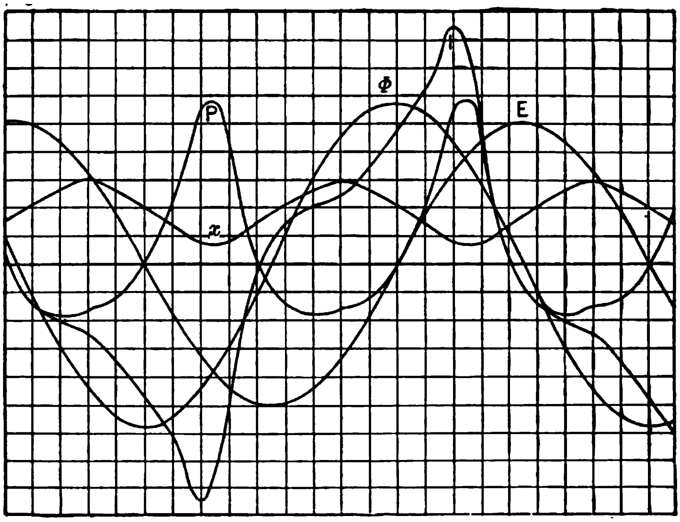


FIG. 208. — Variable Reactance, Reaction Machine.

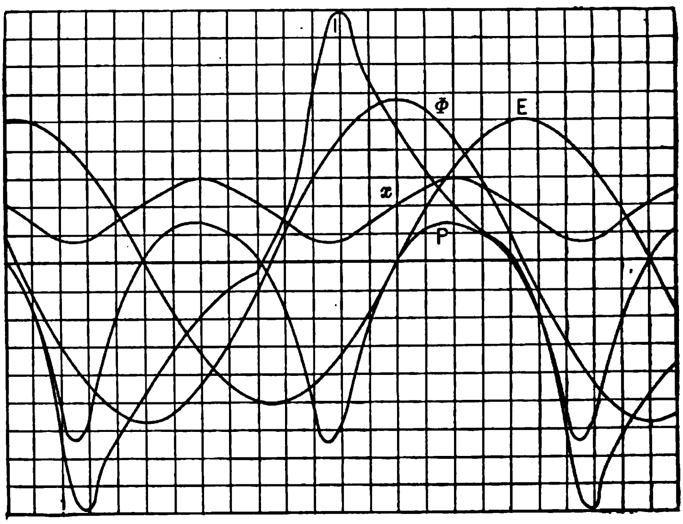


FIG. 209. — Variable Reactance, Reaction Machine.

The effect of this periodic variation of reluctance is a distortion of the wave of e.m.f., or of the wave of current, or of both. Here again, as before, the distorted wave can be replaced by the equivalent sine wave, or sine wave of equal effective intensity and equal power.

The instantaneous value of magnetism produced by the armature current — which magnetism generates in the armature conductor the e.m.f. of self-induction — is proportional to the instantaneous value of the current divided by the instan-

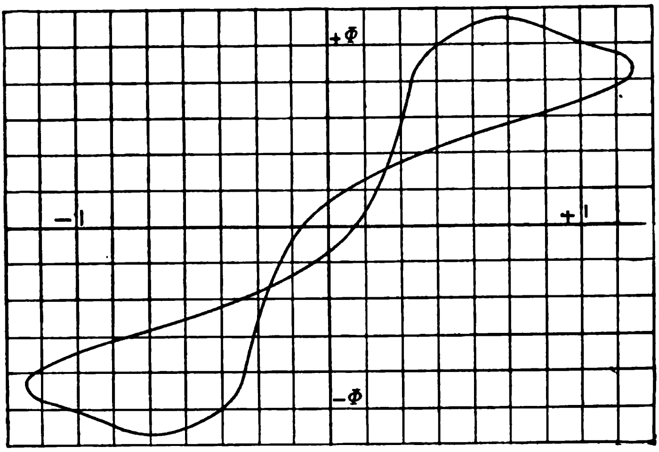


FIG. 210. — Hysteretic Loop of Reaction Machine.

taneous value of the reluctance. Since the extreme values of the reluctance coincide with the symmetrical positions of the armature with regard to the field poles — that is, with zero and maximum value of the generated e.m.f., E_0, of the machine — it follows that, if the current is in phase or in quadrature with the generated e.m.f., E_0, the reluctance wave is symmetrical to the current wave, and the wave of magnetism therefore symmetrical to the current wave also. Hence the equivalent sine wave of magnetism is of equal phase with the current wave; that is, the e.m.f. of self-induction lags 90° behind the current, or is wattless.

Thus at no-phase displacement, and at 90° phase displacement, a reaction machine can neither produce electrical power nor mechanical power.

322. If, however, the current wave differs in phase from the wave of e.m.f. by less than 90°, but more than zero degrees, it is unsymmetrical with regard to the reluctance wave, and the reluctance will be higher for rising current than for decreasing current, or it will be higher for decreasing than for rising current, according to the phase relation of current with regard to generated e.m.f., E_0.

In the first case, if the reluctance is higher for rising, lower for decreasing, current, the magnetism, which is proportional to current divided by reluctance, is higher for decreasing than for

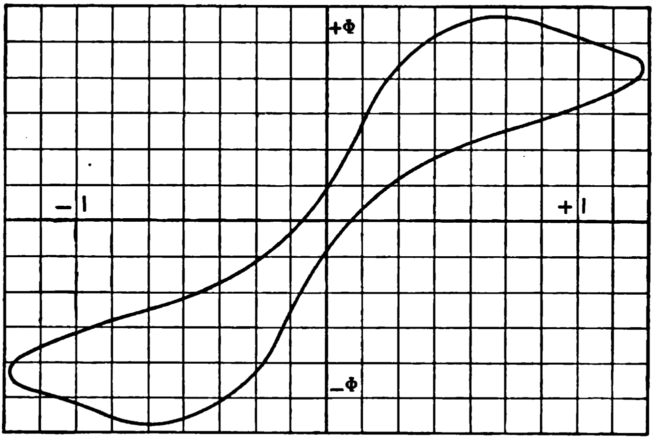


FIG. 211. — Hysteretic Loop of Reaction Machine.

rising current; that is, its equivalent sine wave lags behind the sine wave of current, and the e.m.f. or self-induction will lag more than 90° behind the current; that is, it will consume electrical power, and thereby deliver mechanical power, and do work as a synchronous motor.

In the second case, if the reluctance is lower for rising, and higher for decreasing, current, the magnetism is higher for rising than for decreasing current, or the equivalent sine wave of magnetism leads the sine wave of the current, and the counter e.m.f. of self-induction lags less than 90° behind the current; that is, yields electric power as generator, and thereby consumes mechanical power.

In the first case the reactance will be represented by $X = h - jx$, as in the case of hysteresis; while in the second case the reactance will be represented by $X = -h - jx$.

323. The influence of the periodical variation of reactance will obviously depend upon the nature of the variation, that is, upon the shape of the reactance curve. Since, however, no matter what shape the wave has, it can always be resolved in a series of sine waves of double frequency, and its higher harmonics, in first approximation the assumption can be made that the reactance or the reluctance varies with double frequency of the main current; that is, is represented in the form,

$$x = a + b \cos 2\beta.$$

Let the inductance be represented by

$$\begin{aligned} L &= l + l' \cos 2\beta, \\ &= l\,(1 + \gamma \cos 2\beta); \end{aligned}$$

where γ = amplitude of variation of inductance.

Let

θ = angle of lag of zero value of current behind maximum value of the inductance L.

Then, assuming the current as sine wave, or replacing it by the equivalent sine wave of effective intensity, I, current,

$$i = I\sqrt{2} \sin (\beta - \theta).$$

The magnetism produced by this current is

$$\Phi = \frac{Li}{n},$$

where n = number of turns.

Hence, substituted,

$$\Phi = \frac{lI\sqrt{2}}{n} \sin (\beta - \theta)\,(1 + \gamma \cos 2\beta),$$

or, expanded,

$$\Phi = \frac{lI\sqrt{2}}{n} \left\{ \left(1 - \frac{\gamma}{2}\right) \cos \theta \sin \beta - \left(1 + \frac{\gamma}{2}\right) \sin \theta \cos \beta \right\},$$

when neglecting the term of triple frequency as wattless.

Thus the e.m.f. generated by this magnetism is

$$e = -n\frac{d\Phi}{dt}$$

$$= -2\pi f n\frac{d\Phi}{d\beta};$$

hence, expanded,

$$e = -2\pi f l I\sqrt{2}\left\{\left(1-\frac{\gamma}{2}\right)\cos\theta\cos\beta + \left(1+\frac{\gamma}{2}\right)\sin\theta\sin\beta\right\},$$

and the effective value of e.m.f.,

$$E = 2\pi f l I\sqrt{\left(1-\frac{\gamma}{2}\right)^2\cos^2\theta + \left(1+\frac{\gamma}{2}\right)^2\sin^2\theta}$$

$$= 2\pi f l I\sqrt{1+\frac{\gamma^2}{4}-\gamma\cos 2\theta}.$$

Hence, the apparent power, or the volt-amperes,

$$P_{a_0} = IE = 2\pi f l I^2\sqrt{1+\frac{\gamma^2}{4}-\gamma\cos 2\theta}$$

$$= \frac{E^2}{2\pi f l\sqrt{1+\frac{\gamma^2}{4}-\gamma\cos 2\theta}}.$$

The instantaneous value of power is

$$p = ei$$

$$= -4\pi f l I^2\sin(\beta-\theta)\left\{\left(1-\frac{\gamma}{2}\right)\cos\theta\cos\beta + \left(1+\frac{\gamma}{2}\right)\sin\theta\sin\beta\right\};$$

and, expanded,

$$p = -2\pi f l I^2\left\{\left(1+\frac{\gamma}{2}\right)\sin 2\theta\sin^2\beta - \left(1-\frac{\gamma}{2}\right)\sin 2\theta\cos^2\beta + \sin 2\beta\left(\cos 2\theta - \frac{\gamma}{2}\right)\right\}.$$

Integrated, the effective value of poweris

$$P = -\pi f l I^2\gamma\sin 2\theta;$$

hence, negative, that is, the machine consumes electrical, and produces mechanical, power, as synchronous motor, if $\theta > 0$, that is, with lagging current; positive, that is, the machine produces electrical, and consumes mechanical power, as generator, if $\theta > 0$, that is, with leading current.

The power-factor is

$$p = \frac{P}{P_{a_0}} = \frac{\gamma \sin 2\theta}{2\sqrt{1 + \frac{\gamma^2}{4} - \gamma \cos 2\theta}};$$

hence, a maximum, if

$$\frac{df}{d\theta} = 0;$$

or, expanded,

$$\cos 2\theta = \frac{1}{\gamma} + \frac{\gamma}{4} \pm \frac{1}{4}\sqrt{8 + \gamma^2}.$$

The power, P, is a maximum at given current, I, if

$$\sin 2\theta = 1;$$

that is,

$$\theta = 45°;$$

at given e.m.f., E, the power is

$$P = -\frac{E^2 \gamma \sin 2\theta}{4\pi f l\left(1 + \frac{\gamma^2}{4} - \gamma \cos 2\theta\right)};$$

hence, a maximum at

$$\frac{dP}{d\theta} = 0;$$

or, expanded,

$$\cos 2\theta = \frac{\pm\gamma}{1 + \frac{\gamma^2}{4}}.$$

324. We have thus, at impressed e.m.f., E, and negligible resistance, if we denote the mean value of reactance

$$x = 2\pi f l.$$

Current

$$I = \frac{E}{x\sqrt{1 + \frac{\gamma^2}{4} - \gamma \cos 2\theta}}.$$

Volt-amperes,

$$P_{a_0} = \frac{E^2}{x\sqrt{1 + \frac{\gamma^2}{4} - \gamma \cos 2\theta}}.$$

Power,

$$P = -\frac{E^2 \gamma \sin 2\theta}{2x\left(1 + \frac{\gamma^2}{4} - \gamma \cos 2\theta\right)}.$$

Power-factor,

$$p = \cos(E, I) = \frac{\gamma \sin 2\theta}{2\sqrt{1 + \frac{\gamma^2}{4} - \gamma \cos 2\theta}}.$$

Maximum power at

$$\cos 2\theta = \frac{\gamma}{1 + \frac{\gamma^2}{4}}.$$

Maximum power-factor at

$$\cos 2\theta = \frac{1}{\gamma} + \frac{\gamma}{4} \pm \frac{1}{4}\sqrt{8 + \gamma^2}.$$

$\theta > 0$: synchronous motor, with lagging current,
$\theta < 0$: generator, with leading current.

As an example is shown in Fig. 212, with angle θ as abscissas, the values of current, power, and power-factor, for the constants, $E = 110$, $x = 3$, and $\gamma = 0.8$.

$$I = \frac{41}{\sqrt{1.45 - \cos 2\theta}},$$

$$P = \frac{-2017 \sin 2\theta}{1.45 - \cos 2\theta},$$

$$p = \cos(E, I) = \frac{0{:}447 \sin 2\theta}{\sqrt{1.45 - \cos 2\theta}}.$$

As seen from Fig. 212, the power-factor, p, of such a machine is very low — does not exceed 40 per cent in this instance.

Very similar to the reaction machine in principle and character of operation are the synchronous induction motor, section 208,

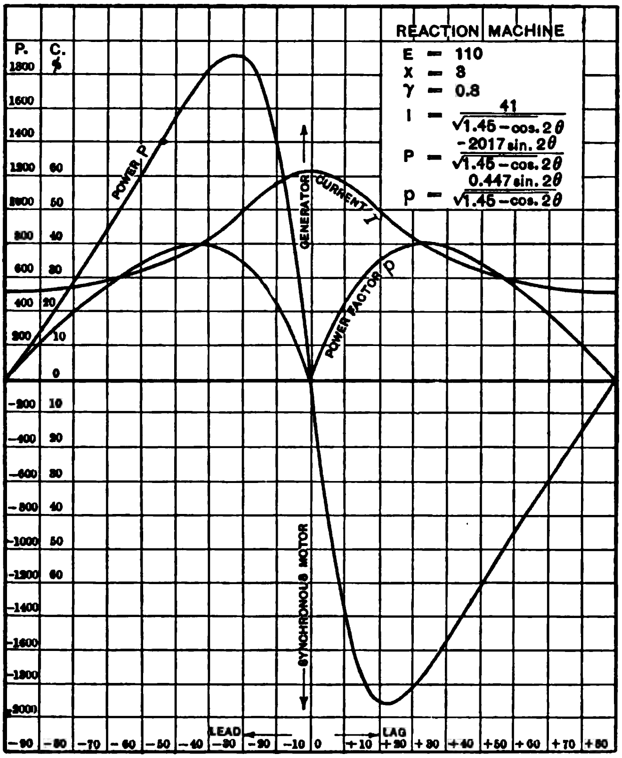


FIG. 212. — Reaction Machine.

and the hysteresis motor, section 209, either of which is a generator above synchronism, and at synchronism can be motor as well as generator, depending on the relative position between stator field and rotor.

CHAPTER XXIX.

DISTORTION OF WAVE-SHAPE AND ITS CAUSES.

325. In the preceding chapters we have considered the alternating currents and alternating e.m.fs. as sine waves or as replaced by their equivalent sine waves.

While this is sufficiently exact in most cases, under certain circumstances the deviation of the wave from sine shape becomes of importance, and with certain distortions it may not be possible to replace the distorted wave by an equivalent sine wave, since the angle of phase displacement of the equivalent sine wave becomes indefinite. Thus it becomes desirable to investigate the distortion of the wave, its causes and its effects.

Since, as stated before, any alternating wave can be represented by a series of sine functions of odd orders, the investigation of distortion of wave-shape resolves itself in the investigation of the higher harmonics of the alternating wave.

In general we have to distinguish between higher harmonics of e.m.f. and higher harmonics of current. Both depend upon each other in so far as with a sine wave of impressed e.m.f. a distorting effect will cause distortion of the current wave, while with a sine wave of current passing through the circuit, a distorting effect will cause higher harmonics of e.m.f.

326. In a conductor revolving with uniform velocity through a uniform and constant magnetic field, a sine wave of e.m.f. is generated. In a circuit with constant resistance and constant reactance, this sine wave of e.m.f. produces a sine wave of current. Thus distortion of the wave-shape or higher harmonics may be due to lack of uniformity of the velocity of the revolving conductor; lack of uniformity or pulsation of the magnetic field; pulsation of the resistance or pulsation of the reactance.

The first two cases, lack of uniformity of the rotation or of the magnetic field, cause higher harmonics of e.m.f. at open

circuit. The last, pulsation of resistance and reactance, causes higher harmonics only when there is current in the circuit, that is, under load.

Lack of uniformity of the rotation is of no practical interest as a cause of distortion, since in alternators, due to mechanical momentum, the speed is always very nearly uniform during the period.

Thus as causes of higher harmonics remain:

1st. Lack of uniformity and pulsation of the magnetic field, causing a distortion of the generated e.m.f. at open circuit as well as under load.

2d. Pulsation of the reactance, causing higher harmonics under load.

3d. Pulsation of the resistance, causing higher harmonics under load also.

Taking up the different causes of higher harmonics we have: —

Lack of Uniformity and Pulsation of the Magnetic Field.

327. Since most of the alternating-current generators contain definite and sharply defined field poles covering in different types different proportions of the pitch, in general the magnetic flux interlinked with the armature coil will not vary as a sine wave, of the form

$$\Phi \cos \beta,$$

but as a complex harmonic function, depending on the shape and the pitch of the field-poles, and the arrangement of the armature conductors. In this case, the magnetic flux issuing from the field-pole of the alternator can be represented by the general equation,

$$\begin{aligned}\Phi = A_0 &+ A_1 \cos \beta + A_2 \cos 2\beta + A_3 \cos 3\beta + \ldots \\ &+ B_1 \sin \beta + B_2 \sin 2\beta + B_3 \sin 3\beta + \ldots\end{aligned}$$

If the reluctance of the armature is uniform in all directions, so that the distribution of the magnetic flux at the field-pole face does not change by the rotation of the armature, the rate of cutting magnetic flux by an armature conductor is Φ, and the e.m.f. generated in the conductor thus equal thereto in

wave-shape. As a rule A_0, A_2, A_4 . . . B_2, B_4 equal zero; that is, successive field-poles are equal in strength and distribution of magnetism, but of opposite polarity. In some types of machines, however, especially induction alternators, this is not the case.

The e.m.f. generated in a full-pitch armature turn — that is, armature conductor and return conductor distant from former by the pitch of the armature pole (corresponding to the distance from field-pole center to pole center) is

$$\begin{aligned}\delta e &= \Phi_0 - \Phi_{180} \\ &= 2\ \{A_1 \cos\beta + A_3 \cos 3\beta + A_5 \cos 5\beta + \ldots \\ &\qquad + B_1 \sin\beta + B_3 \sin 3\beta + B_5 \sin 5\beta + \ldots\}\end{aligned}$$

Even with an unsymmetrical distribution of the magnetic flux in the air-gap, the e.m.f. wave generated in a full-pitch armature coil is symmetrical; the positive and negative half-waves equal, and correspond to the mean flux distribution of adjacent poles. With fractional pitch-windings — that is, windings whose turns cover less than the armature pole-pitch — the generated e.m.f. can be unsymmetrical with unsymmetrical magnetic field, but as a rule is symmetrical also. In unitooth alternators the total generated e.m.f. has the same shape as that generated in a single turn.

With the conductors more or less distributed over the surface of the armature, the total generated e.m.f. is the resultant of several e.m.fs. of different phases, and is thus more uniformly varying; that is, more sinusoidal, approaching sine shape, to within 3 per cent or less, as for instance the curves Fig. 213 and Fig. 214 show, which represent the no-load and full-load wave of e.m.f. of a three-phase multitooth alternator. The principal term of these harmonics is the third harmonic, which consequently appears more or less in all alternator waves. As a rule these harmonics can be considered together with the harmonics due to the varying reluctance of the magnetic circuit.

In iron-clad alternators with few slots and teeth per pole, the passage of slots across the field-poles causes a pulsation of the magnetic reluctance, or its reciprocal, the magnetic reactance of the circuit. In consequence thereof the magnetism per field-pole, or at least that part of the magnetism passing through the

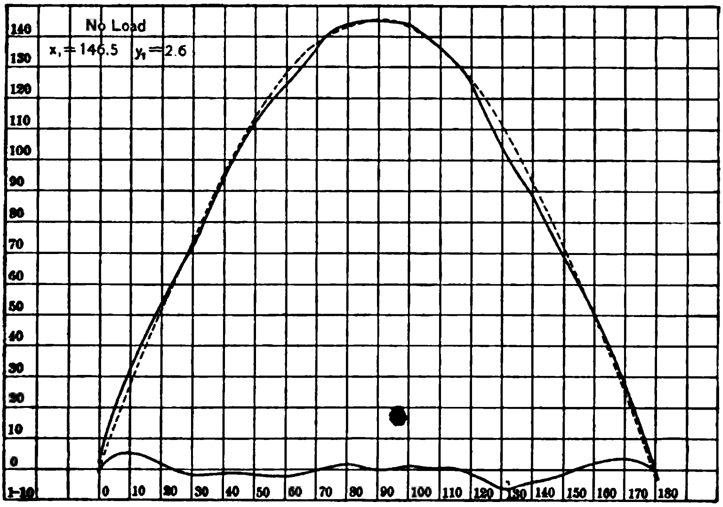


FIG. 213. — No-load Wave of E.M.F. of Multitooth Three-phaser.

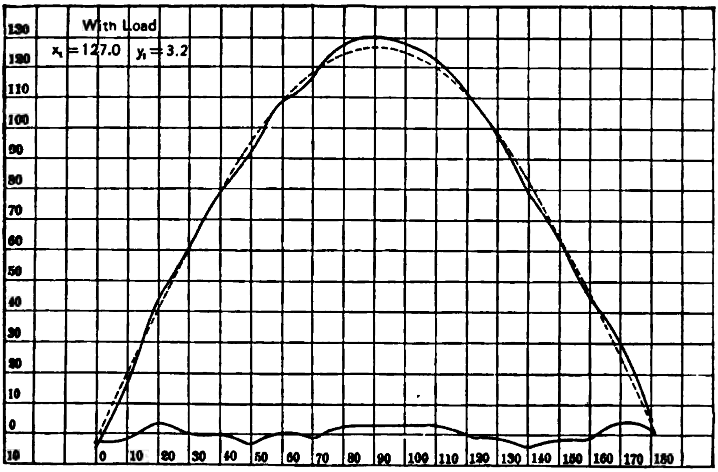


FIG. 214. — Full-Load Wave of E.M.F. of Multitooth Three-phaser.

armature, will pulsate with a frequency, 2γ, if γ = number of slots per pole.

Thus, in a machine with one slot per pole, the instantaneous magnetic flux interlinked with the armature conductors can be expressed by the equation

$$\phi = \Phi \cos\beta \left\{1 + \varepsilon \cos[2\beta - \theta]\right\}$$

where Φ = average magnetic flux,

ε = amplitude of pulsation,

and θ = phase of pulsation.

In a machine with γ slots per pole, the instantaneous flux interlinked with the armature conductors will be

$$\phi = \Phi \cos\beta \left\{1 + \varepsilon \cos[2\gamma\beta - \theta]\right\},$$

if the assumption is made that the pulsation of the magnetic flux follows a simple sine law, as first approximation.

In general the instantaneous magnetic flux interlinked with the armature conductors will be

$$\phi = \Phi \cos\beta \left\{1 + \varepsilon_1 \cos(2\beta - \theta_1) + \varepsilon_2 \cos(4\beta - \theta_2) + \ldots\right\},$$

where the term, ε_γ, is predominating if γ = number of armature slots per pole. This general equation includes also the effect of lack of uniformity of the magnetic flux.

In case of a pulsation of the magnetic flux with the frequency, 2γ, due to an existence of γ slots per pole in the armature, the instantaneous value of magnetism interlinked with the armature coil is

$$\phi = \Phi \cos\beta \left\{1 + \varepsilon \cos[2\gamma\beta - \theta]\right\}.$$

Hence the e.m.f. generated thereby,

$$\begin{aligned} e &= - n \frac{d\phi}{dt} \\ &= - \sqrt{2}\,\pi f n \Phi \frac{d}{d\beta} \left\{\cos\beta\,(1 + \varepsilon \cos[2\gamma\beta - \theta])\right\}. \end{aligned}$$

And, expanded,

$$e = \sqrt{2}\,\pi f n \Phi \left\{ \sin \beta + \varepsilon \frac{2\,\gamma - 1}{2} \sin\left[(2\,\gamma - 1)\,\beta - \theta\right] \right.$$
$$\left. + \,\varepsilon \frac{2\,\gamma + 1}{2} \sin\left[(2\,\gamma + 1)\,\beta - \theta\right] \right\}.$$

Hence, the pulsation of the magnetic flux with the frequency, $2\,\gamma$, as due to the existence of γ slots per pole, introduces two harmonics, of the orders $(2\,\gamma - 1)$ and $(2\,\gamma + 1)$.

328. If $\gamma = 1$ it is

$$e = \sqrt{2}\,\pi f n \Phi \left\{ \sin \beta + \frac{\varepsilon}{2} \sin\,(\beta - \theta) + \frac{3\,\varepsilon}{2} \sin\,(3\,\beta - \theta) \right\};$$

that is, in a unitooth single-phaser a pronounced triple harmonic may be expected, but no pronounced higher harmonics.

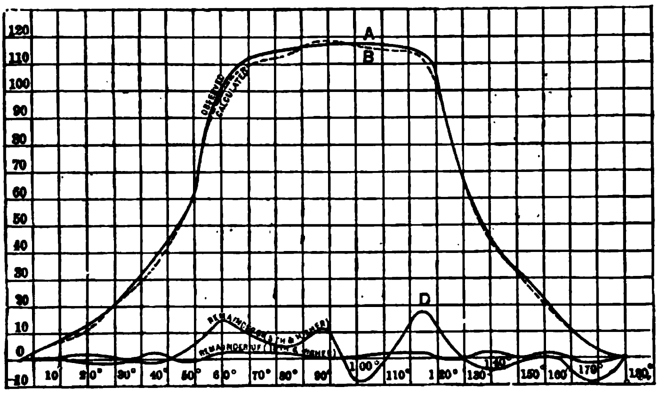


Fig. 215.—No-load Wave of E.M.F. of Unitooth Monocyclic Alternator.

Fig. 215 shows the wave of e.m.f. of the main coil of a monocyclic alternator at no load, represented by,

$$e = E\,\{\sin \beta - 0.242 \sin\,(3\,\beta - 6.3) - 0.046 \sin\,(5\,\beta - 2.6)$$
$$+ 0.068 \sin\,(7\,\beta - 3.3) - 0.027 \sin\,(9\,\beta - 10.0) - 0.018$$
$$\sin\,(11\,\beta - 6.6) + 0.029 \sin\,(13\,\beta - 8.2)\};$$

hence giving a pronounced triple harmonic only, as expected.

If $\gamma = 2$, it is,

$$e = \sqrt{2}\pi f n \Phi \left\{ \sin \beta + \frac{3\varepsilon}{2} \sin (3\beta - \theta) + \frac{5\varepsilon}{2} \sin (5\beta - \theta) \right\},$$

the no-load wave of a unitooth quarter-phase machine, having pronounced triple and quintuple harmonics.

If $\gamma = 3$, it is,

$$e = \sqrt{2}\pi f n \Phi \left\{ \sin \beta + \frac{5\varepsilon}{2} \sin (5\beta - \theta) + \frac{7\varepsilon}{2} \sin (7\beta - \theta) \right\}.$$

That is, in a unitooth three-phaser, a pronounced quintuple and septuple harmonic may be expected, but no pronounced triple harmonic.

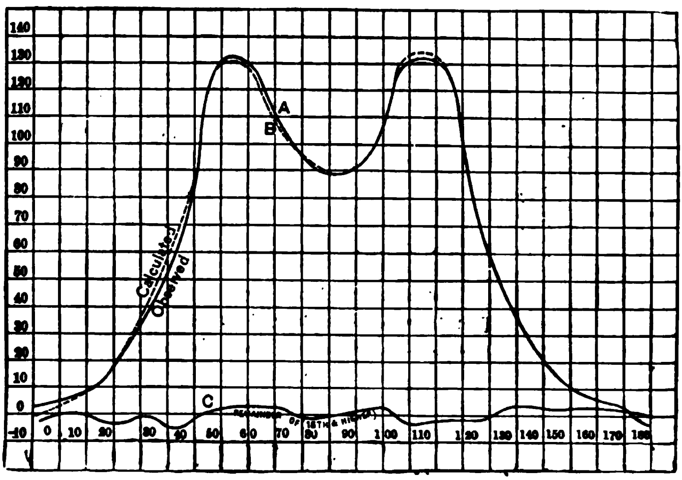


FIG. 216.—No-load Wave of E.M.F. of Unitooth Three-Phase Alternator.

Fig. 216 shows the wave of e.m.f. of a unitooth three-phaser at no-load, represented by

$$e = E \{ \sin \beta - 0.12 \sin (3\beta - 2.3) - 0.23 \sin (5\beta - 1.5) + 0.134 \sin (7\beta - 6.2) - 0.002 \sin (9\beta + 27.7) - 0.046 \sin (11\beta - 5.5) + .031 \sin (13\beta - 61.5) \}.$$

Thus giving a pronounced quintuple and septuple and a lesser triple harmonic, probably due to the deviation of the field

from uniformity, as explained above, and deviation of the pulsation of reluctance from sine-shape. In some especially favorable cases, harmonics as high as the 23d and 25th have been observed, caused by pulsation of the reluctance, and even still higher harmonics.

In general, if the pulsation of the magnetic reactance is denoted by the general expression

$$1 + \sum_1^\infty {}^\gamma \epsilon_\gamma \cos (2 \gamma\beta - \theta_\gamma),$$

the instantaneous magnetic flux is

$$\begin{aligned}\phi &= \Phi \cos \beta \left\{1 + \sum_1^\infty {}^\gamma \epsilon_\gamma \cos (2 \gamma\beta - \theta_\gamma)\right\}\\ &= \Phi \left\{ \cos \beta + \frac{\epsilon_1}{2} \cos (\beta - \theta_1) + \sum_1^\infty {}^\gamma \left[\frac{\epsilon_\gamma}{2} \cos [(2 \gamma + 1)\right.\right.\\ &\qquad \left.\left.\beta - \theta_\gamma] + \frac{\epsilon_{\gamma+1}}{2} \cos [(2 \gamma + 1) \beta - \theta_{\gamma+1}]\right]\right\};\end{aligned}$$

hence, the e.m.f.,

$$\begin{aligned}e &= \sqrt{2}\, \pi f n \Phi \left\{ \sin \beta + \frac{\epsilon_1}{2} \sin (\beta - \theta_1) + \sum_1^\infty {}^\gamma \frac{2 \gamma + 1}{2}\right.\\ &\quad \left.\left[\epsilon_\gamma \sin [(2 \gamma + 1) \beta - \theta_\gamma] + \epsilon_{\gamma+1} \sin [(2 \gamma + 1) \beta - \theta_{\gamma+1}]\right]\right\}.\end{aligned}$$

Pulsation of Reactance.

329. The main causes of a pulsation of reactance are magnetic saturation and hysteresis, and synchronous motion. Since in an iron-clad magnetic circuit the magnetism is not proportional to the m.m.f., the wave of magnetism and thus the wave of e.m.f. will differ from the wave of current. As far as this distortion is due to the variation of permeability, the distortion is symmetrical and the wave of generated e.m.f. represents no power. The distortion caused by hysteresis, or the lag of the magnetism behind the m.m.f., causes an unsymmetrical distortion of the wave which makes the wave of generated e.m.f. differ by more than 90° from the current wave and thereby represents power — the power consumed by hysteresis.

In practice both effects are always superimposed; that is, in a ferric inductive reactance, a distortion of wave-shape takes place due to the lack of proportionality between magnetism and m.m.f. as expressed by the variation in the hysteretic cycle.

This pulsation of reactance gives rise to a distortion consisting mainly of a triple harmonic. Such current waves distorted by hysteresis, with a sine wave of impressed e.m.f., are shown in Figs. 86 and 87, Chapter XIII, on Hysteresis. Inversely, if the current is a sine wave, the magnetism and the e.m.f. will differ from sine-shape.

For further discussion of this distortion of wave-shape by hysteresis, Chapter XIII may be consulted.

330. Distortion of wave-shape takes place also by the pulsation of reactance due to synchronous rotation, as discussed in the chapter on Reaction Machines.

In Figs. 208 and 209, at a sine wave of impressed e.m.f., the distorted current waves have been constructed.

Inversely, if a sine wave of current,

$$i = I \cos \beta,$$

exists through a circuit of synchronously varying reactance, as for instance, the armature of a unitooth alternator or synchronous motor — or, more general, an alternator whose armature reluctance is different in different positions with regard to the field-poles — and the reactance is expressed by

$$X = x \left\{ 1 + \varepsilon \cos (2 \beta - \theta) \right\};$$

or, more general,

$$X = x \left\{ 1 + \sum_1^\infty {}_\gamma \, \varepsilon_\gamma \cos (2 \gamma \beta - \theta_\gamma) \right\};$$

the wave of magnetism is

$$\phi = \frac{X}{2 \pi f n} \cos \beta = \frac{x}{2 \pi f n} \left\{ \cos \beta + \sum_1^\infty {}_\gamma \, \varepsilon_\gamma \cos \beta \cos (2 \gamma \beta - \theta_\gamma) \right\}$$

$$= \frac{x}{2 \pi f n} \left\{ \cos \beta + \frac{\varepsilon_1}{2} \cos (\beta - \theta_1) + \sum_1^\infty {}_\lambda \left[\frac{\varepsilon_\gamma}{2} \cos [(2 \gamma + 1) \beta - \theta_\gamma] + \frac{\varepsilon_\lambda + 1}{2} \cos [(2 \gamma + 1) \beta - \theta_\lambda + 1] \right] \right\};$$

hence the wave of generated e.m.f.,

$$e = -n\frac{d\phi}{dt} = -2\pi f n\frac{d\phi}{d\beta}$$

$$= x\left\{\sin\beta + \frac{\varepsilon_1}{2}\sin(\beta - \theta_1) + \sum_1^{\infty}\gamma\frac{2\gamma+1}{2}\left[\varepsilon_\gamma \sin[(2\gamma+1)\right.\right.$$

$$\left.\left.\beta - \theta_\gamma] + \varepsilon_\gamma + 1\sin[(2\gamma+1)\beta - \theta_\gamma + 1]\right]\right\};$$

that is, the pulsation of reactance of frequency, 2γ, introduces two higher harmonics of the order $(2\gamma - 1)$ and $(2\gamma + 1)$.

If $$X = x\{1 + \varepsilon\cos(2\beta - \theta)\},$$

it is

$$\phi = \frac{x}{2\pi f n}\left\{\cos\beta + \frac{\varepsilon}{2}\cos(\beta - \theta) + \frac{\varepsilon}{2}\cos(3\beta - \theta)\right\};$$

$$e = x\left\{\sin\beta + \frac{\varepsilon}{2}\sin(\beta - \theta) + \frac{3\varepsilon}{2}\sin(3\beta - \theta)\right\}.$$

Since the pulsation of reactance due to magnetic saturation and hysteresis is essentially of the frequency, $2f$ — that is, describes a complete cycle for each half-wave of current — this shows why the distortion of wave-shape by hysteresis consists essentially of a triple harmonic.

The phase displacement between e and i, and thus the power consumed or produced in the electric circuit, depends upon the angle, θ, as discussed before.

331. In case of a distortion of the wave-shape by reactance, the distorted waves can be replaced by their equivalent sine waves, and the investigation with sufficient exactness for most cases be carried out under the assumption of sine waves, as done in the preceding chapters.

Similar phenomena take place in circuits containing polarization cells, leaky condensers, or other apparatus representing a synchronously varying negative reactance. Possibly dielectric hysteresis in condensers causes a distortion similar to that due to magnetic hysteresis.

Pulsation of Resistance.

332. To a certain extent the investigation of the effect of synchronous pulsation of the resistance coincides with that of reactance; since a pulsation of reactance, when unsymmetrical with regard to the current wave, introduces a power component which can be represented by an "effective resistance."

Inversely, an unsymmetrical pulsation of the ohmic resistance introduces a wattless component, to be denoted by "effective reactance."

A typical case of a synchronously pulsating resistance is represented in the alternating arc.

The apparent resistance of an arc depends upon the current through the arc; that is, the apparent resistance of the arc = $\frac{\text{potential difference between electrodes}}{\text{current}}$, is high for small currents, low for large currents. Thus in an alternating arc the apparent resistance will vary during every half-wave of current between a maximum value at zero current and a minimum value at maximum current, thereby describing a complete cycle per half-wave of current.

Let the effective value of current through the arc be represented by I.

Then the instantaneous value of current, assuming the current wave as sine wave, is represented by

$$i = I\sqrt{2}\sin\beta;$$

and the apparent resistance of the arc, in first approximation, by

$$R = r\,(1 + \varepsilon\cos 2\,\beta);$$

thus the potential difference at the arc is

$$\begin{aligned} e = iR &= I\sqrt{2}\,r\sin\beta\,(1 + \varepsilon\cos 2\,\beta) \\ &= rI\sqrt{2}\left\{\left(1 - \frac{\varepsilon}{2}\right)\sin\beta + \frac{\varepsilon}{2}\sin 3\,\beta\right\}.\end{aligned}$$

Hence the effective value of potential difference,

$$E = rI\sqrt{\left(1 - \frac{\varepsilon}{2}\right)^2 + \frac{\varepsilon^2}{4}}$$

$$= rI\sqrt{1 - \varepsilon + \frac{\varepsilon^2}{2}},$$

and the apparent resistance of the arc,

$$r_0 = \frac{E}{I} = r\sqrt{1 - \varepsilon + \frac{\varepsilon^2}{2}}.$$

The instantaneous power consumed in the arc is

$$ie = 2\,rI^2\left\{\left(1 - \frac{\varepsilon}{2}\right)\sin^2\beta + \frac{\varepsilon}{2}\sin\beta\sin 3\,\beta\right\}.$$

Hence the effective power,

$$P = rI^2\left(1 - \frac{\varepsilon}{2}\right).$$

The apparent power, or volt-amperes consumed by the arc,

$$IE = rI^2\sqrt{1 - \varepsilon + \frac{\varepsilon^2}{2}}.$$

Thus the power-factor of the arc,

$$p = \frac{P}{IE} = \frac{1 - \frac{\varepsilon}{2}}{\sqrt{1 - \varepsilon + \frac{\varepsilon}{2}}};$$

that is, less than unity.

333. We find here a case of a circuit in which the power-factor — that is, the ratio of watts to volt-amperes — differs from unity without any displacement of phase; that is, while current and e.m.f. are in phase with each other, but are distorted, the alternating wave cannot be replaced by an equivalent sine wave, since the assumption of equivalent sine wave would introduce a phase displacement,

$$\cos\theta = p$$

of an angle, θ, whose sign is indefinite.

As an example are shown, in Fig. 214, for the constants, $I = 12$, $r = 3$, $\varepsilon = 0.9$, the resistance,

$$R = 3\,(1 + 0.9 \cos 2\,\beta);$$

the current,

$$i = 17 \sin \beta;$$

the potential difference,

$$e = 28\,(\sin \beta + 0.82 \sin 3\,\beta).$$

In this case the effective e.m.f. is

$$E = 25.5;$$

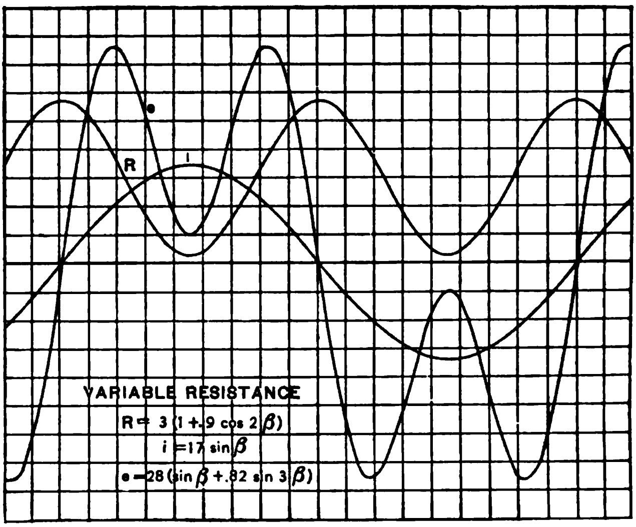


FIG. 217. — Periodically Varying Resistance.

the apparent resistance,

$$r_0 = 2.13;$$

the power,

$$P = 244;$$

the apparent power,

$$EI = 307;$$

the power-factor,

$$p = 0.796.$$

As seen, with a sine wave of current the e.m.f. wave in an alternating arc will become double-peaked, and rise very abruptly near the zero values of current. Inversely, with a sine wave of e.m.f. the current wave in an alternating arc will become peaked, and very flat near the zero values of e.m.f.

334. In reality the distortion is of more complex nature, since the pulsation of resistance in the arc does not follow a simple sine law of double frequency, but varies much more abruptly near the zero value of current, making thereby the variation of e.m.f. near the zero value of current much more abruptly, or, inversely, the variation of current more flat.

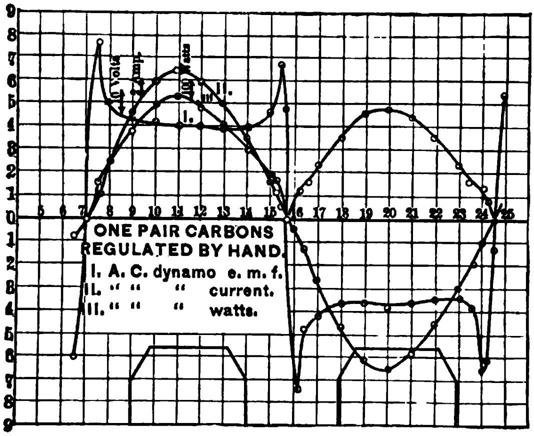


FIG. 218. — Electric Arc.

A typical wave of potential difference, with a sine wave of current through the arc, is given in Fig. 218.*

335. The value of ε, the amplitude of the resistance pulsation, largely depends upon the nature of the electrodes and the steadiness of the arc, and with soft carbons and a steady arc is small, and the power-factor, p, of the arc near unity. With hard carbons and an unsteady arc, ε rises greatly, higher harmonics appear in the pulsation of resistance, and the power-factor, p, falls, being in extreme cases even as low as 0.6.

* From American Institute of Electrical Engineers, Transactions, 1890, p. 376. Tobey and Walbridge, on the Stanley Alternate Arc Dynamo.

This double-peaked appearance of the voltage wave, as shown by Figs. 217 and 218, is characteristic of the arc to such an extent that when in the investigation of an electric circuit by oscillograph such a wave-shape is found, the existence of an arc or arcing ground somewhere in the circuit may usually be suspected. This is of importance as in high-voltage systems arcs are liable to cause dangerous voltages.

The pulsation of the resistance in an arc, as shown in Fig. 218 for hard carbons, is usually very far from sinoidal, as assumed in Fig. 217. It is due to the feature of the arc that the voltage consumed in the arc flame decreases with increase of current — approximately inversely proportional to the square root of the current — and so is lowest at maximum current.

Approximately, the volt-ampere characteristic of the arc can be represented by,

$$e = e_0 + \frac{c}{\sqrt{i}}; \tag{1}$$

where e_0 is a constant of the electrode material (mainly), c a constant depending also upon the electrode material and on the arc length, and approximately proportional thereto.

This equation would give $e = \infty$, for $i = 0$. This obviously is not feasible. However, besides the arc conduction as given by above equation — which depends upon mechanical motion of the vapor stream — a slight conduction also takes place through the residual vapor between the electrodes, as a path of high resistance, r, and near zero current, where the voltage is not sufficient to maintain an arc, this latter conduction carries the current.

The characteristic of the alternating-current arc therefore consists of the combination of two curves; the arc characteristic, (1), and the resistance characteristic,

$$e = ri. \tag{2}$$

The phenomenon then follows that curve which gives the lowest voltage; that is, for high values of current, is represented by equation (1), for low values of current, by equation (2).

336. As an example are shown in Fig. 219 the calculated curves of an alternating arc between hard carbons (or carbides); for the constants,

$$e_0 = 30 \text{ volts},$$
$$c = 40,$$
$$r = 70 \text{ ohms}.$$

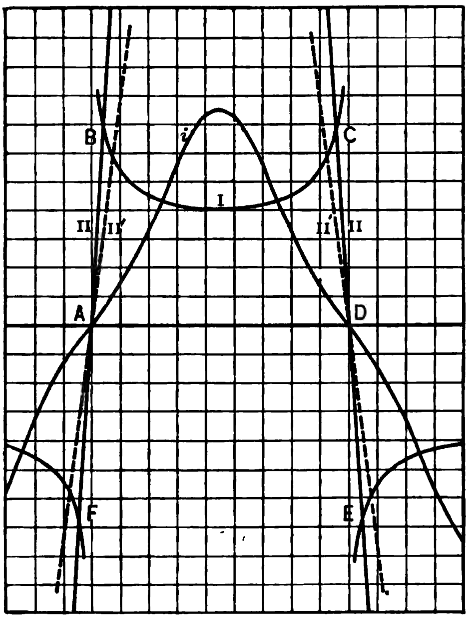


FIG. 219.

The curve, *I*, represents the arc conduction, following equation (1),

$$e = 30 + \frac{40}{\sqrt{i}},$$

and the curve, II, represents the conduction through the (stationary) residual vapor, by equation (2), near the zero points, *A* and *D*, of the current,

$$e = 70\, i.$$

As seen, from *F* to *B* the voltage varies approximately proportionally with the current. At *B* the arc starts, and the volt-

age drops with the further increase of current, and then rises again with the decreasing current, until at C, the intersection point between curves I and II, the arc extinguishes, and the voltage follows curve II, until at E the arc starts again. The two sharp peaks of the curve thus represent respectively the starting and the extinction of the arc.

Since the high values of voltage near zero current lower, and the low values of voltage near maximum current raise the value of the current, the current wave does not remain a sine wave, if the arc voltage is an appreciable part of the total voltage, but the current wave becomes peaked, with flat zero, as expressed approximately by a third harmonic in phase with the fundamental. The current wave in Fig. 211 so has been assumed as

$$i = 13 \cos \phi + 2 \cos 3 \phi.$$

From Fig. 219 follows:

effective value of current, 9.30 amperes,

effective value of voltage, 47.2 volts,

hence, volt-amperes consumed by the arc, 439 volt-amperes; and, by averaging the products of the instantaneous values of volts and amperes,

power consumed in the arc, 388 watts;

hence,

power-factor, 77 per cent.

If the resistance, r, of the residual arc-vapor is lower, as by the use of softer carbons, for instance, given by

$$r = 30 \text{ ohms},$$

as shown by the dotted curve, II′, in Fig. 219, the voltage peaks are greatly cut down, giving a lesser wave-shape distortion, and so,

effective value of voltage, 43.1 volts,

volt-amperes in arc. 395 volt-amperes,

watts in arc, 335 watts,

hence, power-factor, 85 per cent.

Comparing Fig. 219 with 218 shows that 219 fairly well approximates 218, except that in Fig. 218 the second peak is lower than the first. This is due to the lower resistance, r, of the residual vapor, immediately after the passage of the arc, than before the starting of the arc. Fig. 218 also shows a decrease of resistance, r, immediately before starting, or after extinction of the arc, which may be represented by some expression like

$$r = r_0 i^{-b},$$

where $$b < 1,$$

but which has not been considered in Fig. 219.

The softer the carbons, the more is the latter effect appreciable, and the peaks rounded off, thus causing the curve to approach the appearance of Fig. 217, while with metal arcs, where r is very high, the peaks, especially the first, become very sharp and high, frequently reaching values of several thousand volts.

337. One of the most important sources of wave-shape distortion is the presence of iron in a magnetic circuit. The magnetic induction in iron, and therewith the magnetic flux, is not proportional to the magnetizing force or the exciting current, but the magnetic induction and the magnetizing force are related to each other by the hysteresis cycle of the iron, as discussed in Chapter XIII. In an iron-clad magnetic circuit, the magnetic flux and the current, therefore, cannot both be sine waves; if the magnetic flux and therefore the generated e.m.f. are sine waves, the current differs from sine wave-shape, while if a sine wave of current is sent through the circuit, the magnetic flux and the generated e.m.f. cannot be sine waves.

A. *Sine Wave of Voltage.*

Let a sine wave of e.m.f. be impressed upon an iron-clad reactance coil, or a primary coil of a transformer with open secondary circuit. Neglecting the ohmic resistance of the circuit, that is, assuming the generated e.m.f. as equal, or practically equal to the impressed e.m.f., the voltage consumed by the generated e.m.f., and therewith the magnetic flux, are sine waves, as represented by E and B in Fig. 220. The current, which produces this magnetic flux, B, and so the voltage,

E, then is derived point by point from B, by the hysteresis cycle of the iron. With the hysteresis cycle given in Fig. 221, the current then has the wave-shape given as I in Fig. 220, that is,

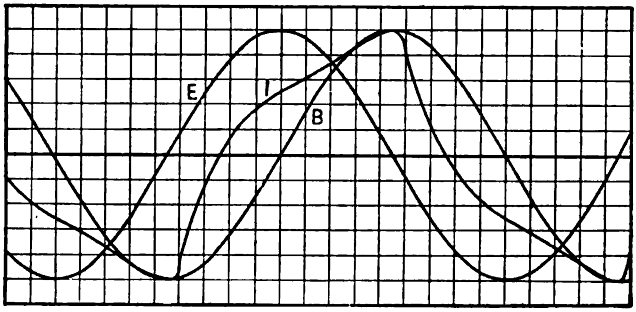


Fig. 220.

greatly differs from a sine wave. This distortion of the current wave is mainly due to the bend of the magnetic characteristic, that is, the magnetic saturation, and not to the energy loss, or

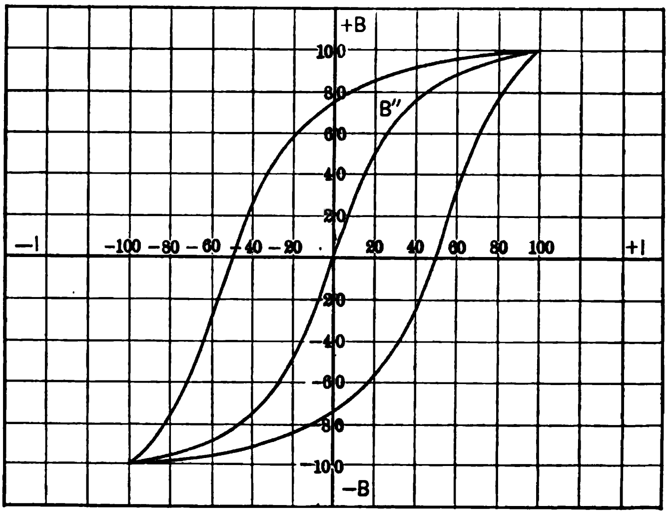


Fig. 221.

the area of the curve. This is seen by resolving the current wave, I, into two components, an energy component, i', in phase with the e.m.f., $e = E \sin \phi$, and a wattless component, i'', in quadra-

ture with E, and in phase with B. These components are calculated as

$$i' = \tfrac{1}{2}\,\{i_\phi + i_{\pi-\phi}\},$$

and

$$i'' = \tfrac{1}{2}\,\{i_\phi - i_{\pi-\phi}\};$$

where i_ϕ and $i_{\pi-\phi}$ are the instantaneous values of the current, I, at the angles ϕ and $\pi - \phi$, respectively.

These components, the *hysteresis power current*, i', and the reactive *magnetizing current*, i'', are plotted in Fig. 222 and

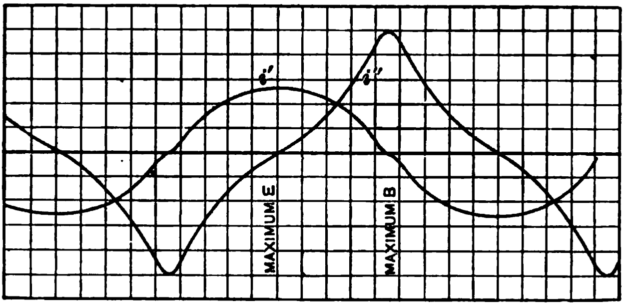


FIG. 222.

show that i' is nearly a sine wave, while i'' is greatly distorted, and peaked.

The total current, I, derived by the hysteresis cycle, Fig. 221, from the magnetic flux,

$$B = B_0 \cos \phi,$$

can be resolved into an infinite series of harmonic waves, that is, a trigonometric or Fourier series of the form:

$$\begin{aligned} i = a_1 \cos \phi + a_3 \cos 3\,\phi + a_5 \cos 5\,\phi + \ldots + a_n \cos n\phi + \ldots \\ + b_1 \sin \phi + b_3 \sin 3\,\phi + b_5 \sin 5\,\phi + \ldots + b_n \sin n\phi + \ldots \end{aligned}$$

or of the form:

$$\begin{aligned} i = c_1 \cos (\phi - \theta_1) + c_3 \cos (3\,\phi - \theta_3) + c_5 \cos (5\,\phi - \theta_5) \\ + \ldots + c_n \cos (n\phi - \theta_n) + \ldots \end{aligned}$$

where

$$c_n = \sqrt{a_n^2 + b_n^2},$$

$$\tan \theta_n = \frac{b_n}{a_n}.$$

The coefficients a_n and b_n are determined by the definite integrals: *

$$a_n = \frac{2}{\pi}\int_0^{\pi} i \cos n\,\phi \, d\phi = 2 \times avg\,(i \cos n\phi)_0^{\pi},$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} i \sin n\phi d\phi = 2 \times avg\,(i \sin n\phi)_0^{\pi};$$

that is, by multiplying the instantaneous values of i, as given numerically, by $\cos n\phi$ and $\sin n\phi$, respectively, and then averaging.

Just as in most investigations dealing with alternating currents, not the fundamental sine wave, but the fundamental sine wave together with all its higher harmonics, that is, the total wave, is of importance, so also when dealing with the higher harmonics, frequently not the individual higher harmonic sine wave is of importance, but the higher harmonic together with all of its higher harmonics. For instance, when dealing with the disturbances caused by the third harmonic in a three-phase system, the third harmonic together with all its higher harmonics or overtones, as the ninth, fifteenth, twenty-first, etc., comes in consideration, that is, all the components which repeat after one-third cycle. The higher harmonic then appears as a distorted wave, including its higher harmonics.

To determine, from the instantaneous values of a distorted wave, the instantaneous values of its nth-harmonic distorted wave, that is, the nth harmonic together with its overtones, of order $3\,n$, $5\,n$, $7\,n$, etc., the average is taken of n instantaneous values of the total wave (or any component thereof, which includes the nth harmonic), differing from each other in phase by $\frac{1}{n}$ period. That is, it is

$$i_n = \sum_0^{n-1}{}_\kappa\, i_{\phi + \frac{2\kappa\pi}{n}}.$$

This method is based on the relations

$$\sum_0^{n-1}{}_\kappa \cos\left(m\phi + \frac{2\,\kappa\pi}{n}\right) = n \cos m\phi,$$

$$\sum_0^{n-1}{}_\kappa \sin\left(m\phi + \frac{2\,\kappa\pi}{n}\right) = n \sin m\phi;$$

* See "Engineering Mathematics."

if $m = n$, or if m is a multiple of n, otherwise these sums $= 0$, where m and n are integer numbers.

338. In this manner, the wave of exciting current, I, of Fig. 220 is resolved, in Fig. 223, into the fundamental sine wave, i_1,

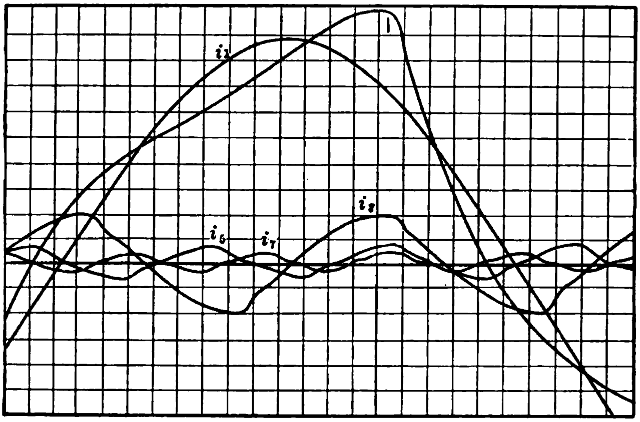


FIG. 223.

and the higher harmonics, i_3, i_5, i_7, which are general waves, that is, include their higher harmonics.

Analytically, it can be represented by

$$
\begin{aligned}
i = {} & 8.857 \cos (\phi + 37.6^\circ) + 1.898 \cos 3 (\phi + 4.1^\circ) \\
& + 0.585 \cos 5 (\phi - 1.7^\circ) + 0.319 \cos 7 (\phi - 3.2^\circ) \\
& + 0.158 \cos 9 (\phi - 2.5^\circ) + \ldots
\end{aligned}
$$

where $$B = 10{,}000 \cos \phi$$

is the wave of magnetic induction.

The equivalent sine wave of above current wave is

$$i_0 = 9.104 \cos (\phi - 36.3^\circ).$$

In this case, of the distortion of a current wave by an iron-clad reactance coil or transformer, with a sine wave of impressed e.m.f., it is, from the above equation of the current wave,

Effective value of the total current	6.423
Effective value of its fundamental sine wave . . .	6.27
Effective value of the sum of all its higher harmonics	1.43.

That is, the effective value of all the harmonics is 22.3 per cent of the effective value of the total current.

B. *Sine Wave of Current.*

339. If a sine wave of current exists through an iron-clad magnetic circuit, as for instance, an iron-clad reactance coil or transformer connected in series to a circuit traversed by a sine wave, the potential difference at the terminals of the reactance cannot be a sine wave, but contains higher harmonics.

From the sine wave of current

$$i = I \cos \phi,$$

follows by the hysteresis cycle Fig. 221 the wave of magnetism. This is not a sine wave, but hollowed out on the rising, humped on the decreasing side, that is, has a distortion about opposite from that of the current wave in Fig. 220; the wave of magnetism has the maximum at the same angle, ϕ, as the current, but passes the zero much later than the current.

From the wave of magnetism follows the wave of generated e.m.f., and so (approximately, that is, neglecting resistance) of terminal voltage e at the reactance, since e is proportional to $\frac{dB}{d\phi}$. It is plotted as E in Fig. 224, and resolved into its harmonics in the same manner as the current wave in A.

As seen, with a sine wave of current traversing an iron-clad reactance, the e.m.f. wave is very greatly distorted, and the maximum value of the distorted e.m.f. wave is more than twice the maximum of its fundamental sine wave.

Denoting the current wave by,

$$i = 10 \sin (\phi + 30°),$$

the e.m.f. wave in Fig. 224 is represented by

$$\begin{aligned} e = {} & 11.67 \cos (\phi + 2.5°) + 6.64 \cos 3 (\phi - 1.13°) \\ & + 3.24 \cos 5 (\phi - 2.4°) + 1.8 \cos 7 (\phi - 1.53°) + \\ & 1.16 \cos 9 (\phi - 0.5°) + 0.80 \cos 11 (\phi - 2°) \\ & + 0.53 \cos 13 (\phi - 2°) + 0.19 \cos 15 (\phi - 1°) + \dots \end{aligned}$$

that is, all the harmonics are nearly in phase with each other, so accounting for the very steep peak. It is

Effective value of total wave.	9.91
Effective value of its fundamental sine wave	8.25
Effective value of the sum of all its higher harmonics .	5.48

that is, the effective value of all the higher harmonics is 55.3 per cent of the effective value of the total wave.

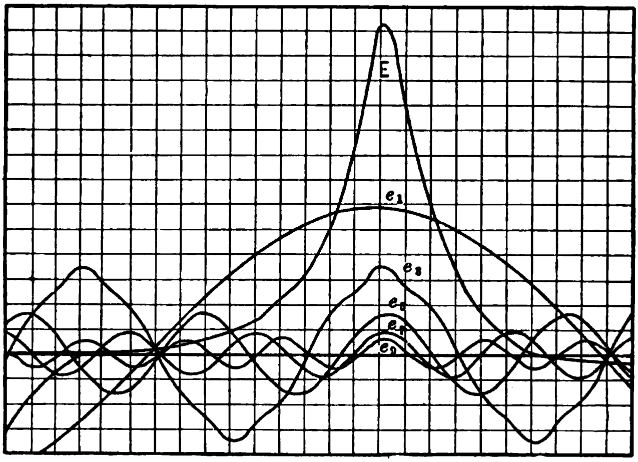


FIG. 224.

The impedance of this iron-clad reactance, with a sine wave current of 7.07 effective, so is

$$z = \frac{9.91}{7.07} = 1.40,$$

while the same reactance, with a sine wave e.m.f. of 7.07 effective, in A, gives the impedance,

$$z = \frac{7.07}{6.42} = 1.10.$$

The conclusion is that an iron-clad magnetic circuit is not suitable for a reactor, since even below saturation (as above assumed) it produces very great wave-shape distortion.

As discussed before, the insertion of even a small air-gap into the magnetic circuit makes the current wave nearly coincide in phase and in shape with the wave of magnetism.

C. *Three-phase Circuits.*

340. The wave-shape distortion in an iron-clad magnetic circuit has an important bearing on transformer connections in three-phase circuits.

The e.m.fs. and the currents in a three-phase system are displaced from each other in phase by one-third of a period or 120 degrees. Their third harmonics, therefore, differ by $3 \times 120°$, or a complete period, that is, are in phase with each other. That is, whatever third harmonics of e.m.f. and of current may exist in a three-phase system, must be in phase with each other in all three phases, or in other words, for the third harmonics, the three-phase system is single-phase.

The sum of the three e.m.fs. between the lines of a three-phase system (Δ voltages) is zero. Since their third harmonic would be in phase with each other, and so add up, it follows:

The voltages between the lines of a three-phase system, or Δ voltages, cannot contain any third harmonic or their overtones (ninth, fifteenth, twenty-first, etc., harmonics).

Since in a three-wire, three-phase system the sum of the three currents in the line is zero, but their third harmonics would be in phase with each other, and their sum, therefore, not zero, it follows:

The currents in the lines of a three-wire, three-phase system, or Y currents, cannot contain any third harmonic.

Third harmonics, however, can exist in the Y voltage or voltage between line and neutral of the system, and since the third harmonics are in phase with each other, in this case, a potential difference of triple frequency exists between the neutral of the system and all three phases as the other terminal, that is, the whole system pulsates against the neutral, at triple frequency.

Third harmonics can also exist in the currents between the lines, or Δ currents. Since the two currents from one line to the other two lines are displaced 60 degrees from each other, their third harmonics are in opposition, and, therefore, neutralize. That is, the third harmonics in the Δ currents of a three-phase system do not exist in the Y currents in the lines, but exist only in a local closed-circuit.

Third harmonics can exist in the line currents in a four-wire, three-phase system, as a system with grounded neutral. In this

case, the third harmonics of currents in the lines return jointly over the fourth or neutral wire, and even with balanced load on the three phases, the neutral wire carries a current which is of triple frequency.

341. With a sine wave of impressed e.m.f. the current in an iron-clad circuit, as the exciting current of a transformer, must contain a strong third harmonic, otherwise the e.m.f. cannot be a sine wave. Since in the lines of a three-phase system the third harmonics of current cannot exist, interesting wave-shape distortions thus result in transformers, when connected to a three-phase system in such a manner that the third harmonic of the exciting current would have to enter the line as Y current, and so is suppressed.

For instance, connecting three iron-clad reactors, as the primary coils of three transformers — with their secondaries open-circuited — in star or Y connection into a three-phase system, with a sine wave of e.m.f., e, impressed upon the lines. Normally, the voltage of each transformer should be a sine wave also, and equal $\frac{e}{\sqrt{3}}$. This, however, would require that the current taken by the transformer as exciting current contains a third harmonic. As such a third harmonic cannot exist in a three-phase circuit, the wave of magnetism cannot be a sine wave, but must contain a third harmonic, about opposite to that which was suppressed in the exciting current. The e.m.f. generated by this magnetism, and therewith the potential difference at the transformer, or Y voltage, therefore, must also contain a third harmonic, and its overtones, three times as great as that of the magnetism, due to the triple frequency.

With three transformers connected in Y into a three-phase system, with open secondary circuit, we have, then, with a sine wave of e.m.f. impressed between the three-phase lines, the conditions:

The voltage at the transformers, or Y voltage, cannot be a sine wave, but must contain a third harmonic and its overtones, but can contain no other harmonics, since the other harmonics, as the fifth, seventh, etc., would not eliminate by combining two Y voltages to the Δ voltage or line voltage, and the latter was assumed as sine wave.

The exciting current in the transformers cannot contain any third harmonic or its overtones, but can contain all other harmonics.

The magnetic flux is not a sine wave, but contains a third harmonic and its overtones, corresponding to those of the Y voltage, but contains no other harmonics, and is related to the exciting current by the hysteresis cycle.

Herefrom then the wave-shapes of currents, magnetism and voltage can be constructed. Obviously, since the relation between current and magnetism is merely empirical, given by the hysteresis cycle, this cannot be done analytically, but only by the calculation or construction of the instantaneous values of the curves.

342. For the hysteresis cycle in Fig. 221, and for a system of transformers connected in Y, with open secondary circuit, into a three-phase system with a sine wave of e.m.f. between the lines, the curves of exciting current, magnetic flux and voltage per transformer, or between lines and neutral, are constructed in Fig. 225.

i is the exciting current of the transformer, and contains all the harmonics, except the third and its multiples. It is given by the equation:

$$i = 8.28 \sin (\phi + 30.8°) - 0.71 \sin (5\phi - 17.2°) + \dots$$

B is the magnetic flux density in the transformer. It contains only the third harmonic and its multiples, but no other harmonics, and is given by the equation:

$$B = 10.0 \sin \phi + 1.38 \sin (3\phi - 9.2°) + 0.045 \sin 9\phi + \dots$$

e is the potential difference of the transformer terminals, or voltage between the three-phase lines and the transformer neutral. It contains the third harmonic and its multiples, but no other harmonics, and is given by the equation:

$$e = 10.0 \cos \phi + 4.14 \cos (3\phi - 9.2°) + 0.405 \cos 9\phi + \dots$$

The effective value of the voltage is 0.625 e, and the maximum value is 1.175 E, where E = supply voltage or Δ voltage.

While with a sine wave the effective value would be

$$\frac{E}{\sqrt{3}} = 0.577\ E,$$

and the maximum value

$$\frac{E\sqrt{2}}{\sqrt{3}} = 0.815\ E;$$

that is, by the suppression of the third harmonic of exciting current in the three-phase system, the effective value of the voltage per transformer, or voltage between three-phase lines and neutral

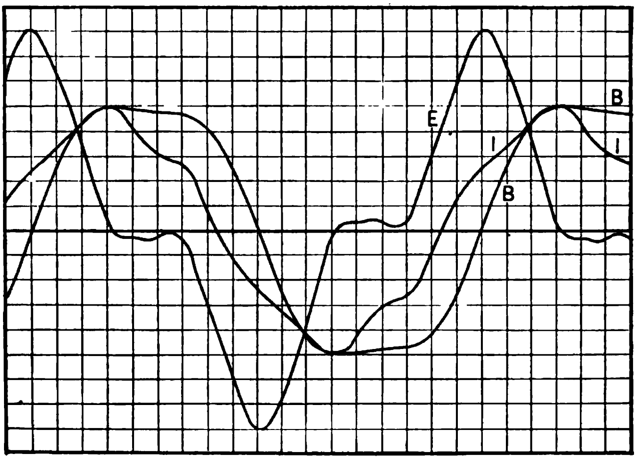


FIG. 225.

(or ground, if the neutral is grounded) has been increased by 8.5 per cent, the maximum value by 44.6 per cent, and the voltage wave has become very peaked, by a pronounced third harmonic, of an effective value of 0.24 E — that is, 38.5 per cent of the effective value of the total wave.

The very high peak of e.m.f. produced by this wave-shape distortion is liable to be dangerous in high-potential, three-phase systems by increasing the strain on the insulation between lines and ground, and leading to resonance phenomena with the third harmonic.

The maximum value of the distorted wave of magnetism is 8.89, while with a sine wave it would be 10.0, that is, the maxi-

mum of the wave of magnetism has been reduced by 11.1 per cent, and the core loss of the transformer so by about 17 per cent.

343. Assuming now that in such transformers, connected with their primaries in Y into a three-phase circuit, the secondaries are connected in Δ. The third harmonics of e.m.f., generated in the three transformer secondaries, then are in series in short-circuit, thus produce a local current in the secondary transformer triangle. This current is of triple frequency, and hence supplies the third harmonic of exciting current, which was suppressed in the primary, and thereby eliminates the third harmonic of magnetism, and of e.m.f., which results from the suppression of the third harmonic of exciting current, and so limits itself. That is, connecting the transformer secondaries in Δ, the wave-shape distortion disappears, and voltage and magnetism are again sine waves, and the exciting current is that corresponding to a sine wave of magnetism, except that it is divided between primary and secondary; the third harmonic of the exciting current does not exist in the primary, but is produced by induction in the secondary circuit. Obviously, in this case, the magnetic flux and the voltage are not perfect sine waves, but contain a slight third harmonic, which produces in the secondary the triple-frequency exciting current.

If the primary neutral of the transformers is connected to a fourth wire, in a four-wire, three-phase system, or three-phase system with grounded neutral, and this fourth wire leads back to the generator neutral, or a neutral of a transformer in which the triple-frequency current can exist, that is, in which the secondary is connected in Δ — the wave-shape distortion also disappears.

It follows herefrom that in the three-phase system attention must be paid to provide a path for the third harmonic of the transformer exciting current, either directly or inductively, otherwise a serious distortion of the e.m.f. wave of the transformers occurs.

CHAPTER XXX.

EFFECTS OF HIGHER HARMONICS.

344. To elucidate the variation in the shape of alternating waves caused by various harmonics, in Figs. 226 and 227 are

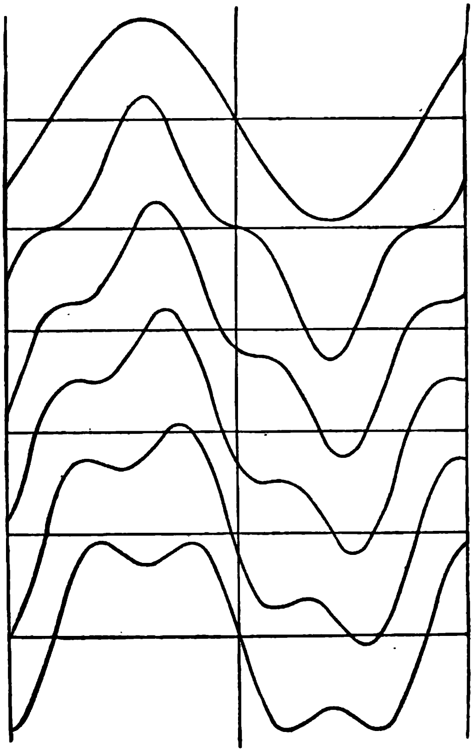


Fig. 226.

shown the wave-forms produced by the superposition of the triple and the quintuple harmonic upon the fundamental sine wave.

In Fig. 226 is shown the fundamental sine wave and the com-

plex waves produced by the superposition of a triple harmonic of 30 per cent the amplitude of the fundamental, under the relative phase displacements of 0°, 45°, 90°, 135°, and 180°, represented by the equations:

$$\sin\beta$$
$$\sin\beta - 0.3\sin 3\beta$$
$$\sin\beta - 0.3\sin(3\beta - 45°)$$
$$\sin\beta - 0.3\sin(3\beta - 90°)$$
$$\sin\beta - 0.3\sin(3\beta - 135°)$$
$$\sin\beta - 0.3\sin(3\beta - 180°).$$

As seen, the effect of the triple harmonic is, in the first figure, to flatten the zero values and point the maximum values of the wave, giving what is called a peaked wave. With increasing phase displacement of the triple harmonic, the flat zero rises and gradually changes to a second peak, giving ultimately a flat-top or even double-peaked wave with sharp zero. The intermediate positions represent what is called a saw-tooth wave.

In Fig. 227 are shown the fundamental sine wave and the complex waves produced by superposition of a quintuple harmonic of 20 per cent the amplitude of the fundamental, under the relative phase displacement of 0°, 45°, 90°, 135°, 180°, represented by the equations:

$$\sin\beta$$
$$\sin\beta - 0.2\sin 5\beta$$
$$\sin\beta - 0.2\sin(5\beta - 45°)$$
$$\sin\beta - 0.2\sin(5\beta - 90°)$$
$$\sin\beta - 0.2\sin(5\beta - 135°)$$
$$\sin\beta - 0.2\sin(5\beta - 180°).$$

The quintuple harmonic causes a flat-topped or even double-peaked wave with flat zero. With increasing phase displacement the wave becomes of the type called saw-tooth wave also. The flat zero rises and becomes a third peak, while of the two former peaks, one rises, the other decreases, and the wave gradually

changes to a triple-peaked wave with one main peak, and a sharp zero.

As seen, with the triple harmonic, flat-top or double-peak coincides with sharp zero, while the quintuple harmonic flat-top or double-peak coincides with flat zero.

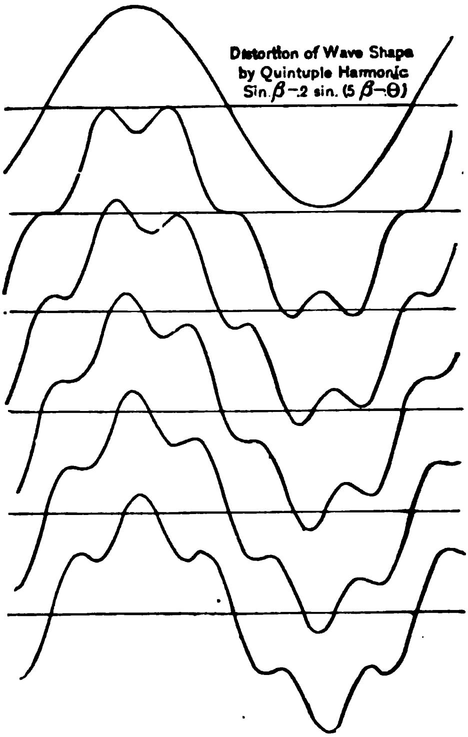


FIG. 227.

Sharp peak coincides with flat zero in the triple, with sharp zero in the quintuple harmonic. With the triple harmonic, the saw-tooth shape appearing in case of a phase difference between fundamental and harmonic is single, while with the quintuple harmonic it is double.

Thus in general, from simple inspection of the wave-shape, the existence of these first harmonics can be discovered. Some characteristic shapes are shown in Fig. 228.

Flat top with flat zero,

$$\sin \beta - 0.15 \sin 3\beta - 0.10 \sin 5\beta.$$

Flat top with sharp zero,

$$\sin \beta - 0.225 \sin (3\beta - 180°) - 0.05 \sin (5\beta - 180°).$$

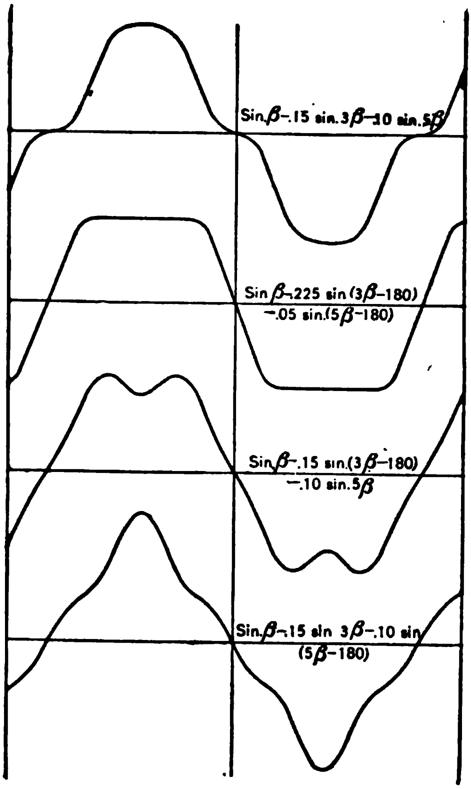


FIG. 228.—Some Characteristic Wave-Shapes.

Double peak, with sharp zero,

$$\sin \beta - 0.15 \sin (3\beta - 180°) - 0.10 \sin 5\beta.$$

Sharp peak with sharp zero,

$$\sin \beta - 0.15 \sin 3\beta - 0.10 \sin (5\beta - 180°).$$

345. Since the distortion of the wave-shape consists in the superposition of higher harmonics, that is, waves of higher frequency, the phenomena taking place in a circuit supplied by such a wave will be the combined effect of the different waves.

Thus in a non-inductive circuit the current and the potential difference across the different parts of the circuit are of the same shape as the impressed e.m.f. If inductive reactance is inserted in series with a non-inductive circuit, the self-inductive reactance consumes more e.m.f. of the higher harmonics, since the reactance is proportional to the frequency, and thus the current and the e.m.f. in the non-inductive part of the circuit show the higher harmonics in a reduced amplitude. That is, self-inductive reactance in series with a non-inductive circuit reduces the higher harmonics or smooths out the wave to a closer resemblance to sine-shape. Inversely, capacity in series to a non-inductive circuit consumes less e.m.f. at higher than at lower frequency, and thus makes the higher harmonics of current and of potential difference in the non-inductive part of the circuit more pronounced — intensifies the harmonics.

Self-induction and capacity in series may cause an increase of voltage due to complete or partial resonance with higher harmonics, and a discrepancy between volt-amperes and watts, without corresponding phase displacement, as will be shown hereafter.

346. In long-distance transmission over lines of noticeable inductive and condensive reactance, rise of voltage due to resonance may occur with higher harmonics, as waves of higher frequency, while the fundamental wave is usually of too low a frequency to cause resonance.

An approximate estimate of the possible rise by resonance with various harmonics can be obtained by the investigation of a numerical example. Let in a long-distance line, fed by step-up transformers at 60 cycles,

The resistance drop in the transformers at full load = 1 per cent.

The reactance voltage in the transformers at full load = 5 per cent with the fundamental wave.

The resistance drop in the line at full load = 10 per cent.

The reactance voltage in the line at full load = 20 per cent with the fundamental wave.

The capacity or charging current of the line = 20 per cent of the full-load current, I, at the frequency of the fundamental.

The line capacity may approximately be represented by a condenser shunted across the middle of the line. The e.m.f. at the generator terminals, E, is assumed as maintained constant.

The e.m.f. consumed by the resistance of the circuit from generator terminals to condenser is

$$Ir = 0.06\,E,$$

or,

$$r = 0.06\,\frac{E}{I}\,.$$

The reactance e.m.f. between generator terminals and condenser is, for the fundamental frequency,

$$Ix = 0.15\,E,$$

or,

$$x = 0.15\,\frac{E}{I}\,;$$

thus the reactance corresponding to the frequency $(2\,k-1)\,f$ of the higher harmonic is

$$x\,(2\,k-1) = 0.15\,(2\,k-1)\,\frac{E}{I}\,.$$

The capacity current at fundamental frequency is,

$$i = 0.2\,I;$$

hence, at the frequency $(2\,k-1)\,f$,

$$i = 0.2\,(2\,k-1)\,e'\,\frac{I}{E}\,,$$

if

e' = e.m.f. of the $(2\,k-1)$th harmonic at the condenser,
e = e.m.f. of the $(2\,k-1)$th harmonic at the generator terminals.

The e.m.f. at the condenser is

$$e' = \sqrt{e^2 - i^2r^2} + ix\,(2\,k-1);$$

hence, substituted,

$$a = \frac{e'}{e} = \frac{1}{\sqrt{1 - .059856\,(2\,k-1)^2 + .0009\,(2\,k-1)^4}},$$

the rise of voltage by inductive and condensive reactance.

Substituting,

$k =$	1	2	3	4	5	6
or, $2k - 1 =$	1	3	5	7	9	11
and $a =$	1.03	1.36	3.76	2.18	.70	.38

That is, the fundamental will be increased at open circuit by 3 per cent, the triple harmonic by 36 per cent, the quintuple harmonic by 276 per cent, the septuple harmonic by 118 per cent, while the still higher harmonics are reduced.

The maximum possible rise will take place for

$$\frac{da}{d\,(2\,k-1)} = 0, \text{ or, } 2\,k - 1 = 5.77;$$

that is, at a frequency, $f = 346$, and $a = 14.4$.

That is, complete resonance will appear at a frequency between quintuple and septuple harmonic, and would raise the voltage at this particular frequency 14.4-fold.

If the voltage shall not exceed the impressed voltage by more than 100 per cent, even at coincidence of the maximum of the harmonic with the maximum of the fundamental,

the triple harmonic must be less than 70 per cent of the fundamental,
the quintuple harmonic must be less than 26.5 per cent of the fundamental,
the septuple harmonic must be less than 46 per cent of the fundamental.

The voltage will not exceed twice the normal, even at a frequency of complete resonance with the higher harmonic, if none of the higher harmonics amounts to more than 7 per cent of the fundamental. Herefrom it follows that the danger of resonance in high-potential lines is frequently overestimated, since the conditions assumed in this example are rather more severe than found in lines of moderate length, the capacity current of the line very seldom reaching 20 per cent of the main current.

347. The power developed by a complex harmonic wave in a non-inductive circuit is the sum of the powers of the individual

harmonics. Thus if upon a sine wave of alternating e.m.f. higher harmonic waves are superposed, the effective e.m.f., and the power produced by this wave in a given circuit or with a given effective current, are increased. In consequence hereof alternators and synchronous motors of iron-clad unitooth construction — that is, machines giving waves with pronounced higher harmonics — may give with the same number of turns on the armature, and the same magnetic flux per field-pole at the same frequency, a higher output than machines built to produce sine waves.

348. This explains an apparent paradox:

If in the three-phase star-connected generator with the magnetic field constructed as shown diagrammatically in Fig. 229, the magnetic flux per pole $= \Phi$, the number of turns in series per circuit $= n$, the frequency $= f$, the e.m.f. between any two collector rings is

$$E = \sqrt{2}\, \pi f\, 2\, n \Phi\, 10^{-8},$$

since $2\,n$ armature-turns simultaneously interlink with the magnetic flux, Φ.

The e.m.f. per armature circuit is

$$e = \sqrt{2}\, \pi f n \Phi\, 10^{-8};$$

hence the e.m.f. between collector rings, as resultant of two e.m.fs., e, displaced by 60° from each other, is

$$E = e\sqrt{3} = \sqrt{2}\, \pi f \sqrt{3}\, n \Phi\, 10^{-8},$$

while the same e.m.f. was found from the number of turns, the magnetic flux, and the frequency by direct calculation to be equal to $2\,e$; that is, the two values found for the same e.m.f. have the proportion $\sqrt{3} : 2 = 1 : 1.154$.

This discrepancy is due to the existence of more pronounced higher harmonics in the wave, e, than in the wave, $E = e \times \sqrt{3}$, which have been neglected in the formula

$$e = \sqrt{2}\, \pi f n \Phi\, 10^{-8}.$$

Hence it follows that, while the e.m.f. between two collector rings in the machine shown diagrammatically in Fig. 229 is only $e \times \sqrt{3}$, by massing the same number of turns in one slot instead of in two slots, we get the e.m.f., $2e$, or 15.4 per cent higher e.m.f., that is, larger output.

It follows herefrom that the distorted e.m.f. wave of a unitooth alternator is produced by lesser magnetic flux per pole — that is,

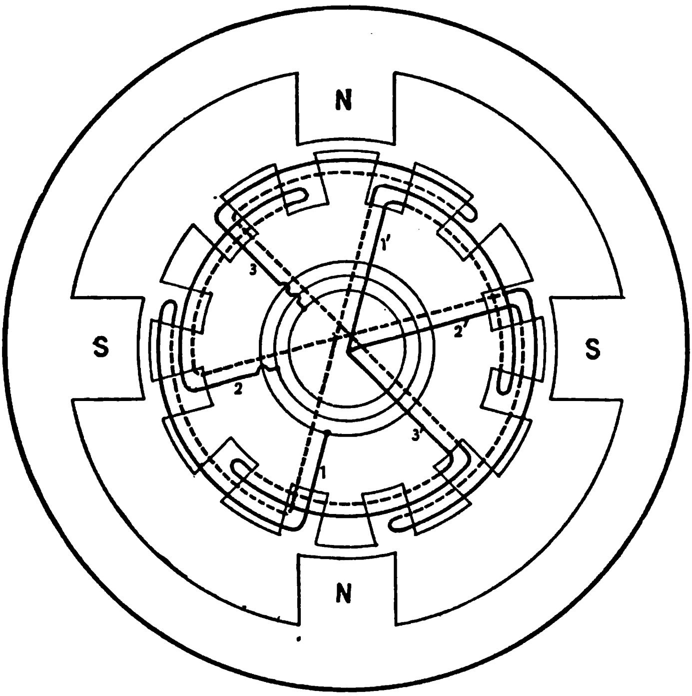


FIG. 229.— Three-phase Star-connected Alternator.

in general, at a lesser hysteretic loss in the armature or at higher efficiency — than the same effective e.m.f. would be produced with the same number of armature turns if the magnetic disposition were such as to produce a sine wave.

349. Inversely, if such a distorted wave of e.m.f. is impressed upon a magnetic circuit, as, for instance, a transformer, the wave of magnetism in the primary will repeat in shape the wave of

magnetism interlinked with the armature coils of the alternator, and consequently, with a lesser maximum magnetic flux, the same effective counter e.m.f. will be produced, that is, the same power converted in the transformer. Since the hysteretic loss in the transformer depends upon the maximum value of magnetism, it follows that the hysteretic loss in a transformer is less with a distorted wave of a unitooth alternator than with a sine wave.

350. From another side the same problem can be approached: If upon a transformer a sine wave of e.m.f. is impressed, the wave of magnetism will be a sine wave also. If now upon the sine wave of e.m.f.. higher harmonics, as sine waves of triple, quintuple, etc., frequency are superposed in such a way that the corresponding higher harmonic sine waves of magnetism do not increase the maximum value of magnetism, or even lower it by a coincidence of their negative maxima with the positive maximum of the fundamental — in this case all the power represented by these higher harmonics of e.m.f. will be transformed without an increase of the hysteretic loss, or even with a decreased hysteretic loss.

Obviously, if the maximum of the higher harmonic wave of magnetism coincides with the maximum of the fundamental, and thereby makes the wave of magnetism more pointed, the hysteretic loss will be increased more than in proportion to the increased power transformed, i.e., the efficiency of the transformer will be lowered.

That is, some distorted waves of e.m.f. are transformed at a lesser, some at a larger, hysteretic loss than the sine wave, if the same effective e.m.f. is impressed upon the transformer.

The unitooth alternator wave and the first wave in Fig. 226 belong to the former class; the waves derived from continuous-current machines, tapped at two equi-distant points of the armature, frequently, to the latter class.

351. Regarding the loss of energy by Foucault or eddy currents, this loss is not affected by distortion of wave-shape, since the e.m.f. of eddy currents, like the generated e.m.f., is proportional to the secondary e.m.f.; and thus at constant impressed primary e.m.f., the power consumed by eddy currents bears a constant relation to the output of the secondary circuit, as obvious, since

the division of power between the two secondary circuits — the eddy-current circuit, and the useful or consumer circuit — is unaffected by wave-shape or intensity of magnetism.

352. In high-potential lines, distorted waves whose maxima are very high above the effective values, as peaked waves, may be objectionable by increasing the strain on the insulation. It is, however, not settled yet beyond doubt whether the striking-distance of a rapidly-alternating potential depends upon the maximum value or upon some value between effective and maximum. Since disruptive phenomena do not always take place immediately after application of the potential, but the time element plays an important part, it is possible that insulation-strain and striking-distance is, in a certain range and with some materials, dependent upon the effective potential, and thus independent of the wave-shape.

In this respect it is quite likely that different insulating materials show a different behavior, and homogeneous solid substances, as paraffin, depend in their disruptive strength upon the maximum value of the potential difference, while heterogeneous materials, as mica, laminated organic substances, air, etc., that is, substances in which the disruptive strength decreases with the time application of the potential difference, are less affected by very high peaks of e.m.f. of very short duration.

CHAPTER XXXI.

SYMBOLIC REPRESENTATION OF GENERAL ALTERNATING WAVES.

353. The vector representation,

$$A = a^1 + ja^{11} = a\,(\cos\theta + j\sin\theta)$$

of the alternating wave,

$$A = a_0 \cos(\phi - \theta)$$

applies to the sine wave only.

The general alternating wave, however, contains an infinite series of terms, of odd frequencies,

$$A = A_1 \cos(\phi - \theta_1) + A_3 \cos(3\phi - \theta_3) + A_5 \cos(5\phi - \theta_5) +$$

thus cannot be directly represented by one complex vector quantity.

The replacement of the general wave by its equivalent sine wave, as before discussed, that is, a sine wave of equal effective intensity and equal power, while sufficiently accurate in many cases, completely fails in other cases, especially in circuits containing capacity, or in circuits containing periodically (and in synchronism with the wave) varying resistance or reactance (as alternating arcs, reaction machines, synchronous induction motors, oversaturated magnetic circuits, etc.).

Since, however, the individual harmonics of the general alternating wave are independent of each other, that is, all products of different harmonics vanish, each term can be represented by a complex symbol, and the equations of the general wave then are the resultants of those of the individual harmonics.

This can be represented symbolically by combining in one formula symbolic representations of different frequencies, thus,

$$\dot{A} = \sum_1^{\infty}{}^{2n-1}\,(a_n{}^1 + j_n a_n{}^{11}),$$

where

$$j_n = \sqrt{-1},$$

and the index of the j_n merely denotes that the j's of different indices n, while algebraically identical, physically represent different frequencies, and thus cannot be combined.

The general wave of e.m.f. is thus represented by

$$\dot{E} = \sum_1^\infty{}_{2n-1} (e_n^1 + j_n e_n^{11}),$$

the general wave of current by

$$\dot{I} = \sum_1^\infty{}_{2n-1} (i_n^1 + j_n i_n^{11}).$$

If

$$Z_1 = r - j\,(x_m + x_0 + x_c)$$

is the impedance of the fundamental harmonic, where

x_m is that part of the reactance which is proportional to the frequency (inductance, etc.).

x_0 is that part of the reactance which is independent of the frequency (mutual inductance, synchronous motion, etc.).

x_c is that part of the reactance which is inversely proportional to the frequency (capacity, etc.).

The impedance for the nth harmonic is

$$Z = r - j_n\left(nx_m + x_0 + \frac{x_c}{n}\right).$$

This term can be considered as the general symbolic expression of the impedance of a circuit of general wave-shape.

Ohm's law, in symbolic expression, assumes for the general alternating wave the form

$$\dot{I} = \frac{\dot{E}}{Z} \text{ or, } \sum_1^\infty{}_{2n-1} (i_n^1 + j_n i_n^{11}) = \sum_1^\infty{}_{2n-1} \frac{e_n^1 + j_n e_n^{11}}{r - j_n\left(nx_m + x_0 + \frac{x_c}{n}\right)};$$

$$\dot{E} = \dot{I} Z \text{ or, } \sum_1^\infty{}_{2n-1} (e_n^1 + j_n e_n^{11}) = \sum_1^\infty{}_{2n-1} \left[r - j_n\left(nx_m + x_0 + \frac{x_c}{n}\right)\right] (i_n^1 + j_n i_n^{11});$$

$$Z = \frac{\dot{E}}{\dot{I}} \text{ or } Z_n = r - j_n\left(nx_m + x_0 + \frac{x_c}{n}\right) = \frac{e_n + j_n e_n^{11}}{i_n^1 + j_n i_n^{11}}.$$

The symbols of multiplication and division of the terms, $\dot{E}$, $\dot{I}$, Z, thus represent, not algebraic operation, but multiplication and division of corresponding terms of $\dot{E}$, $\dot{I}$, Z, that is, terms of the same index, n, or, in algebraic multiplication and division of the series, $\dot{E}$, $\dot{I}$, all compound terms, that is, terms containing two different n's, vanish.

354. The effective value of the general wave,

$a = A_1 \cos (\phi - \theta_1) + A_3 \cos (3\phi - \theta_3) + A_5 \cos (5\phi - \theta_5) + \ldots$

is the square root of the sum of mean squares of individual harmonics,

$$A = \sqrt{\tfrac{1}{2}\left\{A_1^2 + A_3^2 + A_5^2 + \ldots\right\}}.$$

Since, as discussed above, the compound terms of two different indices, n, vanish, the absolute value of the general alternating wave,

$$\dot{A} = \sum_1^\infty{}_{2n-1} \frac{a_n^1 + j_n a_n^{11}}{b_n^1 + j_n b_n^{11}},$$

is thus,

$$A = \sqrt{\sum_1^\infty{}_{2n-1} \frac{a_n^{1^2} + a_n^{11^2}}{b_n^{1^2} + b_n^{11^2}}},$$

which offers an easy means of reduction from symbolic to absolute values.

Thus, the absolute value of the e.m.f.,

$$\dot{E} = \sum_1^\infty{}_{2n-1} (e_n^1 + j_n e_n^{11}),$$

is

$$E = \sqrt{\sum_1^\infty{}_{2n-1} (e_n^{1^2} + e_n^{11^2})},$$

the absolute value of the current,

$$\dot{I} = \sum_1^\infty{}_{2n-1} (i_n^1 + j_n i_n^{11}),$$

is

$$I = \sqrt{\sum_1^\infty{}_{2n-1} (i_n^{1^2} + i_n^{11^2})}.$$

355. The double frequency power (torque, etc.) equation of the general alternating wave has the same symbolic expression as with the sine wave,

$$
\begin{aligned}
\underset{\cdot}{P} &= [\underset{\cdot}{E}\underset{\cdot}{I}] \\
&= P^1 + jP^j \\
&= [\underset{\cdot}{E}\underset{\cdot}{I}]^1 + j\,[EI]^j \\
&= \sum_1^{\infty}{}_{2n-1}\,(e_n^{1}i_n^{1} + e_n^{11}i_n^{11}) + \sum_1^{\infty}{}_{2n-1}\;j_n\,(e_n^{11}i_n^{1} - e_n^{1}i_n^{11}),
\end{aligned}
$$

where

$$P^1 = [\underset{\cdot}{E}\underset{\cdot}{I}]^1 = \sum_1^{\infty}{}_{2n-1}\,(e_n^{1}i_n^{1} + e_n^{11}i_n^{11}),$$

$$P^j = [\underset{\cdot}{E}\underset{\cdot}{I}]^j = \sum_1^{\infty}{}_{2n-1}\frac{j_n}{j}\,(e_n^{11}i_n^{1} - e_n^{1}i_n^{11}).$$

The j_n enters under the summation sign of the reactive or "wattless power," P^j, so that the wattless powers of the different harmonics cannot be algebraically added.

Thus,

The total "true power" of a general alternating-current circuit is the algebraic sum of the powers of the individual harmonics.

The total "reactive power" of a general alternating-current circuit is not the algebraic, but the absolute sum of the wattless powers of the individual harmonics.

Thus, regarding the reactive power as a whole, in the general alternating circuit no distinction can be made between lead and lag, since some harmonics may be leading, others lagging.

The apparent power, or total volt-amperes, of the circuit is

$$P_a = EI = \sqrt{\sum_1^{\infty}{}_{2n-1}\,(e_n^{1^2} + e_n^{11^2})\sum_1^{\infty}{}_{2n-1}\,(i_n^{1^2} + i_n^{11^2})}.$$

The power-factor of the circuit is,

$$p = \frac{P^1}{P_a} = \frac{\sum_1^{\infty}{}_{2n-1}\,(e_n^{1}i_n^{1} + e_n^{11}i_n^{11})}{\sqrt{\sum_1^{\infty}{}_{2n-1}\,(e_n^{1^2} + e_n^{11^2})\;\sum_1^{\infty}{}_{2n-1}\,(i_n^{1^2} + i_n^{11^2})}}.$$

The term "inductance factor," however, has no meaning any more, since the reactive powers of the different harmonics are not directly comparable.

The quantity

$$q_0 = \sqrt{1 - p^2}$$

has no physical significance, and is not $\dfrac{\text{reactive power}}{\text{total apparent power}}$.

The term

$$\frac{P^j}{EI}$$

$$= \sum_1^\infty {}_{2n-1} \frac{j_n}{j} \frac{e_n^{11} i_n^1 - e_n^{11} i_n^1}{EI}$$

$$= \sum_1^\infty {}_{2n-1} \frac{j_n}{j} q_n,$$

where

$$q_n = \frac{e_n^{11} i_n^1 - e_n^1 i_n^{11}}{EI},$$

consists of a series of inductance factors, q_n, of the individual harmonics.

As a rule, if

$$q^2 = \sum_1^\infty {}_{2n-1} q_n^2,$$

$$p^2 + q^2 < 1,$$

for the general alternating wave, that is, q differs from

$$q_0 = \sqrt{1 - p^2}.$$

The complex quantity,

$$\dot{V} = \frac{\dot{P}}{P_a} = \frac{[\dot{E}\dot{I}]}{EI} = \frac{[\dot{E}\dot{I}]^1 + j[\dot{E}\dot{I}]^j}{EI}$$

$$= \frac{\sum_1^\infty {}_{2n-1} (e_n^1 i_n^1 + e_n^{11} i_n^{11}) + \sum_1^\infty {}_{2n-1} j_n (e_n^{11} i_n^1 - e_n^1 i_n^{11})}{\sqrt{\sum_1^\infty {}_{2n-1} (e_n^{1^2} + e_n^{11^2}) \sum_1^\infty {}_{2n-1} (i_n^{1^2} + i_n^{11^2})}}.$$

$$= p + \sum_1^\infty {}_{2n-1} j_n q_n,$$

takes in the circuit of the general alternating wave the same position as power-factor and inductance factor with the sine wave.

$\dot{V} = \dfrac{\dot{P}}{P_a}$ may be called the "*circuit-factor.*"

It consists of a real term, p, the power-factor, and a series of imaginary terms, $j_n q_n$, the inductance factors of the individual harmonics.

The absolute value of the circuit-factor,

$$v = \sqrt{p^2 + \sum_1^{\infty}{}_{2n-1} q_n^2},$$

as a rule, is < 1.

356. Some applications of this symbolism will explain its mechanism and its usefulness more fully.

1st *Example:* Let the e.m.f.,

$$\underset{\cdot}{E} = \sum_1^{5}{}_{2n-1} (e_n^1 + j_n e^{11}),$$

be impressed upon a circuit of the impedance,

$$Z = r - j_n\left(n x_m - \frac{x_c}{n}\right);$$

that is, containing resistance r, inductive reactance x_m and condensive reactance x_c in series.

Let $e_1^1 = 720$ $\quad e_1^{11} = 540$
$e_3^1 = 283$ $\quad e_3^{11} = -283$
$e_5^1 = -104$ $\quad e_5^{11} = 138$

or,

$e_1 = 900$ $\quad \tan\theta_1 = 0.75$
$e_3 = 400$ $\quad \tan\theta_3 = -1$
$e_5 = 173$ $\quad \tan\theta_5 = -1.33$

It is thus in symbolic expression,

$Z_1 = 10 + 80\,j_1$ $\quad z_1 = 80.6$
$Z_3 = 10$ $\quad z_3 = 10$
$Z_5 = 10 - 32\,j_5$ $\quad z_5 = 33.5,$

and e.m.f.,

$$\underset{\cdot}{E} = (720 + 540\,j_1) + (283 - 283\,j_3) + (-104 + 138\,j_5),$$

or absolute,

$$E = 1000,$$

and current,

$$\dot{I} = \frac{\dot{E}}{Z} = \frac{720 + 540\, j_1}{10 + 80\, j_1} + \frac{283 - 283\, j_3}{10} + \frac{-104 + 138\, j_5}{10 - 32\, j_5}$$
$$= (7.76 - 8.04\, j_1) + (28.3 - 28.3\, j_3) + (-4.86 - 1.73\, j_5)$$

or, absolute

$$I = 41.85,$$

of which is of fundamental frequency, $I_1 = 11.15$
of which is of triple frequency, $I_3 = 40$
of which is of quintuple frequency, $I_5 = 5.17.$

The total apparent power of the circuit is

$$P_a = EI = 41{,}850.$$

The true power of the circuit is,

$$P^1 = [EI]^1 = 1240 + 16{,}000 + 270,$$
$$= 17{,}510,$$

the reactive power,

$$jP^j = j\,[EI]^j = 10{,}000\, j_1 - 850\, j_5;$$

thus, the total power,

$$\dot{P} = 17{,}510 + 10{,}000\, j_1 - 850\, j_5.$$

That is, the reactive power of the first harmonic is leading, that of the third harmonic zero, and that of the fifth harmonic lagging.

$$17{,}510 = I^2 r, \text{ as obvious.}$$

The circuit-factor is,

$$\dot{V} = \frac{\dot{P}}{Pa} = \frac{[\dot{E}\dot{I}]}{EI}$$
$$= 0.418 + .239\, j_1 - 0.0203\, j_5,$$

or, absolute,

$$v = \sqrt{0.418^2 + 0.239^2 + 0.0203^2}.$$
$$= 0.482.$$

The power-factor is

$$p = 0.418.$$

The inductance factor of the first harmonic is $q_1 = 0.239$, that of the third harmonic $q_3 = 0$, and of the fifth harmonic $q_5 = -0.0203$.

Considering the waves as replaced by their equivalent sine waves, from the sine wave formula,

$$p^2 + q_0^2 = 1,$$

the inductance factor would be,

$$q_0 = 0.914,$$

and the phase angle,

$$\tan\theta = \frac{q_0}{p} = \frac{0.914}{0.418} = 2.8, \qquad \theta = 65.4°,$$

giving apparently a very great phase displacement, while in reality, of the 41.85 amperes total current, 40 amperes (the current of the third harmonic) are in phase with their e.m.f.

We thus have here a case of a circuit with complex harmonic waves which cannot be represented by their equivalent sine waves. The relative magnitudes of the different harmonics in the wave of current and of e.m.f. differ essentially, and the circuit has simultaneously a very low power-factor and a very low inductance factor; that is, a low power-factor exists without corresponding phase displacement, the circuit-factor being less than one-half.

Such circuits, for instance, are those including alternating arcs, reaction machines, synchronous induction motors, reactances with over-saturated magnetic circuit, high potential lines in which the maximum difference of potential exceeds the voltage at which brush discharges begin, polarization cells, and in general electrolytic conductors above the dissociation voltage of the electrolyte, etc. Such circuits cannot correctly, and in many cases not even approximately, be treated by the theory of the equivalent sine waves, but require the symbolism of the complex harmonic wave.

357. *2d Example:* A condenser of capacity $C_0 = 20$ mf. is connected into the circuit of a 60-cycle alternator giving a wave of the form,

$$e \doteq E\,(\cos\phi - 0.10\cos 3\,\phi - 0.08\cos 5\,\phi + 0.06\cos 7\,\phi),$$

or, in symbolic expression,

$$\dot{E} = e\,(1_1 - 0.10_3 - 0.08_5 + 0.06_7).$$

The synchronous impedance of the alternator is

$$Z_0 = r_0 - \dot{j}_n n x_0 = 0.3 - 5\,n\dot{j}_n.$$

What is the apparent capacity, C, of the condenser (as calculated from its terminal volts and amperes) when connected directly with the alternator terminals, and when connected thereto through various amounts of resistance and inductive reactance?

The condensive reactance of the condenser is

$$x_c = \frac{10^6}{2\,\pi f C_0} = 132 \text{ ohms},$$

or, in symbolic expression,

$$+\,\dot{j}_n \frac{x_c}{n} = \frac{132}{n}\dot{j}_n.$$

Let $Z_1 = r - \dot{j}_n n x$ = impedance inserted in series with the condenser.

The total impedance of the circuit is then

$$Z = Z_0 + Z_1 + \dot{j}_n \frac{x_c}{n} = (0.3 + r) - \dot{j}_n\left([5 + x]\,n - \frac{132}{n}\right).$$

The current in the circuit is

$$\dot{I} = \frac{\dot{E}}{Z} = e\left[\frac{1}{(0.3 + r) - \dot{j}\,(x - \underset{127}{\cancel{132}})} - \frac{.1}{(0.3 + r) - \dot{j}_3\,(3\,x - 29)}\right.$$
$$\left. - \frac{.08}{(0.3 + r) - \dot{j}_5\,(5\,x - 1.4)} + \frac{.06}{(0.3 + r) - \dot{j}_7\,(7\,x + 16.1)}\right],$$

and the e.m.f. at the condenser terminals,

$$\dot{E}_1 = j_n \frac{x_c \dot{I}}{n} = e \left[\frac{132\, j_1}{(0.3 + r) - j_1 (x - 132)} - \frac{4.4\, j_3}{(0.3 + r) - j_3 (3x - 29)} - \frac{2.11\, j_5}{(0.3 + r) - j_5 (5x - 1.4)} + \frac{1.13\, j_7}{(0.3 + r) - j_7 (7x + 16.1)} \right];$$

thus the apparent condensive reactance of the condenser is

$$x_1 = \frac{E_1}{I},$$

and the apparent capacity,

$$C = \frac{10^6}{2\pi f x_1}.$$

(*a*) $x = 0$: Resistance, r, in series with the condenser. Reduced to absolute values it is

$$\frac{1}{x_1^2} = \frac{\frac{1}{(.3+r)^2+17424} + \frac{0.01}{(.3+r)^2+841} + \frac{0.0064}{(.3+r)^2+1.96} + \frac{0.0036}{(.3+r)^2+259}}{\frac{17424}{(.3+r)^2+17424} + \frac{19.4}{(.3+r)^2+841} + \frac{4.45}{(.3+r)^2+1.96} + \frac{1.28}{(.3+r)^2+259}}.$$

(*b*) $r = 0$: Inductive reactance x in series with the condenser. Reduced to absolute values it is

$$\frac{1}{x_1^2} = \frac{\frac{1}{.09+(x-132)^2} + \frac{.01}{.09+(3x-29)^2} + \frac{.0064}{.09+(5x-1.4)^2} + \frac{.0036}{.09+(7x+16.1)^2}}{\frac{17424}{.09+(x-132)^2} + \frac{19.4}{.09+(3x-29)^2} + \frac{4.45}{.09+(5x-1.4)^2} + \frac{1.28}{.09+(7x+16.1)^2}}.$$

From $\frac{1}{x_1^2}$ are derived the values of apparent capacity,

$$C = \frac{10^6}{2\pi f x_1},$$

and plotted in Fig. 230 for values of r and x respectively varying from 0 to 22 ohms.

As seen, with neither additional resistance nor reactance in series to the condenser, the apparent capacity with this generator wave is 84 mf., or 4.2 times the true capacity, and gradually decreases with increasing series resistance, to $C = 27.5$ mf. $= 1.375$ times the true capacity at $r = 13.2$ ohms, or $\frac{1}{10}$ the true capacity reactance. With $r = 132$ ohms, or with an additional resistance equal to the condensive reactance, $C = 20.5$ mf. or

only 2.5 per cent in excess of the true capacity C_0, and at $r = \infty$, $C = 20.3$ mf. or 1.5 per cent in excess of the true capacity.

With reactance x, but no additional resistance, r, in series, the apparent capacity, C, rises from 4.2 times the true capacity at $x = 0$, to a maximum of 5.03 times the true capacity, or $C = 100.6$ mf. at $x = 0.28$, the condition of resonance of the fifth harmonic then decreases to a minimum of 27 mf., or 35 per cent

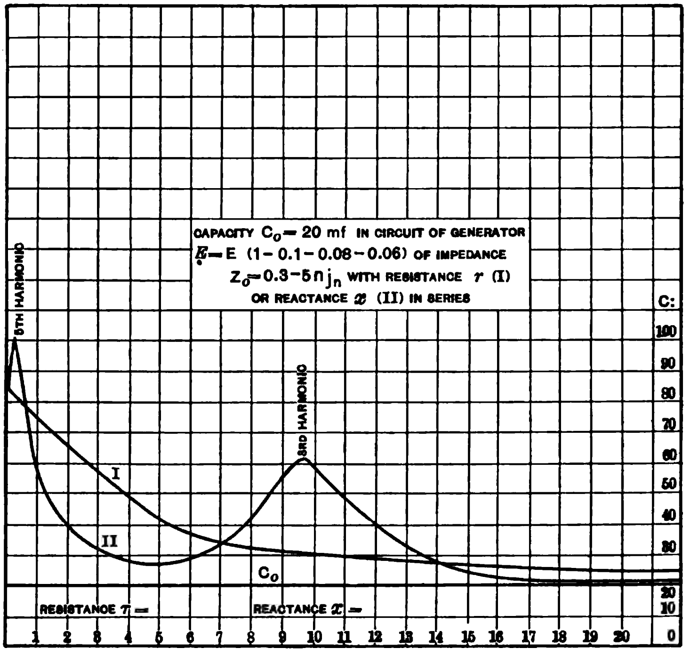


FIG. 230.

in excess of the true capacity, rises again to 60.2 mf., or 3.01 times the true capacity at $x = 9.67$, the condition of resonance with the third harmonic, and finally decreases, reaching 20 mf., or the true capacity at $x = 132$, or an inductive reactance equal to the condensive reactance, then increases again to 20.2 mf. at $x = \infty$.

This rise and fall of the apparent capacity is within certain limits independent of the magnitude of the higher harmonics of the generator wave of e.m.f., but merely depends upon their

presence. That is, with such a reactance connected in series as to cause resonance with one of the higher harmonics, the increase of apparent capacity is approximately the same, whatever the value of the harmonic, whether it equals 25 per cent of the fundamental or less than 5 per cent, provided the resistance in the circuit is negligible. The only effect of the amplitude of the higher harmonic is that when it is small a lower resistance makes itself felt by reducing the increase of apparent capacity below the value it would have were the amplitude greater.

It thus follows that the true capacity of a condenser cannot even approximately be determined by measuring volts and amperes if there are any higher harmonics present in the generator wave, except by inserting a very large resistance or reactance in series to the condenser.

358. 3*d* *Example:* An alternating-current generator of the wave,

$$E_0 = 2000\,[1_1 + 0.12_3 - 0.23_5 - 0.13_7],$$

and of synchronous impedance,

$$Z_0 = 0.3 - 5\,nj_n,$$

feeds over a line of impedance,

$$Z_1 = 2 - 4\,nj_n,$$

a synchronous motor of the wave,

$$\dot{E}_1 = 2250\,[(\cos\theta + j_1 \sin\theta) + 0.24\,(\cos 3\,\theta + j_3 \sin 3\,\theta)],$$

and of synchronous impedance,

$$Z_2 = 0.3 - 6\,nj_n.$$

The total impedance of the system is then,

$$Z = Z_0 + Z_1 + Z_2 = 2.6 - 15\,nj_n,$$

thus the current,

$$\dot{I} = \frac{\dot{E}_0 - \dot{E}_1}{Z}$$

$$= \frac{2000 - 2250\cos\theta - 2250\,j_1\sin\theta}{2.6 - 15\,j_1} + \frac{240 - 540\cos 3\,\theta - 540 j_3 \sin 3\,\theta}{2.6 - 45\,j_3}$$

$$- \frac{460}{2.6\ - 75\,j_5} - \frac{260}{2.6\ - 105\,j_7}$$

$$= (a_1{}^1 + j_1 a_1{}^{11}) + (a_3{}^1 + j_3 a_3{}^{11}) + (a_5{}^1 + j_5 a_5{}^{11}) + (a_7{}^1 + j_7 a_7{}^{11});$$

where

$$\begin{aligned}
a_1^{1} &= 22.5 - 25.2 \cos \theta + 146 \sin \theta, \\
a_3^{1} &= .306 - 0.69 \cos 3\,\theta + 11.9 \sin 3\,\theta, \\
a_5^{1} &= 0.213, \\
a_7^{1} &= -0.061, \\
a_1^{11} &= 130 - 146 \cos \theta - 25.2 \sin \theta, \\
a_3^{11} &= 5.3 - 11.9 \cos 3\,\theta - 0.69 \sin 3\,\theta. \\
a_5^{11} &= -6.12, \\
a_7^{11} &= -2.48,
\end{aligned}$$

or, absolute,

first harmonic,

$$a_1 = \sqrt{a_1^{1^2} + a_1^{11^2}},$$

third harmonic,

$$a_3 = \sqrt{a_3^{1^2} + a_3^{11^2}},$$

fifth harmonic,

$$a_5 = 6.12,$$

seventh harmonic,

$$a_7 = 2.48,$$

$$I = \sqrt{a_1^2 + a_3^2 + a_5^2 + a_7^2};$$

while the total current of higher harmonics is

$$I_0 = \sqrt{a_3^2 + a_5^2 + a_7^2}.$$

The true input of the synchronous motor is

$$\begin{aligned}
P^1 &= [E_1 I]^1 \\
&= (2250\, a_1^{1} \cos \theta + 2250\, a_1^{11} \sin \theta) + (540\, a_3^{1} \cos 3\,\theta + 540\, a_3^{11} \sin 3\,\theta \\
&= P_1^{1} + P_3^{1}
\end{aligned}$$

$$P_1^{1} = 2250\,(a_1^{1} \cos \theta + a_1^{11} \sin \theta),$$

is the power of the fundamental wave,

$$P_3^{1} = 540\,(a_3^{1} \cos 3\,\theta + a_3^{11} \sin 3\,\theta),$$

the power of the third harmonic.

The fifth and seventh harmonics do not give any power, since they are not contained in the synchronous motor wave. Substituting now different numerical values for θ, the phase angle

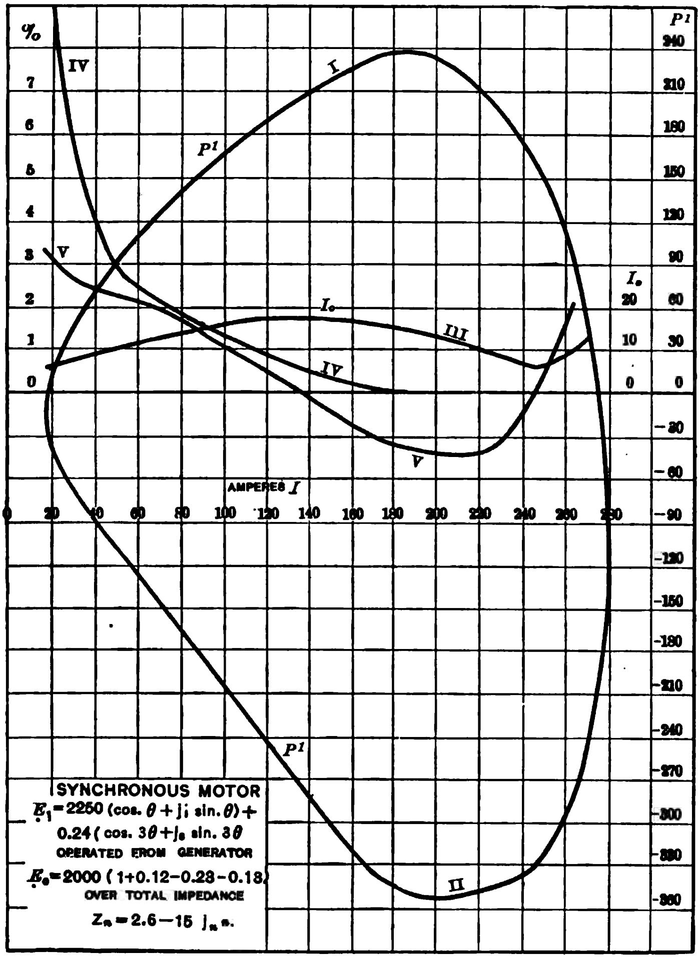


FIG. 231. — Synchronous Motor.

between generator e.m.f. and synchronous motor counter e.m.f., corresponding values of the currents, I, I_0, and the powers, P^1, P_1^1, P_3^1, are derived. These are plotted in Fig. 231 with the

total current, I, as abscissas. To each value of the total current, I, correspond two values of the total power, P^1, a positive value plotted as Curve I — synchronous motor — and a negative value plotted as Curve II — alternating-current generator —. Curve III gives the total current of higher frequency, I_0, Curve IV the difference between the total current and the current of fundamental frequency, $I - I_1$, in percentage of the total current, I, and V the power of the third harmonic, P_3^1, in percentage of the total power, P^1.

Curves III, IV, and V correspond to the positive or synchronous motor part of the power curve, P^1. As seen, the increase of current due to the higher harmonics is small, and entirely disappears at about 180 amperes. The power of the third harmonic is positive, that is, adds to the power of the synchronous motor up to about 140 amperes, or near the maximum output of the motor, and then becomes negative.

It follows herefrom that higher harmonics in the e.m.f. waves of generators and synchronous motors do not represent a mere waste of current, but may contribute more or less to the output of the motor. Thus at 75 amperes total current, the percentage of increase of power due to the higher harmonic is equal to the increase of current, or in other words the higher harmonics of current do work with the same efficiency as the fundamental wave.

359. *4th Example:* In a small three-phase induction motor, the constants per delta circuit are

$$\text{primary admittance} \qquad Y = 0.002 + 0.03\,j,$$

$$\text{self-inductive impedance} \qquad Z_0 = Z_1 = 0.6 - 2.4\,j, :$$

and a sine wave of e.m.f., $e_0 = 110$ volts, is impressed upon the motor.

The power output, P, current input I_s, and power-factor p, as function of the slip s, are given in the first columns of the following table, calculated in the manner as described in the chapter on Induction Motors.

To improve the power-factor of the motor and bring it to unity at an output of 500 watts, a condenser capacity is required

giving 4.28 amperes leading current at 110 volts, that is, neglecting the power loss in the condenser, capacity susceptance

$$\frac{4.28}{110} = 0.039.$$

In this case, let $\dot{I}_s$ = current input into the motor per delta circuit at slip s, as given in the following table.

The total current supplied by the circuit with a sine wave of impressed e.m.f. is

$$\dot{I}' = \dot{I}_s - 4.28\,j,$$

and herefrom the power-factor $= \dfrac{\text{power current}}{\text{total current}}$, given in the second columns of the table.

If the impressed e.m.f. is not a sine wave but a wave of the shape,

$$\dot{E}_0 = e_0\,(1_1 + 0.12_3 - 0.23_5 - 0.134_7),$$

to give the same output, the fundamental wave must be the same: e_0 = 110 volts, when assuming the higher harmonics in the motor as wattless, that is,

$$\dot{E}_0 = 110_1 + 13.2_3 - 25.3_5 - 14.7_7 = e_0 + \dot{E}_0^1,$$

where $\dot{E}_0^1 = 13.2_3 - 25.3_5 - 14.7_7$

$=$ component of impressed e.m.f. of higher frequency.

The effective value is

$$E_0 = 114.5 \text{ volts}.$$

The condenser admittance for the general alternating wave is

$$Y_c = -\,0.039\,nj_n.$$

Since the frequency of rotation of the motor is very small compared with the frequency of the higher harmonics, as total impedance of the motor for these higher harmonics can be assumed the stationary impedance, and by neglecting the resistance we have

$$Z^1 = -\,nj_n\,(x_0 + x_1) = -\,4.8\,nj_n.$$

The exciting admittance of the motor, for these higher harmonics, is, by neglecting the conductance,

$$Y^1 = \frac{bj_n}{n} = \frac{0.03\, j_n}{n},$$

and the higher harmonics of counter e.m.f.,

$$\dot{E}^1 = \frac{\dot{E}_0^1}{2}.$$

Thus we have,

current input in the condenser,

$$\dot{I}_c = \dot{E}_0 Y_c = -4.28\, j_1 - 1.54\, j_3 + 4.93\, j_5 + 4.02\, j_7\,;$$

high-frequency component of motor-impedance current,

$$\frac{\dot{E}_0^1}{\dot{Z}^1} = 0.92\, j_3 - 1.06\, j_5 - 0.44\, j_7\,;$$

high-frequency component of motor-exciting current,

$$\dot{E}^1 Y^1 = \frac{\dot{E}_0^1 Y^1}{2} = 0.07\, j_3 - 0.08\, j_5 - 0.03\, j_7:$$

thus, total high-frequency component of motor current,

$$\dot{I}_0^1 = \frac{\dot{E}_0^1}{Z^1} + \dot{E}^1 Y^1 = 0.99\, j_3 - 1.14\, j_5 - 0.47\, j_7,$$

and total current, without condenser,

$$\dot{I}_0 = \dot{I}_s + \dot{I}_0^1 = \dot{I}_s + 0.99\, j_3 - 1.14\, j_5 - 0.47\, j_7,$$

with condenser,

$$\dot{I} = \dot{I}_s + \dot{I}_0^1 - \dot{I}_c = \dot{I}_s - 4.28\, j_1 - 0.55\, j_3 + 3.79\, j_5 + 3.55\, j_7;$$

and herefrom the power-factor.

In the following table and in Fig. 232 are given the values of current and power-factor: —

I. With sine wave of e.m.f., of 110 volts, and no condenser.
II. With sine wave of e.m.f., of 110 volts, and with condenser.
III. With distorted wave of e.m.f., of 114.5 volts, and no condenser.
IV. With distorted wave of e.m.f., of 114.5 volts, and with condenser.

TABLE.

		I.			II.		III.		IV.	
s	P	I_s	I_s	p	I^1	p	I_0	p	I	p
0	0	.24+ 3.10j	3.1	7.8	1.2	20	3.5	6.6	5.2	4.4
.01	160	1.73+ 3.16j	3.6	48	2.1	84	3.9	43	5.5	31
.02	320	3.32+ 3.47j	4.8	69	3.4	97.2	5.1	64	6.1	54
.035	500	5.16+ 4.28j	6.7	77	5.2	100	6.9	72.5	7.2	68
.05	660	6.95+ 5.4j	8.8	79	7.0	98.7	8.9	76	8.6	77
.07	810	8.77+ 7.3j	11.4	77	9.3	94.5	11.5	73.5	10.6	80
.10	885	10.1 + 9.85j	14.1	71.5	11.5	87	14.2	68	12.6	77
.13	900	10.45+11.45j	15.5	67.5	12.7	82	15.6	64.5	13.7	73
.15	890	10.75+12.9j	16.8	64	13.8	78	16.9	61	14.7	70

The curves II and IV with condenser are plotted in dotted lines in Fig. 232. As seen, even with such a distorted wave the

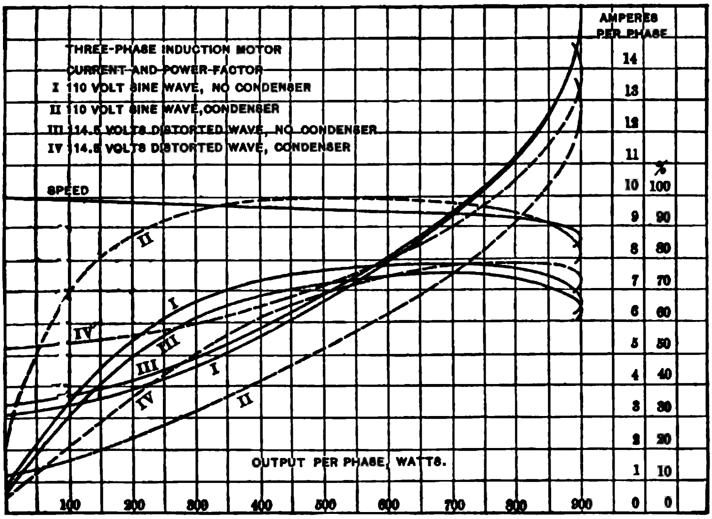


FIG. 232.

current input and power-factor of the motor are not much changed if no condenser is used. When using a condenser in shunt to the motor, however, with such a wave of impressed e.m.f. the increase of the total current, due to higher-frequency currents in the condenser, is greater than the decrease, due to the compensation of lagging currents, and the power-factor is actually lowered by the condenser, over the total range of load up to overload, and especially at light load.

Where a compensator or transformer is used for feeding the condenser, due to the internal self-inductance of the compensator, the higher harmonics of current are still more accentuated, that is, the power-factor still more lowered.

In the preceding the energy loss in the condenser and compensator and that due to the higher harmonics of current in the motor has been neglected. The effect of this energy loss is a slight decrease of efficiency and corresponding increase of power-factor. The power produced by the higher harmonics has also been neglected; it may be positive or negative, according to the index of the harmonic, and the winding of the motor primary. Thus, for instance, the effect of the triple harmonic is negative in the quarter-phase motor, zero in the three-phase motor, etc.; altogether, however, the effect of these harmonics is small.

360. *5th Example:* In the constant potential-constant-current transformation, discussed in Chapter X, the effect of the distortion of the wave of impressed e.m.f., e_0, is to be investigated.

A. *T-Connection or Resonating Circuit.*

Assuming the same denotation as in § 69, we have, for the nth harmonic:

primary inductive reactance,

$$Z_0 = -jnx_0;$$

secondary inductive reactance,

$$Z_1 = -jnx_1;$$

condensive reactance,

$$Z_2 = +\frac{jx_0}{n};$$

when neglecting the energy losses in the reactances,

load

$$Z = r(1 - jnk)$$

therefore, also for the nth harmonic.

$$\begin{aligned} \dot{E} &= r(1 - jnk)\,\dot{I}_1 \\ \dot{E}_1 &= \dot{E} + Z_1\,\dot{I} \\ &= [r(1 - jnk) - jnx_1]\,\dot{I}, \end{aligned}$$

and also

$$E_1 = j\frac{x_0}{n} I_1;$$

hence,

$$I_1 = \frac{n\,[r\,(1 - jnk) - jnx_1]}{jx_0} I,$$

and

$$\begin{aligned} I_0 &= I + I_1 \\ &= \frac{jx_0 - jn^2x_1 + nr\,(1 - jnk)}{jx_0} I; \end{aligned}$$

hence,

$$\begin{aligned} E_0 &= E_1 + Z_0 I_0 \\ &= \{[r\,(1 - jnk) - jnx_1] - n\,[jx_0 - jn^2x_1 + nr\,(1 - jnk)]\}\, I \\ &= \{-(n^2 - 1)\,r\,(1 - jnk) + jnx_1\,(n^2 - 1) - jnx_0\}\, I; \end{aligned}$$

hence,

$$\begin{aligned} I &= \frac{jE_0}{nx_0 - nx_1\,(n^2 - 1) - j\,(n^2 - 1)\,r\,(1 - jnk)} \\ &= \frac{jE_0}{nx_0 - (n^2 - 1)\,[n\,(x_1 + kr) - jr]}; \end{aligned}$$

hence, approximately, for higher values of n,

$$I = -\frac{jE_0}{n^3\,(x_1 + kr)};$$

that is, for larger values of n, $I = 0$, or the higher harmonics in the current wave disappear.

Herefrom, by substituting in the preceding equations, the supply current, I_0, the condenser current, I_1, their respective e.m.fs., etc., are derived.

It is then, in general expression:

If

$$E_0 = \sum_1^\infty (e_n + j_n e_n^1) = \text{impressed e.m.f.},$$

$$\begin{aligned} I &= \sum_1^\infty \frac{j_n\,(e_n + j_n e_n^1)}{nx_0 - (n^2 - 1)\,[n\,(x_1 + kr) - jr]} \\ &= \frac{j\,(e_0 + je_0^1)}{x_0} + \sum_3^\infty \frac{j_n\,(e_n + je_n^1)}{n^3\,(x_1 + kr)}, \end{aligned}$$

the equation of the secondary current.

For instance, let

$$E_0 = 6600 \{1_1 - 0.20_3 - 0.15_5 + 0.06_7 + 0.25\, j_9\}$$
$$= \text{constant-impressed e.m.f.}$$

or, absolute,

$$e_0 = 6600 \sqrt{1 + 0.20^2 + 0.15^2 + 0.06^2 + 0.25^2}$$
$$= 6600 \times 1.062$$
$$= 7010 \text{ volts,}$$

and choosing the same values as before, in § 73,

$$x_0 = 880 \text{ ohms,}$$
$$x_1 = 508 \text{ ohms,}$$
$$r' = 930 \text{ ohms,}$$
$$k = 0.4;$$

it is, substituting,

$$\dot{I} = 7.5\, j + \frac{60.0_3 + 48.8\, j_3 + 8.0\, j_5 - 1.2\, j_7}{508 + 0.4\, r},$$

or, absolute,

$$i = \sqrt{7.5^2 + \frac{60.0^2 + 48.8^2 + 8.0^2 + 1.2^2}{(508 + 0.4\, r)^2}}$$
$$= \sqrt{7.5^2 + \frac{604600}{(508 + 0.4\, r)^2}};$$

hence, at no-load,

$$i = 7.5 \times 1.00021$$

and, at full load,

$$r = 930,$$
$$i = 7.5 \times 1.00003.$$

That is, the current wave is as perfect a sine wave as possible, regardless of the distortion of the impressed e.m.f., which, for instance, in the above example, contains a third harmonic of 32 per cent. Or in other words, in the T-connection or the resonating circuit, all harmonics of e.m.f. are wiped out in the current wave, and this method indeed offers the best and most convenient means of producing perfect sine waves of current from any shape of e.m.f. waves.

361. B. *Monocyclic Square.*

Assuming the same denotation as in § 75, we have for the nth harmonic:
inductive reactance,

$$Z_2 = -jnx_0;$$

condensive reactance,

$$Z_1 = +j\frac{x_0}{n};$$

load,

$$Z = r(1 - jnc);$$

currents,

$$\dot{I}_1 = \frac{\dot{I}_0 - \dot{I}}{2},$$

and
$$\dot{I}_2 = \frac{\dot{I}_0 + \dot{I}}{2};$$

e.m.fs.,

$$\dot{E}_0 = Z_1\dot{I}_1 + Z_2\dot{I}_2,$$

and
$$Z\dot{I} = Z_1\dot{I}_1 - Z_2\dot{I}_2;$$

hence, substituting, we have

$$\dot{E}_0 = jx_0\left(\frac{\dot{I}_1}{n} - n\dot{I}_2\right),$$

$$r(1 - jnk)\dot{I} = jx_0\left(\frac{\dot{I}_1}{n} + n\dot{I}_2\right);$$

thus,

$$\dot{E}_0 = \frac{jx_0}{2}\left\{\dot{I}_0\left(\frac{1}{n} - n\right) - \dot{I}\left(\frac{1}{n} + n\right)\right\},$$

$$r(1 - jnk)\dot{I} = \frac{jx_0}{2}\left\{\dot{I}_0\left(\frac{1}{n} + n\right) - \dot{I}\left(\frac{1}{n} - n\right)\right\};$$

then, combining, we obtain

$$\dot{E}_0\left(n + \frac{1}{n}\right) + r(1 - jnk)\dot{I}\left(n - \frac{1}{n}\right) = -\frac{jx_0}{2}\dot{I}\left\{\left(n + \frac{1}{n}\right)^2 - \left(n - \frac{1}{n}\right)^2\right\}$$
$$= -2jx_0\dot{I},$$

and

$$\dot{I} = \frac{j\dot{E}_0\left(n + \frac{1}{n}\right)}{2\,x_0 - jr\,(1 - jnk)\left(n - \frac{1}{n}\right)}$$

$$= \frac{j\dot{E}_0\,(n^2 + 1)}{2\,nx_0 - jr\,(n^2 - 1)(1 - jnk)},$$

and herefrom $\dot{I}_0$, $\dot{I}_1$, $\dot{I}_2$, etc.

Approximately, for higher values of n, and for high loads, r,

$$\dot{I} = \frac{j\dot{E}_0}{nkr}.$$

That is, the higher harmonics of current decrease proportionally to their order, at heavy loads — that is, large values of r. For light loads, however, or small values of r, and in the extreme case, at no-load, or $r = 0$, it is

$$\dot{I} = \frac{j\dot{E}_0\,(n^2 + 1)}{2\,nx_0},$$

and, approximately,

$$\dot{I} = \frac{j\dot{E}_0 n}{2\,x_0}.$$

That is, the current is increased proportional to the order of the harmonics, or in other words, at no-load, in the monocyclic square, the higher harmonics of impressed e.m.fs. produce increased values of the higher harmonics of current, that is, the wave-shape distortion is increased the more, the higher the harmonics.

In general expression:

If

$$\dot{E}_0 = \sum_1^\infty (e_n + j_n e_n') = \text{impressed e.m.f.},$$

$$\dot{I} = \sum_1^\infty \frac{j_n\,(n^2 + 1)\,(e_n + j_n e_n')}{2\,nx_0 - j_n r(n^2 - 1)\,(1 - j_n nk)},$$

and herefrom $\dot{I}_0$, $\dot{I}_1$, $\dot{I}_2$, etc.

For instance, let

$$\dot{E}_0 = 6600\,\{1_1 - 0.20_3 + 0.25\,j_3{}' - 0.15_5 + 0.06_7\}$$
$$= \text{constant-impressed e.m.f.},$$

or, absolute,

$$e_0 = 7010 \text{ volts},$$

and, choosing the same values as in § 77,

$$x_0 = 880 \text{ ohms},$$
$$r' = 930 \text{ ohms},$$
$$k = 0.4;$$

it is, substituted,

$$\dot{I} = 7.5 - \frac{(2.5 + 2\, j_3)\, 6600}{5280 - (9.6 + 8\, j)r} - \frac{25{,}740}{8800 - (48 + 24\, j_5)r} + \frac{19{,}800}{12{,}320 - (134.4 + 48\, j)r};$$

herefrom follows,

at no-load, $r = 0$,

$$\dot{I} = 7.5 - (3.12 + 2.5\, j_3) - 2.92 + 1.61_7.$$

That is, at no-load, the secondary current contains excessive higher harmonics, for instance, a third harmonic,

$$\sqrt{3.12^2 + 2.5^2} = 4.0, \text{ or } 53.3 \text{ per cent of the fundamental.}$$

Absolute, the no-load current is

$$i = \sqrt{7.5^2 + 3.12^2 + 2.5^2 + 2.92^2 + 1.61^2} = 9.13 \text{ amperes.}$$

At full load, or $r = 930$, it is

$$\dot{I} = 7.5 + (2.18 - 1.07\, j_3) + (0.51 - 0.32\, j_5) - (0.14 - 0.06\, j_7);$$

that is, at full load, the harmonics, while still intensified, are less than at no-load, and decrease with their order, n, more rapidly. The absolute value is

$$i = \sqrt{7.5^2 + 2.18^2 + 1.07^2 + 0.51^2 + 0.32^2 + 0.14^2 + 0.06^2}$$
$$= 7.91 \text{ amperes.}$$

Instead of 7.5 amperes, the value which the current would have at all loads if no higher harmonics were present, the higher harmonics of impressed e.m.f. raise the current to 9.13 amperes, or by 21.7 per cent at no-load, and to 7.91 amperes, or by 2.1 per cent at full load, while the impressed e.m.f. is increased by 6.2 per cent by its higher harmonics.

It follows also that the constant-current regulation of the system is seriously impaired, and between no-load and full load the current decreases from 9.13 to 7.91 amperes, or by 15.4 per cent, which as a rule is too much for an arc circuit.

362. It follows herefrom:

While the *T*-connection of transformation from constant potential to constant current suppresses the higher harmonics of impressed e.m.f. and makes the constant current a perfect sine wave, the monocyclic square intensifies the higher harmonics so that the higher harmonics of impressed e.m.f. appear at greatly increased intensity in the constant-current wave. The increase of the higher harmonics is different for the different harmonics and for different loads, and the distortion of wave-shape produced hereby is far greater at no-load, and the constant current regulation of the system is thereby greatly impaired, and at load the distortion is less, and very high harmonics are fairly well suppressed, and the operation of an arc circuit so feasible.

Assuming, then, that in the monocyclic square of constant-potential, constant-current transformation, with a distorted wave of impressed e.m.f., we insert in series to the monocyclic square into the main circuit, $\dot{I}_0$, two reactances of opposite sign, which are equal to each other for the fundamental frequency, that is, a condensive reactance, $Z_3 = +j\frac{x_3}{n}$, and an inductive reactance, $Z_4 = -jnx_3$. Then for the fundamental, these two reactances together offer no resultant impedance, but neutralize each other, and the only drop of voltage produced by them is that due to the small loss of power in them. At the nth harmonic, however, the resultant reactance is

$$Z_3 + Z_4 = -jx_3\left(n + \frac{1}{n}\right),$$

or, approximately,

$$= -jx_3 n,$$

and two such impedances so obstruct the higher harmonics, the more, the higher their order while passing the fundamental sine wave.

Such a pair of equal reactances of opposite sign so can be called a "*wave screen.*"

Further problems for investigation by the student then are:

(1) The investigation of the effect of the distortion of the wave of impressed e.m.f. on the constant current, with other transforming devices, and also the reverse problem, the investigation of the effect of the distortion of the constant-current wave, as caused by an arc, on the system of transformation.

(2) What must be the value, x_3, of the reactance of a wave screen, to reduce the wave-shape distortion of the secondary current in the monocyclic square to the same percentage as the distortion of the impressed e.m.f. wave, or to any desired percentage, or to reduce the variation of the constant current with the load, as due to the wave-shape distortion, below a given percentage?

(3) Determination of efficiency and regulation in the monocyclic square with interposed wave screen, x_3, assuming again 3 per cent loss in the inductances, 1 per cent loss in the capacities, and choosing x_3 so as to fill given conditions, regarding wave-shape distortion, or regulation, or efficiency, etc.

CHAPTER XXXII.

GENERAL POLYPHASE SYSTEMS.

363. A polyphase system is an alternating-current system in which several e.m.fs. of the same frequency, but displaced in phase from each other, produce several currents of equal frequency, but displaced phases.

Thus any polyphase system can be considered as consisting of a number of single circuits, or branches of the polyphase system, which may be more or less interlinked with each other.

In general the investigation of a polyphase system is carried out by treating the single-phase branch circuits independently.

Thus all the discussions on generators, synchronous motors, induction motors, etc., in the preceding chapters, apply to single-phase systems as well as polyphase systems, in the latter case the total power being the sum of the powers of the individual or branch circuits.

If the polyphase system consists of n equal e.m.fs. displaced from each other by $\frac{1}{n}$ of a period, the system is called a *symmetrical system*, otherwise an *unsymmetrical system.*

Thus the three-phase system, consisting of three equal e.m.fs. displaced by one-third of a period, is a symmetrical system. The quarter-phase system, consisting of two equal e.m.fs. displaced by 90°, or one-quarter of a period, is an unsymmetrical system.

364. The power in a single-phase system is pulsating; that is, the watt curve of the circuit is a sine wave of double frequency, alternating between a maximum value and zero, or a negative maximum value. In a polyphase system the watt curves of the different branches of the system are pulsating also. Their sum, however, or the total power of the system, may be either constant or pulsating. In the first case, the system is called a *balanced system*, in the latter case an *unbalanced system.*

The three-phase system and the quarter-phase system, with

equal load on the different branches, are balanced systems; with unequal distribution of load between the individual branches both systems become unbalanced systems.

The different branches of a polyphase system may be either independent from each other, that is, without any electrical interconnection, or they may be interlinked with each other. In

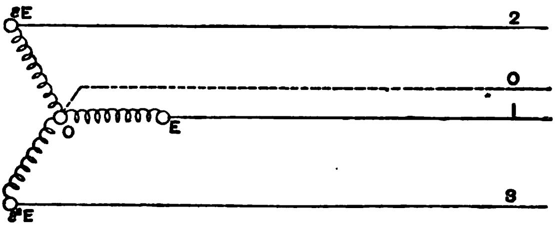


FIG. 233.

the first case the polyphase system is called an *independent system*, in the latter case an *interlinked system*.

The three-phase system with star-connected or ring-connected generator, as shown diagrammatically in Figs. 233 and 234, is an interlinked system.

The four-phase system as derived by connecting four equi-

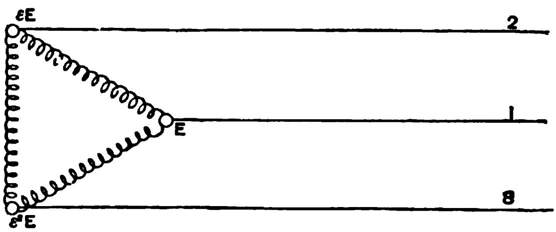


FIG. 234.

distant points of a continuous-current armature with four collector rings, as shown diagrammatically in Fig. 235, is an interlinked system also. The four-wire, quarter-phase system produced by a generator with two independent armature coils, or by two single-phase generators rigidly connected with each other in quadrature, is an independent system. As interlinked system, it is shown in Fig. 236, as star-connected, four-phase system.

365. Thus, polyphase systems can be subdivided into:

Symmetrical systems and unsymmetrical systems.

Balanced systems and unbalanced systems.

Interlinked systems and independent systems.

The only polyphase systems which have found practical application are:

The three-phase system, consisting of three e.m.fs. displaced by one-third of a period, is used exclusively as interlinked system.

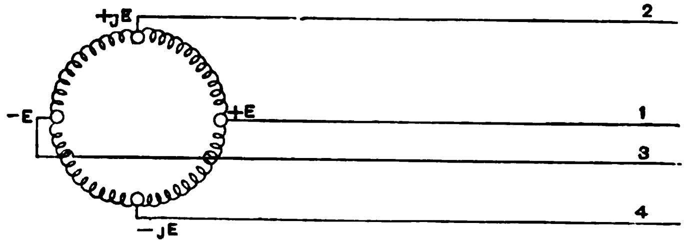


FIG. 235.

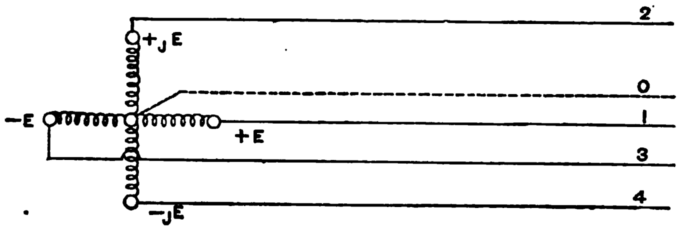


FIG. 236.

The quarter-phase system, consisting of two e.m.fs. in quadrature, and used with four wires, or with three wires, which may be either an interlinked system or an independent system.

The six-phase system, consisting of two three-phase systems in opposition to each other, and derived by transformation from a three-phase system, in the alternating supply circuit of large synchronous converters.

The inverted three-phase system, consisting of two e.m.fs. displaced from each other by 60°, and derived from two phases of a three-phase system by transformation with two transformers, of which the secondary of one is reversed with regard to its primary (thus changing the phase difference from 120° to 180° − 120° = 60°) finds a limited application in low-tension distribution.

CHAPTER XXXIII.

SYMMETRICAL POLYPHASE SYSTEMS.

366. If all the e.m.fs. of a polyphase system are equal in intensity and differ from each other by the same angle of difference of phase, the system is called a symmetrical polyphase system.

Hence, a symmetrical n-phase system is a system of n e.m.fs. of equal intensity, differing from each other in phase by $\frac{1}{n}$ of a period:

$$\begin{aligned}
e_1 &= E \sin \beta; \\
e_2 &= E \sin\left(\beta - \frac{2\pi}{n}\right); \\
e_3 &= E \sin\left(\beta - \frac{4\pi}{n}\right); \\
&\vdots \\
e_n &= E \sin\left(\beta - \frac{2(n-1)\pi}{n}\right).
\end{aligned}$$

The next e.m.f. is, again,

$$e_1 = E \sin(\beta - 2\pi) = E \sin \beta.$$

In the polar diagram the n e.m.fs. of the symmetrical n-phase system are represented by n equal vectors, following each other under equal angles.

Since in symbolic writing rotation by $\frac{1}{n}$ of a period, or angle $\frac{2\pi}{n}$, is represented by multiplication with

$$\cos \frac{2\pi}{n} + j \sin \frac{2\pi}{n} = \epsilon,$$

the e.m.fs. of the symmetrical polyphase system are

$$\dot{E};$$

$$\dot{E}\left(\cos\frac{2\pi}{n} + j\sin\frac{2\pi}{n}\right) = \dot{E}\epsilon;$$

$$\dot{E}\left(\cos\frac{4\pi}{n} + j\sin\frac{4\pi}{n}\right) = \dot{E}\epsilon^2;$$

$$\dot{E}\left(\cos\frac{2(n-1)\pi}{n} + j\sin\frac{2(n-1)\pi}{n}\right) = \dot{E}\epsilon^{n-1}.$$

The next e.m.f. is again,

$$\dot{E}(\cos 2\pi + j\sin 2\pi) = \dot{E}\epsilon^n = \dot{E}.$$

Hence, it is

$$\epsilon = \cos\frac{2\pi}{n} + j\sin\frac{2\pi}{n} = \sqrt[n]{1}.$$

Or in other words:

In a symmetrical n-phase system any e.m.f. of the system is expressed by

$$\epsilon^i\dot{E};$$

where $$\epsilon = \sqrt[n]{1}.$$

367. Substituting now for n different values, we get the different symmetrical polyphase systems, represented by

$$\epsilon^i\dot{E},$$

where $$\epsilon = \sqrt[n]{1} = \cos\frac{2\pi}{n} + j\sin\frac{2\pi}{n}.$$

(1) $n = 1 \quad \epsilon = 1 \quad \epsilon^i\dot{E} = \dot{E},$

the ordinary single-phase system.

(2) $n = 2 \quad \epsilon = -1 \quad \epsilon^i\dot{E} = \dot{E}$ and $-\dot{E}$.

Since $-\dot{E}$ is the return of $\dot{E}$, $n = 2$ gives again the single-phase system.

(3) $n = 3 \quad \epsilon = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = \frac{-1 + j\sqrt{3}}{2}$

$$\epsilon^2 = \frac{-1 - j\sqrt{3}}{2}.$$

The three e.m.fs. of the three-phase system are

$$\varepsilon^i \dot{E} = \dot{E}, \quad \frac{-1 + j\sqrt{3}}{2} \dot{E}, \quad \frac{-1 - j\sqrt{3}}{2} \dot{E}.$$

Consequently the three-phase system is the lowest symmetrical polyphase system.

(4) $n = 4, \quad \varepsilon = \cos \frac{2\pi}{4} + j \sin \frac{2\pi}{4} = j, \; \varepsilon^2 = -1, \varepsilon^3 = -j.$

The four e.m.fs. of the four-phase system are,

$$\varepsilon^i \dot{E} = \dot{E}, \quad j\dot{E}, \quad -\dot{E}, \quad -j\dot{E}.$$

They are in pairs opposite to each other,

$$\dot{E} \text{ and } -\dot{E}; \quad j\dot{E} \text{ and } -j\dot{E}.$$

Hence can be produced by two coils in quadrature with each other, analogous as the two-phase system, or ordinary alternating current system, can be produced by one coil.

Thus the symmetrical quarter-phase system is a four-phase system.

Higher systems than the quarter-phase or four-phase system have not been very extensively used, and are thus of less practical interest. A symmetrical six-phase system, derived by transformation from a three-phase system, has found application in synchronous converters, as offering a higher output from these machines, and a symmetrical eight-phase system proposed for the same purpose.

368. A characteristic feature of the symmetrical n-phase system is that under certain conditions it can produce a rotating m.m.f of constant intensity.

If n equal magnetizing coils act upon a point under equal angular displacements in space, and are excited by the n e.m.fs. of a symmetrical n-phase system, a m.m.f. of constant intensity is produced at this point, whose direction revolves synchronously with uniform velocity.

Let

n' = number of turns of each magnetizing coil.

E = effective value of impressed e.m.f.

I = effective value of current.

Hence,

$\mathfrak{F} = n'I$ = effective m.m.f. of one of the magnetizing coils.

Then the instantaneous value of the m.m.f. of the coil acting in the direction, $\frac{2\pi i}{n}$, is

$$f_i = \mathfrak{F}\sqrt{2}\sin\left(\beta - \frac{2\pi i}{n}\right)$$
$$= n'I\sqrt{2}\sin\left(\beta - \frac{2\pi i}{n}\right).$$

The two rectangular space components of this m.m.f. are

$$f_i' = f_i \cos\frac{2\pi i}{n}$$
$$= n'I\sqrt{2}\cos\frac{2\pi i}{n}\sin\left(\beta - \frac{2\pi i}{n}\right),$$

and

$$f_i'' = f_i \sin\frac{2\pi i}{n}$$
$$= n'I\sqrt{2}\sin\frac{2\pi i}{n}\sin\left(\beta - \frac{2\pi i}{n}\right).$$

Hence the m.m.f. of this coil can be expressed by the symbolic formula

$$f_i = n'I\sqrt{2}\sin\left(\beta - \frac{2\pi i}{n}\right)\left(\cos\frac{2\pi i}{n} + j\sin\frac{2\pi i}{n}\right).$$

Thus the total or resultant m.m.f. of the n coils displaced under the n equal angles is

$$f = \sum_1^n{}_i f_i = n'I\sqrt{2}\sum_1^n{}_i \sin\left(\beta - \frac{2\pi i}{n}\right)\left(\cos\frac{2\pi i}{n} + j\sin\frac{2\pi i}{n}\right),$$

or, expanded,

$$f = n'I\sqrt{2}\left\{\sin\beta\sum_1^n{}_i\left(\cos^2\frac{2\pi i}{n} + j\sin\frac{2\pi i}{n}\cos\frac{2\pi i}{n}\right)\right.$$
$$\left. - \cos\beta\sum_1^n{}_i\left(\sin\frac{2\pi i}{n}\cos\frac{2\pi i}{n} + j\sin^2\frac{2\pi i}{n}\right)\right\}.$$

It is, however,

$$\cos^2\frac{2\pi i}{n}+j\sin\frac{2\pi i}{n}\cos\frac{2\pi i}{n}=\tfrac{1}{2}\left(1+\cos\frac{4\pi i}{n}+j\sin\frac{4\pi i}{n}\right)$$
$$=\tfrac{1}{2}\left(1+\epsilon^{2i}\right),$$

$$\sin\frac{2\pi i}{n}\cos\frac{2\pi i}{n}+j\sin^2\frac{2\pi i}{n}=\frac{j}{2}\left(1-\cos\frac{4\pi i}{n}-j\sin\frac{4\pi i}{n}\right)$$
$$=\frac{j}{2}\left(1-\epsilon^{2i}\right);$$

and, since

$$\sum_1^n{}_i\ \epsilon^{2i}=0,\qquad \sum_1^n{}_i\ \epsilon^{-2i}=0,$$

it is,
$$f=\frac{nn'I\sqrt{2}}{2}\left(\sin\beta-j\cos\beta\right);$$

or,
$$f=\frac{nn'I}{\sqrt{2}}\left(\sin\beta-j\cos\beta\right)$$
$$=\frac{n\mathfrak{F}}{\sqrt{2}}\left(\sin\beta-j\cos\beta\right);$$

the symbolic expression of the m.m.f. produced by the n circuits of the symmetrical n-phase system, when exciting n equal magnetizing coils displaced in space under equal angles.

The absolute value of this m.m.f. is

$$F=\frac{nn'I}{\sqrt{2}}=\frac{n\mathfrak{F}}{\sqrt{2}}=\frac{n\mathfrak{F}_{max}}{2}.$$

Hence constant and equal $\frac{n}{\sqrt{2}}$ times the effective m.m.f. of each coil or $\frac{n}{2}$ times the maximum m.m.f. of each coil.

The phase of the resultant m.m.f. at the time represented by the angle β is

$$\tan\theta=-\cot\beta;\ \text{hence}\ \theta=\beta-\frac{\pi}{2}.$$

That is, the m.m.f. produced by a symmetrical n-phase system revolves with constant intensity,

$$F=\frac{n\mathfrak{F}}{\sqrt{2}},$$

and constant speed, in synchronism with the frequency of the system; and, if the reluctance of the magnetic circuit is constant, the magnetism revolves with constant intensity and constant speed also, at the point acted upon symmetrically by the n m.m.fs. of the n-phase system.

This is a characteristic feature of the symmetrical polyphase system.

369. In the three-phase system, $n = 3$, $F = 1.5\, \mathfrak{F}_{max}$, where $\mathfrak{F}_{max}$ is the maximum m.m.f. of each of the magnetizing coils.

In a symmetrical quarter-phase system, $n = 4$, $F = 2\, \mathfrak{F}_{max}$, where $\mathfrak{F}_{max}$ is the maximum m.m.f. of each of the four magnetizing coils, or, if only two coils are used, since the four-phase m.m.fs. are opposite in phase by two, $F = \mathfrak{F}_{max}$, where $\mathfrak{F}_{max}$ is the maximum m.m.f. of each of the two magnetizing coils of the quarter-phase system.

While the quarter-phase system, consisting of two e.m.fs. displaced by one-quarter of a period, is by its nature an unsymmetrical system, it shares a number of features — as, for instance, the ability of producing a constant-resultant m.m.f. — with the symmetrical system, and may be considered as one-half of a symmetrical four-phase system.

Such systems, consisting of one-half of a symmetrical system, are called *hemisymmetrical systems.*

CHAPTER XXXIV.

BALANCED AND UNBALANCED POLYPHASE SYSTEMS.

370. If an alternating e.m.f.,

$$e = E\sqrt{2} \sin \beta,$$

produces a current,

$$i = I\sqrt{2} \sin (\beta - \theta),$$

where θ is the angle of lag, the power is

$$\begin{aligned} p = ei &= 2\,EI \sin \beta \sin (\beta - \theta) \\ &= EI (\cos \theta - \cos (2\,\beta - \theta)), \end{aligned}$$

and the average value of power,

$$P = EI \cos \theta.$$

Substituting this, the instantaneous value of power is found as

$$p = P \left(1 - \frac{\cos (2\,\beta - \theta)}{\cos \theta}\right).$$

Hence the power, or the flow of energy, in an ordinary single-phase, alternating-current circuit is fluctuating, and varies with twice the frequency of e.m.f. and current, unlike the power of a continuous-current circuit, which is constant,

$$p = ei.$$

If the angle of lag, $\theta = 0$, it is,

$$p = P (1 - \cos 2\,\beta);$$

hence the flow of energy varies between zero and $2\,P$, where P is the average flow of energy or the effective power of the circuit.

If the current lags or leads the e.m.f. by angle θ, the power varies between

$$P\left(1 - \frac{1}{\cos\theta}\right) \text{ and } P\left(1 + \frac{1}{\cos\theta}\right),$$

that is, becomes negative for a certain part of each half-wave. That is, for a time during each half-wave, energy flows back into the generator, while during the other part of the half-wave the generator sends out energy, and the difference between both is the effective power of the circuit.

If $\theta = 90°$, it is

$$p = - EI \sin 2\beta;$$

that is, the effective power $P = 0$, and the energy flows to and fro between generator and receiving circuit.

Under any circumstances, however, the flow of energy in the single-phase system is fluctuating, at least between zero and a maximum value, frequently even reversing.

371. If in a polyphase system

$e_1, e_2, e_3. \ldots$ = instantaneous values of e.m.f.;

$i_1, i_2, i_3, \ldots$ = instantaneous values of current produced thereby,

the total power in the system is

$$p = e_1 i_1 + e_2 i_2 + e_3 i_3 + \ldots.$$

The average power is

$$P = E_1 I_1 \cos\theta_1 + E_2 I_2 \cos\theta_2 + \ldots.$$

The polyphase system is called a balanced system, if the flow of energy

$$p = e_1 i_1 + e_2 i_2 + e_3 i_3 + \ldots$$

is constant, and it is called an unbalanced system if the flow of energy varies periodically, as in the single-phase system; and the ratio of the minimum value to the maximum value of power is called the *balance-factor of the system.*

Hence in a single-phase system on non-inductive circuit, that is, at no-phase displacement, the balance-factor is zero; and it is negative in a single-phase system with lagging or leading current, and becomes equal to -1 if the phase displacement is 90° — that is, the circuit is wattless.

372. Obviously, in a polyphase system the balance of the system is a function of the distribution of load between the different branch circuits.

A balanced system in particular is called a polyphase system, whose flow of energy is constant, if all the circuits are loaded equally with a load of the same character, that is, the same phase displacement.

373. All the symmetrical systems from the three-phase system upward are balanced systems. Many unsymmetrical systems are balanced systems also.

(1) Three-phase system:

Let

$$e_1 = E\sqrt{2}\sin\beta, \qquad \text{and } i_1 = I\sqrt{2}\sin(\beta - \theta),$$
$$e_2 = E\sqrt{2}\sin(\beta - 120), \qquad i_2 = I\sqrt{2}\sin(\beta - \theta - 120),$$
$$e_3 = E\sqrt{2}\sin(\beta - 240), \qquad i_3 = I\sqrt{2}\sin(\beta - \theta - 240),$$

be the e.m.fs. of a three-phase system and the currents produced thereby.

Then the total power is

$$p = 2EI\{\sin\beta\sin(\beta-\theta) + \sin(\beta-120)\sin(\beta-\theta-120) + \sin(\beta-240)\sin(\beta-\theta-240)\}$$
$$= 3EI\cos\theta = P, \text{ or constant.}$$

Hence the symmetrical three-phase system is a balanced system.

(2) Quarter-phase system:

$$\text{Let } e_1 = E\sqrt{2}\sin\beta, \qquad i_1 = I\sqrt{2}\sin(\beta-\theta),$$
$$e_2 = E\sqrt{2}\cos\beta, \qquad i_2 = I\sqrt{2}\cos(\beta-\theta)$$

be the e.m.fs. of the quarter-phase system, and the currents produced thereby.

This is an unsymmetrical system, but the instantaneous value of power is

$$\begin{aligned} p &= 2\,EI\,\{\sin\beta\sin(\beta-\theta)+\cos\beta\cos(\beta-\theta)\} \\ &= 2\,EI\cos\theta = P\text{, or constant.}\end{aligned}$$

Hence the quarter-phase system is an unsymmetrical balanced system.

(3) The symmetrical n-phase system, with equal load and equal phase-displacement in all n branches, is a balanced system. For, let

$$e_i = E\sqrt{2}\sin\left(\beta-\frac{2\pi i}{n}\right) = \text{e.m.f.};$$

$$i_i = I\sqrt{2}\sin\left(\beta-\theta-\frac{2\pi i}{n}\right) = \text{current};$$

the instantaneous value of power is

$$\begin{aligned} p &= \sum_1^n{}^i\, e_i i_i \\ &= 2\,EI\sum_1^n{}^i \sin\left(\beta-\frac{2\pi i}{n}\right)\sin\left(\beta-\theta-\frac{2\pi i}{n}\right) \\ &= EI\left\{\sum_1^n{}^i \cos\theta - \sum_1^n{}^i \cos\left(2\beta-\theta-\frac{4\pi i}{n}\right)\right\};\end{aligned}$$

or $\quad p = nEI\cos\theta = P$, or constant.

374. An unbalanced polyphase system is the so-called inverted three-phase system,* derived from two branches of a three-phase system by transformation by means of two transformers, whose secondaries are connected in opposite direction with respect to their primaries. Such a system takes an intermediate position between the Edison three-wire system and the three-phase system. It shares with the latter the polyphase feature, and with the Edison three-wire system the feature that the potential difference

* Also called "polyphase monocyclic system," since the e.m.f. triangle is similar to that usual in the single-phase monocyclic system.

between the outside wires is higher than between middle wire and outside wire.

By such a pair of transformers the two primary e.m.fs. of 120° displacement of phase are transformed into two secondary e.m.fs., differing from each other by 60°. Thus in the secondary circuit

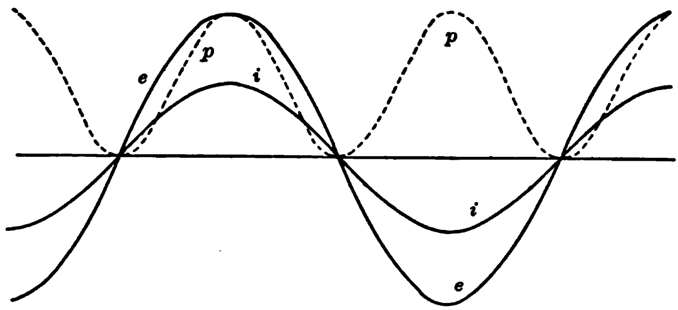


Fig. 237.—Single-phase, Non-inductive Circuit.

the difference of potential between the outside wires is $\sqrt{3}$ times the difference of potential between middle wire and outside wire. At equal load on the two branches, the three currents are equal, and differ from each other by 120°, that is, have the same relative proportion as in a three-phase system. If the load on one

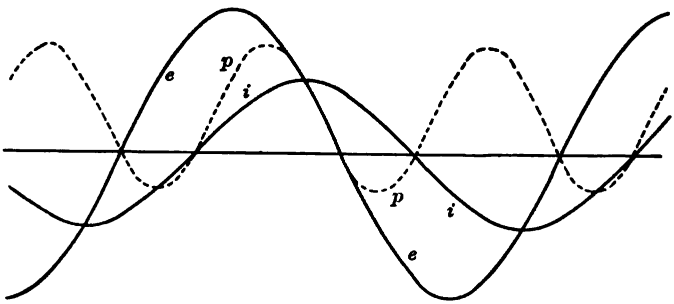


Fig. 238.—Single-phase, 60° Lag.

branch is maintained constant, while the load of the other branch is reduced from equality with that in the first branch down to zero, the current in the middle wire first decreases, reaches a minimum value of $\frac{\sqrt{3}}{2} = .866$ of its original value, and then increases again, reaching at no-load the same value as at full load.

The balance factor of the inverted three-phase system on non-inductive load is 0.333.

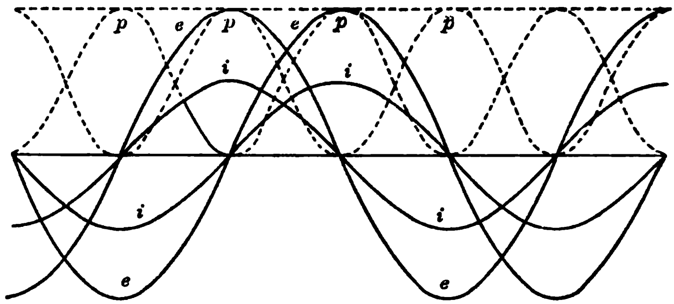


FIG. 239. — Quarter-phase, Non-inductive Circuit.

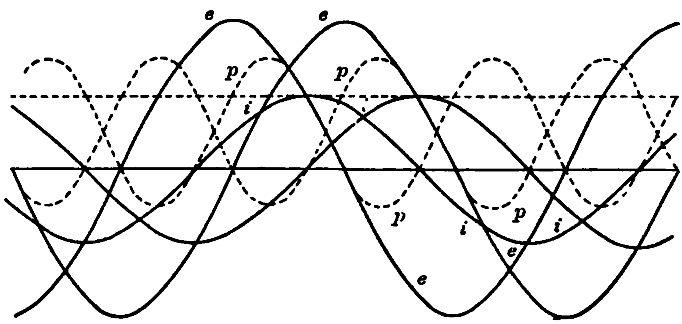


FIG. 240. — Quarter-phase, 60° Lag.

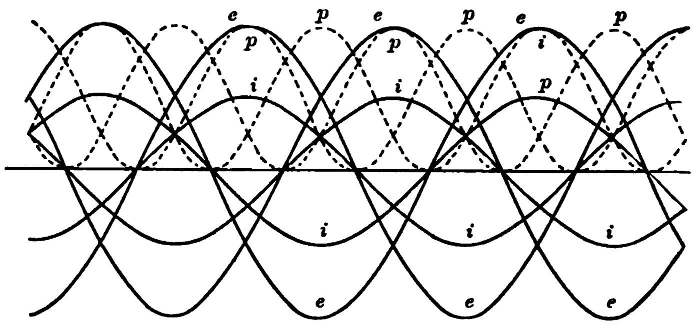


FIG. 241. — Three-phase, Non-inductive Circuit.

375. In Figs. 237 to 244 are shown the e.m.fs., as e and currents as i in full lines, and the power as p in dotted lines, for balance-factor, 0; balance-factor, -0.333; balance-factor, $+1$; balance-

factor, + 1; balance-factor, + 1; balance-factor, + 1; balance-factor, + 0.333, and balance-factor, 0.

376. The flow of energy in an alternating-current system is a most important and characteristic feature of the system, and by its nature the systems may be classified into:

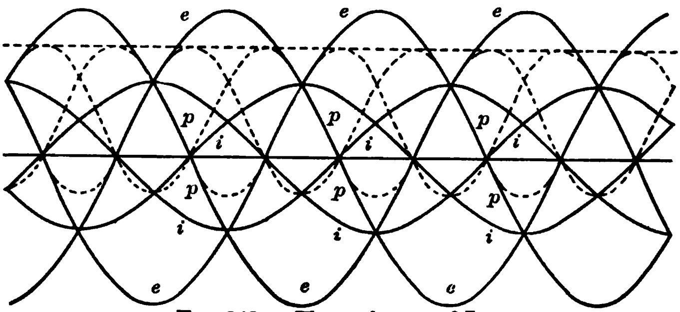


Fig. 242. — Three-phase, 60° Lag.

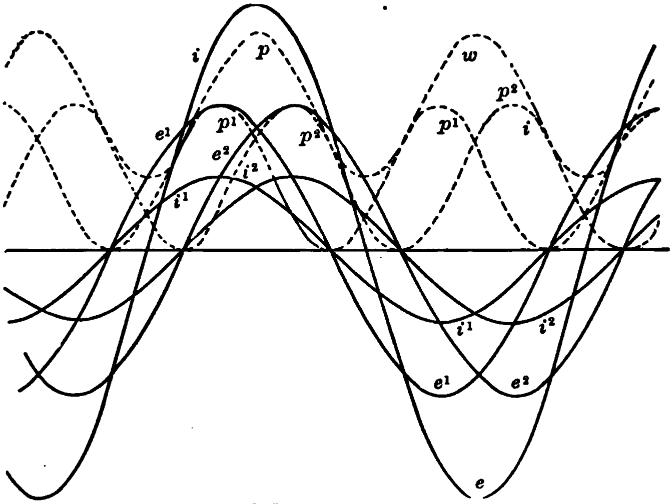


Fig. 243. — Inverted Three-phase, Non-inductive Circuit.

Monocyclic systems, or systems with a balance-factor zero or negative.

Polycyclic systems, with a positive balance-factor.

Balance-factor − 1 corresponds to a wattless single-phase circuit, balance-factor zero to a non-inductive single-phase circuit, balance-factor + 1 to a balanced polyphase system.

377. In polar coordinates the flow of energy of an alternating current system is represented by using the instantaneous value of power as radius vector, with the angle, β, corresponding to the time as amplitude, one complete period being represented by one revolution.

In this way the power of an alternating-current system is represented by a closed symmetrical curve, having the zero point as quadruple point. In the monocyclic systems the zero point is

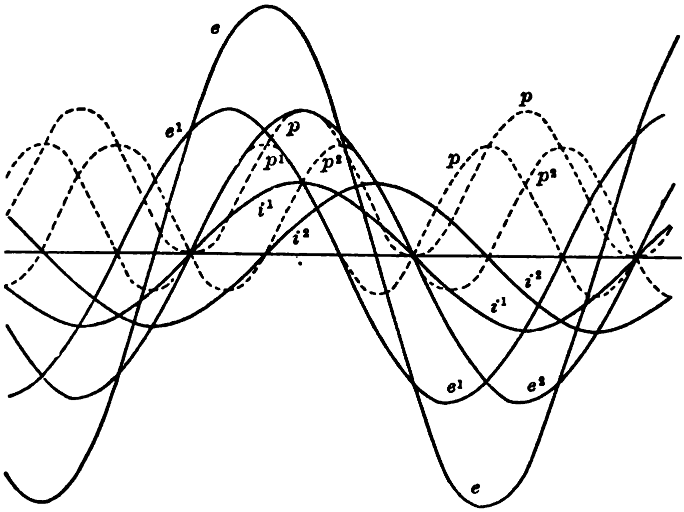


FIG. 244. — Inverted Three-phase, 60° Lag.

quadruple nodal point; in the polycyclic systems quadruple isolated point.

Thus these curves are sextics.

Since the flow of energy in any single-phase branch of the alternating-current system can be represented by a sine wave of double frequency,

$$p = P\left(1 - \frac{\sin(2\beta - \theta)}{\cos\theta}\right),$$

the total flow of energy of the system as derived by the addition of the powers of the branch circuits can be represented in the form

$$p = P(1 + \varepsilon \sin(2\beta - \theta_0)).$$

This is a wave of double frequency also, with ϵ as amplitude of fluctuation of power.

This is the equation of the power characteristics of the system in polar coordinates.

378. To derive the equation in rectangular coordinates we introduce a substitution which revolves the system of coordinates by an angle, $\frac{\theta_0}{2}$, so as to make the symmetry axes of the power characteristic the coordinate axes.

$$p = \sqrt{x^2 + y^2},$$

$$\tan\left(\beta - \frac{\theta_0}{2}\right) = \frac{y}{x};$$

hence,

$$\sin(2\beta - \theta_0) = 2 \sin\left(\beta - \frac{\theta_0}{2}\right)\cos\left(\beta - \frac{\theta_0}{2}\right) = \frac{2xy}{x^2 + y^2},$$

substituted,

$$\sqrt{x^2 + y^2} = P\left\{1 + \frac{2\epsilon xy}{x^2 + y^2}\right\},$$

or, expanded,

$$(x^2 + y^2)^3 - P^2(x^2 + y^2 + 2\epsilon xy)^2 = 0,$$

the sextic equation of the power characteristic.

Introducing

$$a = (1 + \epsilon)P = \text{maximum value of power},$$

$$b = (1 - \epsilon)P = \text{minimum value of power};$$

we have

$$P = \frac{a+b}{2},$$

$$\epsilon = \frac{a-b}{a+b};$$

hence, substituted, and expanded,

$$(x^2 + y^2)^3 - \tfrac{1}{4}\{a(x + y)^2 + b(x - y)^2\}^2 = 0,$$

the equation of the power characteristic, with the *main power axes*, a and b, and the balance-factor, $\frac{b}{a}$.

It is thus:

Single-phase, non-inductive circuit, $p = P\,(1 + \sin 2\,\theta)$, $b = 0$, $a = 2\,P$,

$$(x^2 + y^2)^3 - P^2\,(x + y)^4 = 0, \quad \frac{b}{a} = 0.$$

Single-phase circuit, 60° lag: $p = P\,(1 + 2 \sin 2\,\theta)$, $b = -P$, $a = +3\,P$,

$$(x^2 + y^2)^3 - P^2\,(x^2 + y^2 + 4\,xy)^2 = 0, \quad \frac{b}{a} = -\tfrac{1}{3}.$$

Single-phase circuit, 90° lag: $p = EI \sin 2\,\theta$,

$$b = -EI, \qquad a = +EI,$$

$$(x^2 + y^2)^3 - 4\,(EI)^2\,x^2y^2, \quad \frac{b}{a} = -1.$$

Three-phase non-inductive circuit, $p = P$, $b = 1$, $a = 1$,

$$x^2 + y^2 - P^2 = 0, \text{ circle.} \quad \frac{b}{a} = +1.$$

Three-phase circuit, 60° lag, $p = P$, $b = 1$, $a = 1$,

$$x^2 + y^2 - P^2 = 0, \text{ circle.} \quad \frac{b}{a} = +1.$$

Quarter-phase non-inductive circuit, $p = P$, $b = 1$, $a = 1$,

$$x^2 + y^2 - P^2 = 0, \text{ circle.} \quad \frac{b}{a} = +1.$$

Quarter-phase circuit, 60° lag, $p = P$, $b = 1$, $a = 1$,

$$x^2 + y^2 - P^2 = 0, \text{ circle.} \quad \frac{b}{a} = +1.$$

Inverted three-phase non-inductive circuit,

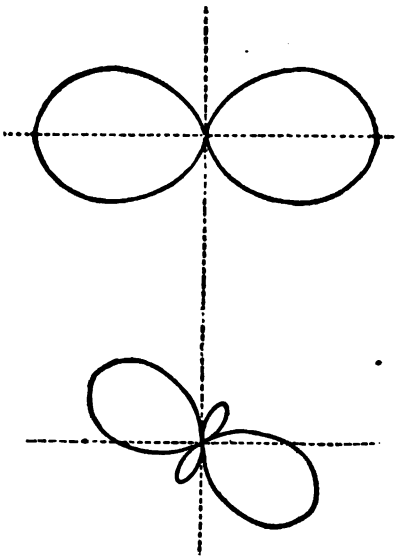
$$p = P\left(1 + \frac{\sin 2\theta}{2}\right), \quad b = \tfrac{1}{2} P, \quad a = \tfrac{3}{2} P,$$

$$(x^2 + y\ )^3 - P^2 (x^2 + y^2 + xy)^2 = 0. \quad \frac{b}{a} = + \tfrac{1}{3}.$$

Inverted three-phase circuit 60° lag, $p = P(1 + \sin 2\theta)$, $b = 0$, $a = 2P$,

$$(x^2 + y^2)^3 - P^2 (x + y)^4 = 0. \quad \frac{b}{a} = 0.$$

a and b are called the main power axes of the alternating-current system, and the ratio, $\frac{b}{a}$, is the balance-factor of the system.

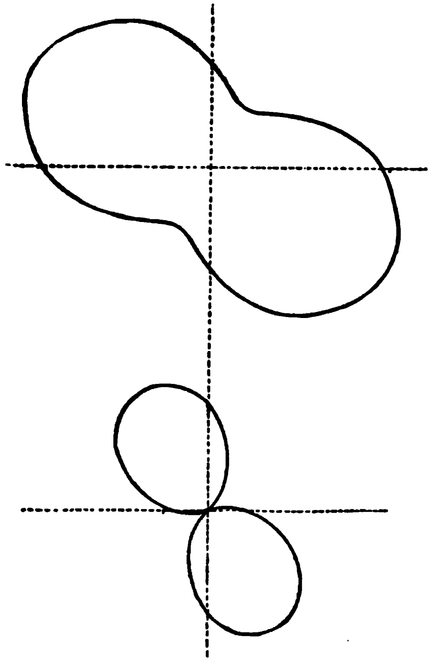


Figs. 245 and 246. — Power Characteristic of Single-phase System, at 0° and 60° Lag.

379. As seen, the flow of energy of an alternating-current system is completely characterized by its two main power axes, a and b.

The power characteristics in polar coordinates, corresponding

to the Figs. 237, 238, 243 and 244 are shown in Figs. 245, 246, 247 and 248.



FIGS. 247 and 248. Power Characteristic of Inverted Three-phase System, at 0° and 60° Lag.

The balanced quarter-phase and three-phase systems give as polar characteristics concentric circles.

CHAPTER XXXV.

INTERLINKED POLYPHASE SYSTEMS.

380. In a polyphase system the different circuits of displaced phases, which constitute the system, may either be entirely separate and without electrical connection with each other, or they may be connected with each other electrically, so that a part of the electrical conductors are in common to the different phases, and in this case the system is called an interlinked polyphase system.

Thus, for instance, the quarter-phase system will be called an independent system if the two e.m.fs. in quadrature with each other are produced by two entirely separate coils of the same, or different, but rigidly connected, armatures, and are connected to four wires which energize independent circuits in motors or other receiving devices. If the quarter-phase system is derived by connecting four equidistant points of a closed-circuit drum or ring-wound armature to the four collector rings, the system is an interlinked quarter-phase system.

Similarly in a three-phase system. Since each of the three currents which differ from each other by one-third of a period is equal to the resultant of the other two currents, it can be considered as the return circuit of the other two currents, and an interlinked three-phase system thus consists of three wires conveying currents differing by one-third of a period from each other, so that each of the three currents is a common return of the other two, and inversely.

381. In an interlinked polyphase system two ways exist of connecting apparatus into the system.

1. The *star connection*, represented diagrammatically in Fig. 249. In this connection the n circuits, excited by currents differing from each other by $\frac{1}{n}$ of a period, are connected with their one end together into a neutral point or common con-

nection, which may either be grounded, or connected with other corresponding neutral points, or insulated.

In a three-phase system this connection is usually called a Y connection, from a similarity of its diagrammatical representation with the letter Y, as shown in Fig. 233.

2. The *ring connection*, represented diagrammatically in Fig. 250, where the n circuits of the apparatus are connected with each other in closed circuit, and the corners or points of connection of adjacent circuits connected to the n lines of the polyphase system. In a three-phase system this connection is

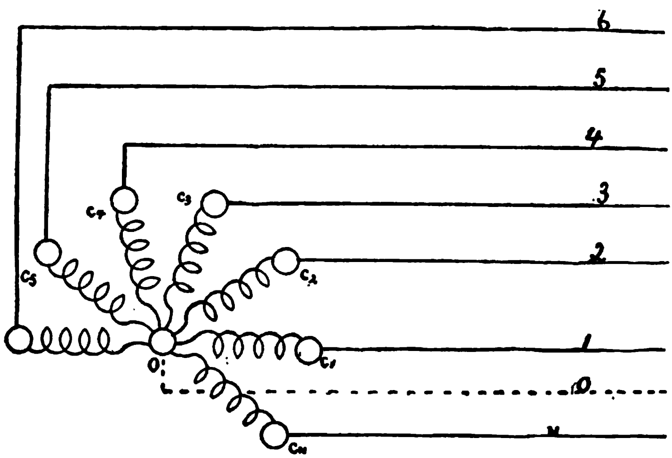


FIG. 249.

called the delta (Δ) connection, from the similarity of its diagrammatic representation with the Greek letter delta, as shown in Fig. 233.

In consequence hereof we distinguish between star-connected and ring-connected generators, motors, etc., or in three-phase systems Y-connected and Δ-connected apparatus.

382. Obviously, the polyphase system as a whole does not differ, whether star connection or ring connection is used in the generators or other apparatus; and the transmission line of a symmetrical n-phase system always consists of n wires carrying current of equal strength, when balanced, differing from each

other in phase by $\frac{1}{n}$ of a period. Since the line wires radiate from the n terminals of the generator, the lines can be considered as being in star connection.

The circuits of all the apparatus, generators, motors, etc., can either be connected in star connection, that is, between one line and a neutral point, or in ring connection, that is, between two adjacent lines.

In general some of the apparatus will be arranged in star

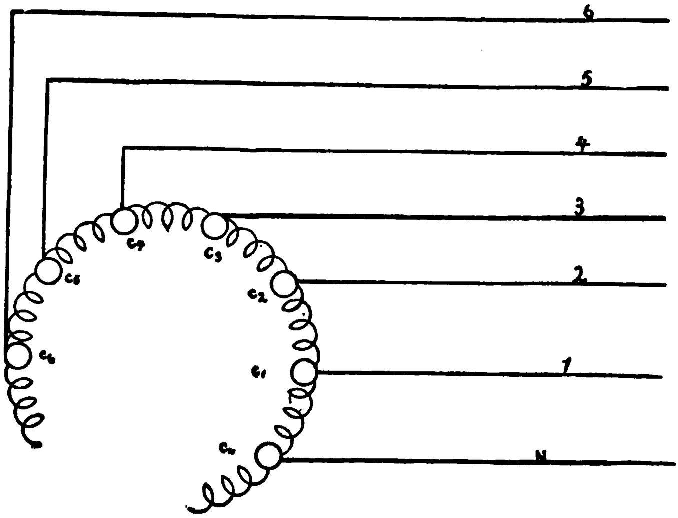


FIG. 250.

connection, some in ring connection, as the occasion may require.

383. In the same way as we speak of star connection and ring connection of the circuits of the apparatus, the terms star potential and ring potential, star current and ring current, etc., are used, whereby as star potential or in a three-phase circuit Y potential, the potential difference between one of the lines and the neutral point, that is, a point having the same difference of potential against all the lines, is understood; that is, the potential as measured by a voltmeter connected into star or Y

connection. By ring or delta potential is understood the difference of potential between adjacent lines, as measured by a voltmeter connected between adjacent lines, in ring or delta connection.

In the same way the star or Y current is the current in a circuit from one line to a neutral point; the ring or delta current, the current in a circuit from one line to the next line.

The current in the transmission line is always the star or Y current, and the potential difference between the line wires, the ring or delta potential.

Since the star potential and the ring potential differ from each other, apparatus requiring different voltages can be connected into the same polyphase mains, by using either star or ring connection.

384. If in a generator with star-connected circuits, the e.m.f. per circuit $= E$, and the common connection or neutral point is denoted by zero, the potentials of the n terminals are

$$\underset{\cdot}{E},\ \epsilon \underset{\cdot}{E},\ \epsilon^2 \underset{\cdot}{E} \ \ldots\ldots\ \epsilon^{n-1} \underset{\cdot}{E};$$

or in general,

$$\epsilon^i \underset{\cdot}{E},$$

at the i^{th} terminal, where,

$$i = 0, 1, 2 \ \ldots\ldots\ n-1,\ \epsilon = \cos\frac{2\pi}{n} + j\sin\frac{2\pi}{n} = \sqrt[n]{1}.$$

Hence the e.m.f. in the circuit from the i^{th} to the k^{th} terminal is

$$\underset{\cdot}{E}_{ki} = \epsilon^k \underset{\cdot}{E} - \epsilon^i \underset{\cdot}{E} = (\epsilon^k - \epsilon^i)\underset{\cdot}{E}.$$

The e.m.f. between adjacent terminals i and $i+1$ is

$$(\epsilon^{i+1} - \epsilon^i)\underset{\cdot}{E} = \epsilon^i(\epsilon - 1)\underset{\cdot}{E}.$$

In a generator with ring-connected circuits, the e.m.f. per circuit

$$\epsilon^i \underset{\cdot}{E},$$

is the ring e.m.f., and takes the place of

$$\epsilon^i(\epsilon - 1)\underset{\cdot}{E};$$

while the e.m.f. between terminal and neutral point, or the star e.m.f., is

$$\frac{\epsilon^i}{\epsilon - 1} \dot{E}.$$

Hence in a star-connected generator with the e.m.f. $\dot{E}$ per circuit, it is:

star e.m.f., $\epsilon^i \dot{E}$,

ring e.m.f., $\epsilon^i (\epsilon - 1) \dot{E}$,

e.m.f. between terminal i and terminal k, $(\epsilon^k - \epsilon^i) \dot{E}$.

In a ring-connected generator with the e.m.f., $\dot{E}$, per circuit, it is

star e.m.f., $\frac{\epsilon^i}{\epsilon - 1} \dot{E}$,

ring e.m.f., $\epsilon^i \dot{E}$,

e.m.f. between terminals i and k, $\frac{\epsilon^k - \epsilon^i}{\epsilon - 1} \dot{E}$.

In a star-connected apparatus, the e.m.f. and the current per circuit have to be the star e.m.f. and the star current. In a ring-connected apparatus the e.m.f. and current per circuit have to be the ring e.m.f. and ring current.

In the generator of a symmetrical polyphase system, if

$\epsilon^i \dot{E}$ are the e.m.fs. between the n terminals and the neutral point, or star e.m.fs.

$\dot{I}_i =$ the currents issuing from terminals i over a line of the impedance, Z_i (including generator impedance in star connection), we have

potential at end of line i,

$$\epsilon^i \dot{E} - Z_i \dot{I}_i,$$

and difference of potential between terminals k and i

$$(\epsilon^k - \epsilon^i) \dot{E} - (Z_k \dot{I}_k - Z_i \dot{I}_i),$$

where $\dot{I}_i$ is the star current of the system, Z_i the star impedance.

The ring potential at the end of the line between terminals i and k is $\dot{E}_{ik}$, and

$$\dot{E}_{ik} = - \dot{E}_{ki}.$$

If now $\dot{I}_{ik}$ denotes the current from terminal i to terminal k, and Z_{ik} impedance of the circuit between terminal i and terminal k, where

$$\dot{I}_{ik} = -\dot{I}_{ki},$$

$$Z_{ik} = Z_{ki},$$

we have

$$\dot{E}_{ik} = Z_{ik}\dot{I}_{ik}.$$

If $\dot{I}_{io}$ denotes the current in the circuit from terminal i to a ground or neutral point, and Z_{io} is the impedance of this circuit between terminal i and neutral point, it is

$$\dot{E}_{io} = \epsilon^i\dot{E} - Z_i\dot{I}_i = Z_{io}\dot{I}_{io}.$$

385. We have thus, by Ohm's law and Kirchhoff's law:

If $\epsilon^i\dot{E}$ is the e.m.f. per circuit of the generator, between the terminal, i, and the neutral point of the generator, or the star e.m.f.

$\dot{I}_i$ = the current at the terminal, i, of the generator, or the star current.

Z_i = the impedance of the line connected to a terminal, i, of the generator, including generator impedance.

$\dot{E}_i$ = the e.m.f. at the end of line connected to a terminal, i, of the generator.

$\dot{E}_{ik}$ = the difference of potential between the ends of the lines, i and k.

$\dot{I}_{ik}$ = the current from line i to line k.

Z_{ik} = the impedance of the circuit between lines i and k.

$\dot{I}_{io}, \dot{I}_{ioo} \ldots$ = the current from line i to neutral points 0, 00, ...

$Z_{io}, Z_{ioo} \ldots$ = the impedance of the circuits between line i and neutral points 0, 00, ...

Then:

(1) $\dot{E}_{ik} = -\dot{E}_{ki}, \quad \dot{I}_{ik} = -\dot{I}_{ki} \quad Z_{ik} = Z_{ki}, \quad \dot{I}_{io} = -\dot{I}_{oi},$
$Z_{io} = Z_{oi}$, etc.

(2) $\dot{E}_i = \epsilon^i\dot{E} - Z_i\dot{I}_i.$

(3) $\dot{E}_i = Z_{io}\dot{I}_{io} = Z_{ioo}\dot{I}_{ioo} = \ldots.$

(4) $\dot{E}_{ik} = \dot{E}_k - \dot{E}_i = (\epsilon^k - \epsilon^i)\dot{E} - (Z_k\dot{I}_k - Z_i\dot{I}_i).$

(5) $\dot{E}_{ik} = Z_{ik}\dot{I}_{ik}.$

(6) $\dot{I}_i = \sum_0^n{}^k \dot{I}_{ik}.$

(7) If the neutral point of the generator does not exist, as in ring connection, or is insulated from the other neutral points:

$$\sum_{1}^{n}{}^{i}\, \dot{I}_{i} = 0;$$

$$\sum_{1}^{n}{}^{i}\, \dot{I}_{io} = 0;$$

$$\sum_{1}^{n}{}^{i}\, \dot{I}_{ioo} = 0, \text{ etc.}$$

Where 0, 00, etc., are the different neutral points which are insulated from each other.

If the neutral point of the generator and all the other neutral points are grounded or connected with each other, we have,

$$\begin{aligned} \sum_{1}^{n}{}^{i}\, \dot{I}_{i} &= \sum_{1}^{n}{}^{i}\, (\dot{I}_{io} + \dot{I}_{ioo} + \dots) \\ &= \sum_{1}^{n}{}^{i}\, \dot{I}_{io} + \sum_{1}^{n}{}^{i}\, \dot{I}_{ioo} + \dots \end{aligned}$$

If the neutral point of the generator or other neutral points are grounded, the system is called a grounded system. If the neutral points are not grounded, the system is an insulated polyphase system, and an insulated polyphase system with equalizing return, if all the neutral points are connected with each other.

(8) The power of the polyphase system is

$$P = \sum_{1}^{n}{}^{i}\, \epsilon^{i} E I_{i} \cos \theta_{i} \text{ at the generator,}$$

$$P = \sum_{0}^{n}{}^{i} \sum_{i}^{n}{}^{k}\, E_{ik} I_{ik} \cos \theta_{ik} \text{ in the receiving circuits.}$$

CHAPTER XXXVI.

TRANSFORMATION OF POLYPHASE SYSTEMS.

386. In transforming one polyphase system into another polyphase system, it is obvious that the primary system must have the same flow of energy as the secondary system, neglecting losses in transformation, and that consequently a balanced system will be transformed again into a balanced system, and an unbalanced system into an unbalanced system of the same balance-factor, since the transformer is not able to store energy, and thereby to change the nature of the flow of energy. The energy stored as magnetism amounts in a well-designed transformer only to a very small percentage of the total energy. This shows the futility of producing symmetrical balanced polyphase systems by transformation from the unbalanced single-phase system without additional apparatus able to store energy efficiently, as revolving machinery, etc.

Since any e.m.f. can be resolved into, or produced by, two components of given directions, the e.m.f. of any polyphase system can be resolved into components or produced from components of two given directions. This enables the transformation of any polyphase system into any other polyphase system of the same balance-factor by two transformers only.

387. Let $E_1, E_2, E_3 \ldots$ be the e.m.fs. of the primary system which shall be transformed into

$$E_1', E_2', E_3' \ldots \text{ the e.m.fs. of the secondary system.}$$

Choosing two magnetic fluxes, $\bar{\Phi}$ and $\bar{\bar{\Phi}}$, of different phases, as magnetic circuits of the two transformers, which generate the e.m.fs., $\bar{e}$ and $\bar{\bar{e}}$, per turn, by the law of parallelogram the e.m.fs., $E_1, E_2, \ldots$ can be resolved into two components, $\bar{E}_1$ and $\bar{\bar{E}}_1$, $\bar{E}_2$ and $\bar{\bar{E}}_2$, $\ldots$ of the phases, $\bar{e}$ and $\bar{\bar{e}}$.

Then

$\bar{E}_1, \bar{E}_2, \ldots$ are the counter e.m.fs. which have to be generated in the primary circuits of the first transformer;

$\bar{\bar{E}}_1, \bar{\bar{E}}_2, \ldots$ the counter e.m.fs. which have to be generated in the primary circuits of the second transformer.

Hence

$\frac{\bar{E}_1}{\bar{e}}, \frac{\bar{E}_2}{\bar{e}} \ldots$ are the numbers of turns of the primary coils of the first transformer.

Analogously

$\frac{\bar{\bar{E}}_1}{\bar{\bar{e}}}, \frac{\bar{\bar{E}}_2}{\bar{\bar{e}}} \ldots$ are the number of turns of the primary coils in the second transformer.

In the same manner as the e.m.fs. of the primary system have been resolved into components in phase with $\bar{e}$ and $\bar{\bar{e}}$, the e.m.fs. of the secondary system, $E_1', E_2', \ldots$ are produced from components, $\overline{E_1'}$ and $\overline{\overline{E_1'}}$, $\overline{E_2'}$ and $\overline{\overline{E_2'}} \ldots$ in phase with $\bar{e}$ and $\bar{\bar{e}}$, and give as numbers of secondary turns, —

$$\frac{\overline{E_1'}}{\bar{e}}, \frac{\overline{E_2'}}{\bar{e}} \ldots \text{ in the first transformer;}$$

$$\frac{\overline{\overline{E_1'}}}{\bar{\bar{e}}}, \frac{\overline{\overline{E_2'}}}{\bar{\bar{e}}} \ldots \text{ in the second transformer.}$$

That means each of the two transformers, $\bar{m}$ and $\bar{\bar{m}}$, contains in general primary turns of each of the primary phases, and secondary turns of each of the secondary phases. Loading now the secondary polyphase system in any desired manner, corresponding to the secondary currents, primary currents will exist in such a manner that the total flow of energy in the primary polyphase system is the same as the total flow of energy in the secondary system, plus the loss of power in the transformers.

388. As an instance may be considered the transformation of the symmetrical balanced three-phase system,

$$E \sin \beta, \quad E \sin (\beta - 120), \quad E \sin (\beta - 240),$$

in an unsymmetrical balanced quarter-phase system,

$$E' \sin \beta, \quad E' \sin (\beta - 90).$$

Let the magnetic flux of the two transformers be chosen in quadrature

$$\Phi \cos \beta \text{ and } \Phi \cos (\beta - 90).$$

Then the e.m.fs. generated per turn in the transformers are

$$e \sin \beta \text{ and } e \sin (\beta - 90);$$

hence, in the primary circuit the first phase, $E \sin \beta$, will give, in the first transformer, $\frac{E}{e}$ primary turns; in the second transformer, 0 primary turns.

The second phase, $E \sin (\beta - 120)$, will give, in the first transformer, $\frac{-E}{2e}$ primary turns; in the second transformer, $\frac{E \times \sqrt{3}}{2e}$ primary turns.

The third phase, $E \sin (\beta - 240)$, will give, in the first transformer, $\frac{-E}{2e}$ primary turns; in the second transformer, $\frac{-E \times \sqrt{3}}{2e}$ primary turns.

In the secondary circuit the first phase, $E' \sin \beta$, will give in the first transformer: $\frac{E'}{e}$ secondary turns; in the second transformer: 0 secondary turns.

The second phase: $E' \sin (\beta - 90)$ will give in the first transformer: 0 secondary turns; in the second transformer, $\frac{E'}{e}$ secondary turns.

Or, if

$$E = 5000, \quad E' = 100, \quad e = 10.$$

	PRIMARY.			SECONDARY.		
	1st.	2d.	3d.	1st.	2d.	Phase.
First transformer	+ 500	− 250	− 250	10	0	
Second transformer	0	+ 433	− 433	0	10	turns.

That means:

Any balanced polyphase system can be transformed by two transformers only, without storage of energy, into any other balanced polyphase system.

Or more generally stated:

Any polyphase system can be transformed by two transformers only, without storage of energy, into any other polyphase system of the same balance factor.

389. Some of the more common methods of transformation between polyphase systems are:

1. The *delta-Y connection* of transformers between three-

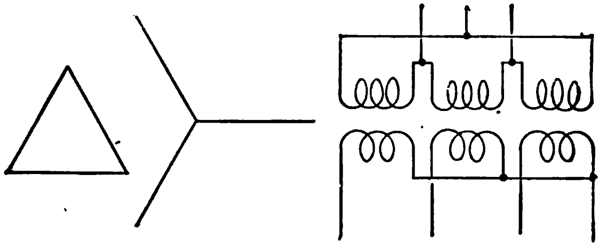


Fig. 251.

phase systems, shown in Fig. 251. One side of the transformers is connected in delta, the other in Y. This arrangement becomes necessary for feeding four-wire three-phase secondary distributions. The Y connection of the secondary allows the bringing out of a neutral wire, while the delta connection of the primary

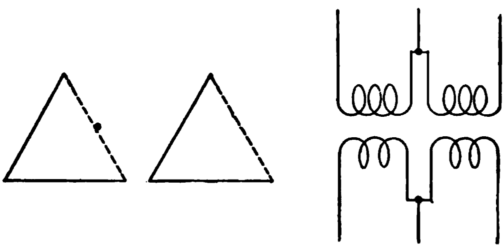


Fig. 252.

maintains the balance, in regard to the voltage between the phases at unequal distribution of load.

The delta-Y connection of step-up transformers is frequently

used in long-distance transmissions, to allow grounding of the high-potential neutral. Under certain conditions—which therefore have to be guarded against — it is liable to induce excessive voltages by resonance with the line capacity.

The reverse thereof, or the *Y-delta connection,* is in general not permissible, since it gives what has been called a "floating neutral;" the three primary *Y* voltages do not remain even approximately constant, at unequal distribution of load on the secondary delta, but the primary voltage corresponding to the heavier loaded secondary, and, therefore, also the corresponding secondary voltage, collapses. Thereby the common connection of the primary shifts towards one corner of the e.m.f. triangle, away from the center of the triangle, and may even fall outside of the triangle. As result thereof the secondary triangle becomes very greatly distorted even at moderate inequality of load, and the system thus loses all ability to maintain constant voltage at unequal distribution of load, that is, becomes inoperative. In high-potential systems in this case excessive voltages may be induced by resonance with the line capacity.

For instance, if only one phase of the secondary triangle is loaded, the other two unloaded, the primary current of the loaded phase must return over the other two transformers, which, at open secondaries, act as very high reactances, thus limiting the current and consuming practically all the voltage, and the loaded primary, and thus its secondary, receive practically no voltage.

Y-delta connection is feasible only if the secondary load is balanced, as induction — or synchronous motors, or if the primary neutral is connected with the generator neutral or the secondary neutral of step-up transformers in which the primaries are connected in delta, and the unbalanced current can return over the neutral. If with *Y*-delta connection, in addition to an unbalanced load, the secondary carries polyphase motors, the motors take different currents in the different phases, so that the total current is approximately the same in all three phases. That is, the motors act as phase converters, and so partially restore the balance of the system.

2. The *delta-delta connection* of transformers between three-phase systems, in which primaries as well as secondaries are connected in the same manner as the primaries in Fig. 251.

Since in this system each phase is transformed by a separate transformer, the voltages of the system remain balanced even at unbalanced load, within the limits of voltage variation due to the internal self-inductive impedance (or short-circuit impedance) of the transformers — which is small, while the exciting impedance (or open-circuit impedance) of the transformers, which causes the unbalancing in the Y-delta connection above discussed is enormous.

3. $Y - Y$ *connection* of transformers between three-phase systems. Primaries and secondaries connected as the secondaries in Fig. 251.

In this case, if the neutral is not fixed by connection with a fixed neutral, either directly or by grounding it, the neutral also is floating, and so abnormal voltages may be produced between the lines and the neutral, without appearing in the voltages between the lines, and may lead to disruptive effects, or to overheating of the transformers, so that it is not safe to use this connection without fixing the neutral.

Where in transformer connections in polyphase systems, a neutral or common connection of the transformers exists, care must, therefore, be taken to have this neutral a fixed voltage point, irrespective of the variation of the load or its distribution, which may occur; otherwise harmful phenomena may result from a "floating" or "unstable" neutral.

In connections (2) and (3), the secondary-e.m.f. triangle is in phase with the primary-e.m.f. triangle, while in (1) it is displaced therefrom by 30°. Therefore, even if the voltages are equal, connection (1) cannot be operated in parallel with (2) or (3), but (2) and (3) can be operated in parallel with each other, and with the connections (4) and (5), provided that the voltages are correct.

4. The *V connection* or *open delta connection* of transformers between three-phase systems, consists in using two sides of the triangle only, as shown in Fig. 252. This arrangement has the disadvantage of transforming one phase by two transformers in series, hence is less efficient, and is liable to unbalance the system by the internal impedance of the transformers. It is convenient for small powers at moderate voltage, since it requires only two transformers, but is dangerous in high potential circuits, being liable to produce destructive voltages by its electrostatic unbalancing.

5. The *main and teaser*, or *T connection* of transformers between three-phase systems, is shown in Fig. 253. One of the two transformers is wound for $\frac{\sqrt{3}}{2}$ times the voltage of the other (the altitude of the equilateral triangle), and connected with one of its ends to the center of the other transformer. From the point one-third inside of the teaser transformer, a neutral wire can be brought out in this connection.

6. The *monocyclic connection*, transforming between three-phase and inverted three-phase or polyphase monocyclic, by

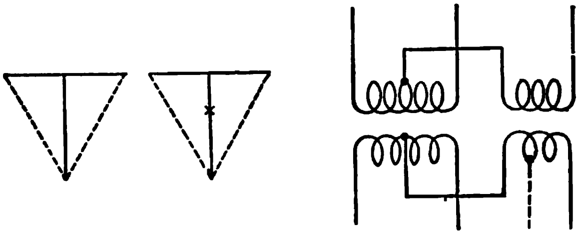


FIG. 253.

two transformers, the secondary of one being reversed regarding its primary, as shown in Fig. 254.

7. The *L connection* for transformation between quarter-phase and three-phase as described in the example, paragraph 357.

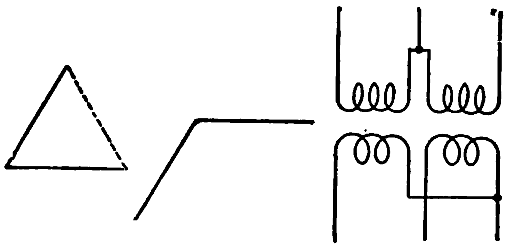


FIG. 254.

8. The *T connection* of transformation between quarter-phase and three-phase, as shown in Fig. 255. The quarter-phase sides of the transformers contain two equal and independent (or interlinked) coils, the three-phase sides two coils with the ratio of turns, $1 \div \frac{\sqrt{3}}{2}$, connected in *T*.

9. The *double delta connection* of transformation from three-phase to six-phase, shown in Fig. 256. Three transformers, with two secondary coils each, are used, one set of secondary coils connected in delta, the other set in delta also, but with reversed terminals, so as to give a reversed e.m.f. triangle. These e.m.fs. thus give topographically a six-cornered star.

10. The *double Y connection* or *diametrical connection* of transformation from three-phase to six-phase, shown in Fig. 257.

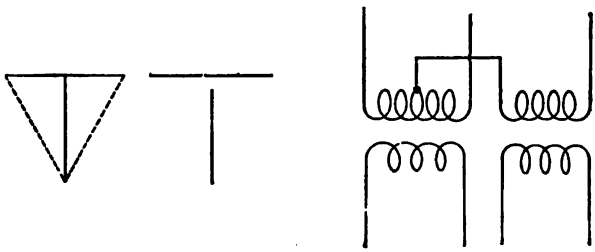


FIG. 255.

It is analogous to (7), the delta connection merely being replaced by the Y connection. The neutrals of the two Y's may be connected together and to an external neutral if desired.

The primaries in 9 and 10 may be connected either delta or

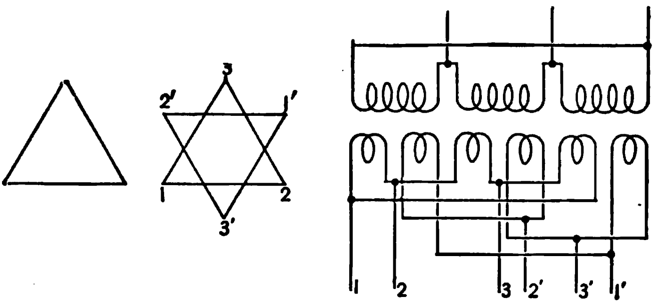


FIG. 256.

Y, and in the latter case a floating neutral must be guarded against.

11. The *double T connection* of transformation from three-

phase to six-phase, shown in Fig. 257. Two transformers are used with two secondary coils which are *T*-connected, but one with reversed terminals. This method also allows a secondary neutral to be brought out.

390. Transformation with a change of the balance-factor of the system is possible only by means of apparatus able to store energy, since the difference of energy between primary and

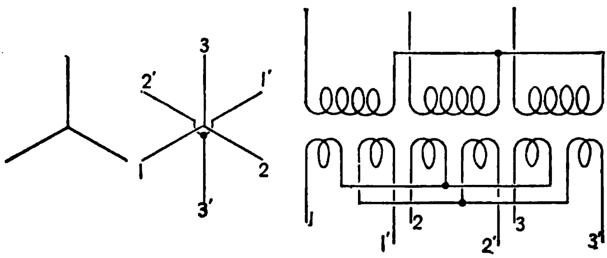


FIG. 257.

secondary circuit has to be stored at the time when the secondary power is below the primary, and returned during the time when the primary power is below the secondary. The

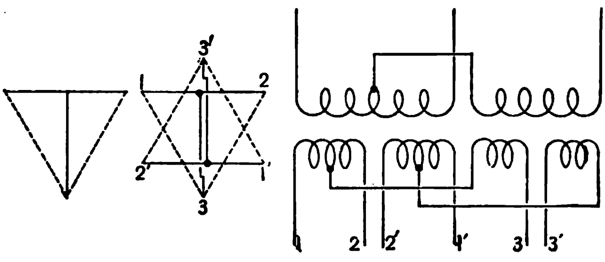


FIG. 258.

most efficient storing device of electric energy is mechanical momentum in revolving machinery. It has, however, the disadvantage of requiring attendance; fairly efficient also are condensive and inductive reactances, but, as a rule, they have the disadvantage of not giving constant potential.

CHAPTER XXXVII.

EFFICIENCY OF SYSTEMS.

391. In electric power transmission and distribution, wherever the place of consumption of the electric energy is distant from the place of production, the conductors which carry the current are a sufficiently large item to require consideration, when deciding which system and what potential is to be used.

In general, in transmitting a given amount of power at a given loss over a given distance, other things being equal, the amount of copper required in the conductors is inversely proportional to the square of the potential used. Since the total power transmitted is proportional to the product of current and e.m.f., at a given power, the current will vary inversely proportionally to the e.m.f., and therefore, since the loss is proportional to the product of current-square and resistance, to give the same loss the resistance must vary inversely proportional to the square of the current, that is, proportional to the square of the e.m.f.; and since the amount of copper is inversely proportional to the resistance, other things being equal, the amount of copper varies inversely proportional to the square of the e.m.f. used.

This holds for any system.

Therefore to compare the different systems, as two-wire single-phase, single-phase three-wire, three-phase and quarter-phase, equality of the potential must be assumed.

Some systems, however, as, for instance, the Edison three-wire system, or the inverted three-phase system, have different potentials in the different circuits constituting the system, and thus the comparison can be made either —

1st. On the basis of the maximum potential difference between any two conductors of the system; or

2nd. On the basis of the maximum potential difference between any conductor of the system and the ground; or

3rd. On the basis of the minimum potential difference in the system, or the potential difference per circuit or phase of the system.

In low-potential circuits, as secondary networks, where the potential is not limited by the insulation strain, but by the potential of the apparatus connected into the system, as incandescent lamps, the proper basis of comparison is equality of the potential per branch of the system, or per phase.

On the other hand, in long-distance transmissions where the potential is not restricted by any consideration of apparatus suitable for a certain maximum potential only, but where the limitation of potential depends upon the problem of insulating the conductors against disruptive discharge, the proper comparison is on the basis of equality of the maximum difference of potential; that is, equal maximum dielectric strain on the insulation.

In this case, the comparison voltage may be either the potential difference between any two conductors of the system, or it may be the potential difference between any conductor of the system and the ground, depending on the character of the circuit.

The dielectric stress is from conductor to conductor, or between any two conductors, in a system which is insulated from the ground, as is mostly the case in medium voltage overhead transmissions, and frequently in underground cables.

In an ungrounded cable system, in which all the conductors are enclosed in the same cable, the insulation stress is mainly from conductor to conductor, and this therefore is the basis of comparison. But even in an underground cable system with grounded neutral, as very commonly used, a direct path exists from conductor to conductor inside of the cables, for a disruptive voltage, and the comparison of systems, therefore, has to be made, in this case, on the basis of maximum potential difference between conductors as well as between conductor and ground.

In an ungrounded overhead system, the disruptive stress is from conductor to ground and back from ground to conductor. If the system is of considerable extent — as is the case where high voltages of serious disruptive strength have to be considered — the neutral of the system is maintained at approxi-

mate ground potential by the capacity of the system, and the normal voltage stress from conductor to ground therefore is that from conductor to neutral, that is, the same as in a system with grounded neutral, and the basis of comparison then would appear to be the voltage from line to ground, and not between lines. Since, however, one conductor of the system may temporarily ground, if it is required to maintain operation even with one conductor of the system grounded, the voltage between conductors must be the basis of comparison, since with one conductor grounded, the disruptive stress between the other conductors and ground is the potential difference between the conductors of the system.

In an overhead system with grounded neutral, frequently used for transmission systems of very high voltage, or in general in a grounded system, the disruptive stress is that due to the potential difference between conductor and ground or neutral, and this then is the basis of comparison.

In moderate-potential power circuits, in considering the danger to life from live wires entering buildings or otherwise accessible, the comparison on the basis of maximum potential also appears appropriate.

Thus the comparison of different systems of long-distance transmission at high potential or power distribution for motors is to be made on the basis of equality of the maximum difference of potential existing in the system; the comparison of low-potential distribution circuits for lighting on the basis of equality of the minimum difference of potential between any pair of wires connected to the receiving apparatus.

392. 1st. *Comparison on the basis of equality of the minimum difference of potential, in low-potential lighting circuits:*

In the single-phase, alternating-current circuit, if e = e.m.f., i = current, r = resistance per line, the total power is $= ei$, the loss of power, $2\,i^2r$.

Using, however, a three-wire system: the potential between outside wires and neutral being given equal to e, the potential between the outside wires is equal to $2\,e$, that is, the distribution takes place at twice the potential, or only one-fourth the copper is needed to transmit the same power at the same loss, if, as it is theoretically possible, the neutral wire has no cross-section.

If, however, the neutral wire is made of the same cross-section as each of the outside wires, three-eighths as much copper as in the two-wire system is needed; if the neutral wire is one-half the cross-section of each of the outside wires, five-sixteenths as much copper is needed. Obviously, a single-phase, five-wire system will be a system of distribution at the potential, $4\,e$, and therefore require only one-sixteenth of the copper of the single-phase system in the outside wires; and if each of the three neutral wires is of one-half the cross-section of the outside wires, seven-sixty-fourths or 10.93 per cent of the copper.

Coming now to the three-phase system with the potential, e, between the lines as delta potential, if $i =$ the current per line or Y current, the current from line to line or delta current $= \frac{i_1}{\sqrt{3}}$; and since three branches are used, the total power is $\frac{3\,e i_1}{\sqrt{3}} = e i_1 \sqrt{3}$.

Hence if the same power has to be transmitted by the three-phase system as with the single-phase system, the three-phase line current must be $i_1 = \frac{i}{\sqrt{3}}$; where $i =$ single-phase current, $r =$ single-phase resistance per line, at equal power and loss: hence if $r_1 =$ resistance of each of the three wires, the loss per wire is $i_1^2 r_1 = \frac{i^2 r_1}{3}$, and the total loss is $i^2 r_1$, while in the single-phase system it is $2\,i^2 r$. Hence, to get the same loss, it must be: $r_1 = 2\,r$, that is, each of the three three-phase lines has twice the resistance — that is, half the copper of each of the two single-phase lines; or in other words, the three-phase system requires three-fourths as much copper as the single-phase system of the same potential.

Introducing, however, a fourth or neutral wire into the three-phase system, and connecting the lamps between the neutral wire and the three outside wires — that is, in Y connection — the potential between the outside wires or delta potential will be $= e \times \sqrt{3}$, since the Y potential $= e$, and the potential of the system is raised thereby from e to $e\sqrt{3}$; that is, only one-third as much copper is required in the outside wires as before — that is one-fourth as much copper as in the single-phase two-wire

system. Making the neutral of the same cross-section as the outside wires, requires one-third more copper, or one-third = 33.3 per cent of the copper of the single-phase system; making the neutral of half cross-section, requires one-sixth more, or $\frac{7}{24}$ = 29.17 per cent of the copper of the single-phase system. The system, however, now is a four-wire system.

The independent quarter-phase system with four wires is identical in efficiency to the two-wire, single-phase system, since it is nothing but two independent single-phase systems in quadrature.

The four-wire, quarter-phase system can be used as two independent Edison three-wire systems also, deriving therefrom the same saving by doubling the potential between the outside wires, and has in this case the advantage, that by interlinkage, the same neutral wire can be used for both phases, and thus one of the neutral wires saved.

In this case the quarter-phase system with common neutral of full cross-section requires $\frac{5}{16}$ or 31.25 per cent, the quarter-phase system with common neutral of one-half cross-section requires $\frac{9}{32}$ or 28.125 per cent of the copper of the two-wire, single-phase system.

In this case, however, the system is a five-wire system, and as such far inferior to the five-wire, single-phase system.

Coming now to the quarter-phase system with common return and potential e per branch, denoting the current in the outside wires by i_2, the current in the central wire is $i_2\sqrt{2}$; and if the same current density is chosen for all three wires, as the condition of maximum efficiency, and the resistance of each outside wire denoted by r_2, the resistance of the central wire $= \frac{r_2}{\sqrt{2}}$, and the loss of power per outside wire is $i_2^2 r_2$, in the central wire $\frac{2\, i_2^2 r_2}{\sqrt{2}}$ $= i_2^2 r_2 \sqrt{2}$; hence the total loss of power is $2\, i_2^2 r_2 + i_2^2 r_2 \sqrt{2}$ $= i_2^2 r_2 \,(2 + \sqrt{2})$. The power transmitted per branch is $i_2 e$, hence the total power, $2\, i_2 e$. To transmit the same power as by a single-phase system of power, ei, it must be $i_2 = \frac{i}{2}$; hence the loss, $\frac{i^2 r_2\,(2 + \sqrt{2})}{4}$. Since this loss shall be the same as the loss,

$2\,i^2r$, in the single-phase system, it must be $2\,r = \dfrac{(2+\sqrt{2})}{4}\,r_2$, or $r_2 = \dfrac{8\,r}{2+\sqrt{2}}$. Therefore each of the outside wires must be $\dfrac{2+\sqrt{2}}{8}$ times as large as each single-phase wire, the central wire $\sqrt{2}$ times larger; hence the copper required for the quarter-phase system with common return bears to the copper required for the single-phase system the relation,

$$\frac{2\,(2+\sqrt{2})}{8} + \frac{(2+\sqrt{2})\,\sqrt{2}}{8} \div 2, \quad \text{or,} \quad \frac{3+2\sqrt{2}}{8} \div 1, \; = 72.9$$

per cent of the copper of the single-phase system.

Hence the quarter-phase system with common return saves 2 per cent more copper than the three-phase system, but is inferior to the single-phase three-wire system.

The inverted three-phase system, consisting of two e.m.fs. e at 60° displacement, and three equal currents i_3 in the three lines of equal resistance r_3, gives the output $2\,ei_3$, that is, compared with the single-phase system, $i_3 = \dfrac{i}{2}$. The loss in the three lines is $3\,i_3^2r_3 = \frac{3}{4}\,i^2r_3$. Hence, to give the same loss, $2\,i^2r$, as the single-phase system, it must be $r_3 = \frac{8}{3}\,r$, that is, each of the three wires must have three-eighths of the copper cross-section of the wire in the two-wire single-phase system; or in other words, the inverted three-phase system requires nine-sixteenths of the copper of the two-wire single-phase system.

Thus if a given power has to be transmitted at a given loss, and a given *minimum* potential, as for instance 110 volts for lighting, the amount of copper necessary is:

2 WIRES:	Single-phase system,	100.0
3 WIRES:	Edison three-wire single-phase system, neutral full section,	37.5
	Edison three-wire single-phase system, neutral half-section,	31.25
	Inverted three-phase system,	56.25
	Quarter-phase system with common return,	72.9
	Three-phase system,	75.0

4 Wires:	Three-phase, with neutral-wire full section,	33.3
	Three-phase, with neutral-wire half-section,	29.17
	Independent quarter-phase system,	100.0
5 Wires:	Edison five-wire, single-phase system, full neutral,	15.625
	Edison five-wire, single-phase system, half-neutral,	10.93
	Four-wire, quarter-phase, with common-neutral full section,	31.25
	Four-wire, quarter-phase, with common-neutral half-section,	28.125

We see herefrom, that in distribution for lighting — that is. with the same minimum potential, and with the same number of wires — the single-phase system is superior to any polyphase system.

The continuous-current system is equivalent in this comparison to the single-phase alternating-current system of the same effective potential, since the comparison is made on the basis of effective potential, and the power depends upon the effective potential also.

393. *Comparison on the Basis of Equality of the Maximum Difference of Potential between any two Conductors of the System, in Long-Distance Transmission, Power Distribution, etc.*

Wherever the potential is so high as to bring the question of the strain on the insulation into consideration, or in other cases, to approach the danger limit to life, the proper comparison of different systems is on the basis of equality of maximum potential in the system.

Hence in this case, since the maximum potential is fixed, nothing is gained by three- or five-wire, Edison systems. Thus, such systems do not come into consideration.

The comparison of the three-phase system with the single-phase system remains the same, since the three-phase system has the same maximum as minimum potential; that is:

The three-phase system requires three-fourths of the copper

of the single-phase system to transmit the same power at the same loss over the same distance.

The four-wire, quarter-phase system requires the same amount of copper as the single-phase system, since it consists of two single-phase systems.

In a quarter-phase system with common return, the potential between the outside wires is $\sqrt{2}$ times the potential per branch, hence to get the same maximum strain on the insulation — that is, the same potential, e, between the outside wires as in the single-phase system — the potential per branch will be $\frac{e}{\sqrt{2}}$, hence the current $i_4 = \frac{i}{\sqrt{2}}$, if i equals the current of the single-phase system of equal power, and $i_4 \sqrt{2} = i$ will be the current in the central wire.

Hence, if r_4 = resistance per outside wire, $\frac{r_4}{\sqrt{2}}$ = resistance of central wire, and the total loss in the system is

$$2\, i_4^2 r_4 + \frac{i_4^2\, 2\, r_4}{\sqrt{2}} = i_4^2 r_4\, (2 + \sqrt{2}) = i^2 r_4\, \frac{(2 + \sqrt{2})}{2}.$$

Since in the single-phase system, the loss $= 2\, i^2 r$, it is

$$r_4 = \frac{4\, r}{2 + \sqrt{2}}.$$

That is, each of the outside wires has to contain $\frac{2 + \sqrt{2}}{4}$ times as much copper as each of the single-phase wires. The central wires have to contain $\frac{2 + \sqrt{2}}{4} \sqrt{2}$ times as much copper; hence the total system contains $\frac{2\,(2 + \sqrt{2})}{4} + \frac{2 + \sqrt{2}}{4} \sqrt{2}$ times as much copper as each of the single-phase wires; that is, $\frac{3 + 2\sqrt{2}}{4}$ times the copper of the single-phase system.

Or, in other words,

A quarter-phase system with common return requires $\frac{3 + 2\sqrt{2}}{4} = 1.457$ times as much copper as a single-phase

system of the same maximum potential, same power, and same loss.

Since the comparison is made on the basis of equal maximum potential, and the maximum potential of an alternating system is $\sqrt{2}$ times that of a continuous-current circuit of equal effective potential, the alternating circuit of effective potential, e, compares with the continuous-current circuit of potential $e\sqrt{2}$, which latter requires only half the copper of the alternating system.

This comparison of the alternating with the continuous-current system is not proper however, since the continuous-current potential may introduce, besides the electrostatic strain, an electrolytic strain on the dielectric which does not exist in the alternating system, and thus may make the action of the continuous-current potential on the insulation more severe than that of an equal alternating potential. Besides, self-induction having no effect on a steady current, continuous-current circuits as a rule have a self-induction far in excess of any alternating circuit. During changes of current, as make and break, and changes of load, especially rapid changes, there are consequently generated in these circuits e.m.fs. far exceeding their normal potentials. At the voltages which came under consideration, the continuous current is usually excluded to begin with.

Thus we get:

If a given power is to be transmitted at a given loss, and a given *maximum* difference of potential in the system, that is, with the same strain on the insulation, the amount of copper required is:

2 Wires:	Single-phase system,	100.0
	[Continuous-current system,	50.0]
3 Wires:	Three-phase system,	75.0
	Quarter-phase system, with common return,	145.7
4 Wires:	Independent Quarter-phase system,	100.0

Hence the quarter-phase system with common return is practically excluded from long-distance transmission.

394. In a different way the same comparative results between single-phase, three-phase, and quarter-phase systems can be derived by resolving the systems into their single-phase branches.

The three-phase system of e.m.f., e, between the lines can be considered as consisting of three single-phase circuits of e.m.f., $\frac{e}{\sqrt{3}}$, and no return; the single-phase system of e.m.f., e, between lines as consisting of two single-phase circuits of e.m.f., $\frac{e}{2}$, and no return. Thus, the relative amount of copper in the two systems being inversely proportional to the square of e.m.f., bears the relation $\left(\frac{\sqrt{3}}{e}\right)^2 : \left(\frac{2}{e}\right)^2 = 3:4$; that is, the three-phase system requires 75 per cent of the copper of the single-phase system.

The quarter-phase system with four equal wires requires the same copper as the single-phase system, since it consists of two single-phase circuits. Replacing two of the four quarter-phase wires by one wire of the same cross-section as each of the wires replaced thereby, the current in this wire is $\sqrt{2}$ times as large as in the other wires, hence, the loss is twice as large — that is, the same as in the two wires replaced by this common wire, or the total loss is not changed — while 25 per cent of the copper is saved, and the system requires only 75 per cent of the copper of the single-phase system, but produces $\sqrt{2}$ times as high a potential between the outside wires. Hence, to give the same maximum potential, the e.m.fs. of the system have to be reduced by $\sqrt{2}$, that is, the amount of copper doubled, and thus the quarter-phase system with common return of the same cross-section as the outside wires requires 150 per cent of the copper of the single-phase system. In this case, however, the current density in the middle wire is higher, thus the copper not used most economically, and transferring a part of the copper from the outside wires to the middle wire, to bring all three wires to the same current density, reduces the loss, and thereby reduces the amount of copper at a given loss, to 145.7 per cent of that of a single-phase system.

395. *Comparison on the basis of equality of the maximum difference of potential between any conductor of the system and the ground, in long-distance, three-phase transmissions with grounded neutral, single-phase systems with ground return, etc.*

A system may be grounded by grounding its neutral point, for the purpose of maintaining constant-potential difference between the conductors and ground, without carrying any current through the ground, or the ground may be used as return conductor. In either case the system can be considered as consisting of and resolved into as many single-phase systems with ground return, as there are overhead conductors, and with zero resistance in the ground.

It immediately follows herefrom, that the copper efficiency of such a system is the same as that of a single-phase system with ground return, of the same voltage as exists between conductor and ground of the system under consideration. If then all the overhead conductors have the same potential difference against ground, as is the case in a three-phase or quarter-phase system with grounded neutral, a single-phase system with grounded neutral, or quarter-phase system with common ground return of both phases, the copper efficiency is the same. That is:

All grounded systems, whether with grounded neutral or with ground return, have the same copper efficiency, provided that all the overhead conductors have the same potential difference against ground.

Hence:

The three-phase system with grounded neutral has no superiority over the single-phase or the quarter-phase system with grounded neutral, in copper efficiency. The advantage of the three-phase system — which causes its practically universal use — over the single-phase system is the greater usefulness of polyphase power, the advantage over the quarter-phase system is the use of three conductors, against four with the quarter-phase system.

No saving in copper results from the use of the ground (of zero resistance) as return circuit, but a single-phase or quarter-phase system with ground return, at equal dielectric strain on the insulation, requires the same amount of copper as a system with grounded neutral, but has a greater self-induction, due to the greater distance between conductor and return conductor

or ground, and has the objection of establishing current through the ground and so disturbing neighboring circuits, by electromagnetic and electrostatic induction.

The apparent saving in copper, in the single-phase system, by replacing one of the conductors by the ground as return, therefore is a fallacy. By doing so, the potential difference of the other conductors against ground becomes twice what it would be with two conductors and grounded neutral, and at the same potential difference between conductors. That is, the single-phase system with ground return requires the same insulation as a single-phase system with grounded neutral, of twice the voltage, and then requires the same copper. A saving results only in the number of insulators required, etc. Only where the amount of power is so small that mechanical strength, and not power loss, determines the size of the conductor, a saving results by replacing one of the conductors by the ground.

The high-tension, direct-current system, whether insulated, or with grounded neutral, or with ground return, appears equal in copper efficiency to a single-phase system of the same character (insulated, or with grounded neutral, or with ground return) and of the same effective voltage, that is, with a sine wave of a maximum voltage $\sqrt{2}$ times that of the direct current. Due to the different character of unidirectional electric stress of the direct-current system, from the alternating stress, a general comparison of the system by a numerical factor appears hardly feasible.

CHAPTER XXXVIII.

THREE-PHASE SYSTEM.

396. With equal load of the same phase displacement in all three branches, the symmetrical three-phase system offers no special features over those of three equally loaded single-phase systems, and can be treated as such; since the mutual reactions between the three phases balance at equal distribution of load, that is, since each phase is acted upon by the preceding phase in an equal but opposite manner as by the following phase.

With unequal distribution of load between the different branches, the voltages and phase differences become more or less unequal. These unbalancing effects are obviously maximum if some of the phases are fully loaded, others unloaded.

Let $\dot{E}$ = e.m.f. between branches 1 and 2 of a three-phaser. Then

$$\varepsilon \dot{E} = \text{e.m.f. between 2 and 3,}$$

$$\varepsilon^2 \dot{E} = \text{e.m.f. between 3 and 1;}$$

where
$$\varepsilon = \sqrt[3]{1} = \frac{-1 + j\sqrt{3}}{2}.$$

Let

Z_1, Z_2, Z_3 = impedances of the lines issuing from generator terminals 1, 2, 3,

and Y_1, Y_2, Y_3 = admittances of the consumer circuits connected between lines 2 and 3, 3 and 1, 1 and 2.

If then,

$\dot{I}_1, \dot{I}_2, \dot{I}_3$, are the currents issuing from the generator terminals into the lines, it is,

$$\dot{I}_1 + \dot{I}_2 + \dot{I}_3 = 0. \qquad (1)$$

If, I_1', I_2', I_3' = currents through the admittances, Y_1, Y_2, Y_3, from 2 to 3, 3 to 1, 1 to 2, it is,

$$\left.\begin{aligned} I_1 &= I_3' - I_2', \text{ or, } I_1 + I_2' - I_3' = 0 \\ I_2 &= I_1' - I_3', \text{ or, } I_2 + I_3' - I_1' = 0 \\ I_3 &= I_2' - I_1', \text{ or, } I_3 + I_1' - I_2' = 0 \end{aligned}\right\} \qquad (2)$$

These three equations (2) added, give (1) as dependent equation.

At the ends of the lines 1, 2, 3, it is:

$$\left.\begin{aligned} E_1' &= E_1 - Z_2 I_2 + Z_3 I_3 \\ E_2' &= E_2 - Z_3 I_3 + Z_1 I_1 \\ E_3' &= E_3 - Z_1 I_1 + Z_2 I_2 \end{aligned}\right\} \qquad (3)$$

the differences of potential, and:

$$\left.\begin{aligned} I_1' &= E_1' Y_1 \\ I_2' &= E_2' Y_2 \\ I_3' &= E_3' Y_3 \end{aligned}\right\} \qquad (4)$$

the currents in the receiver circuits.

These nine equations (2), (3), (4), determine the nine quantities: I_1, I_2, I_3, I_1', I_2', I_3', E_1', E_2', E_3'.

Equations (4) substituted in (2) give:

$$\left.\begin{aligned} I_1 &= E_3' Y_3 - E_2' Y_2 \\ I_2 &= E_1' Y_1 - E_3' Y_3 \\ I_3 &= E_2' Y_2 - E_1' Y_1 \end{aligned}\right\} \qquad (5)$$

These equations (5) substituted in (3), and transposed, give:

since $\left.\begin{aligned} E_1 &= \epsilon E \\ E_2 &= \epsilon^2 E \\ E_3 &= E \end{aligned}\right\}$ as e.m.fs. at the generator terminals.

$$\left.\begin{aligned} \epsilon E - E_1'(1 + Y_1 Z_2 + Y_1 Z_3) + E_2' Y_2 Z_3 + E_3' Y_3 Z_2 &= 0 \\ \epsilon^2 E - E_2'(1 + Y_2 Z_3 + Y_2 Z_1) + E_3' Y_3 Z_1 + E_1' Y_1 Z_3 &= 0 \\ E - E_3'(1 + Y_3 Z_1 + Y_3 Z_2) + E_1' Y_1 Z_2 + E_2' Y_2 Z_1 &= 0 \end{aligned}\right\} \qquad (6)$$

as three linear equations with the three quantities, E_1', E_2', E_3'.

Substituting the abbreviations:

$$\left.\begin{aligned}
K &= \begin{bmatrix} -(1+Y_1Z_2+Y_1Z_3), & Y_2Z_3, & Y_3Z_2 \\ Y_1Z_3, & -(1+Y_2Z_3+Y_2Z_1), & Y_3Z_1 \\ Y_1Z_2, & Y_2Z_1, & -(1+Y_3Z_1+Y_3Z_2) \end{bmatrix} \\
K_1 &= \begin{bmatrix} \varepsilon, & Y_2Z_3, & Y_3Z_2 \\ \varepsilon^2, & -(1+Y_2Z_3+Y_2Z_1), & Y_3Z_1 \\ 1, & Y_2Z_1, & -(1+Y_3Z_1+Y_3Z_2) \end{bmatrix} \\
K_2 &= \begin{bmatrix} -(1+Y_1Z_2+Y_1Z_3), & \varepsilon, & Y_3Z_2 \\ Y_1Z_3, & \varepsilon^2, & Y_3Z_1 \\ Y_1Z_2, & 1, & -(1+Y_3Z_1+Y_3Z_2) \end{bmatrix} \\
K_3 &= \begin{bmatrix} -(1+Y_1Z_2+Y_1Z_3), & Y_2Z_3, & \varepsilon \\ Y_1Z_3, & -(1+Y_2Z_3+Y_2Z_1), & \varepsilon^2 \\ Y_1Z_2, & Y_2Z_1, & 1 \end{bmatrix}
\end{aligned}\right\} \qquad (7)$$

we have:

$$\left.\begin{aligned}
E_1' &= \frac{EK_1}{K} \\
E_2' &= \frac{EK_2}{K} \\
E_3' &= \frac{EK_3}{K}
\end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned}
I_1' &= \frac{Y_1EK_1}{K} \\
I_2' &= \frac{Y_2EK_2}{K} \\
I_3' &= \frac{Y_3EK_3}{K}
\end{aligned}\right\} \qquad (9)$$

$$\left.\begin{aligned}
I_1 &= \frac{Y_3K_3 - Y_2K_2E}{K} \\
I_2 &= \frac{Y_1K_1 - Y_3K_3E}{K} \\
I_3 &= \frac{Y_2K_2 - Y_1K_1E}{K}
\end{aligned}\right\} \qquad (10)$$

hence,
$$\left.\begin{aligned} \dot{E}_1' + \dot{E}_2' + \dot{E}_3' &= 0 \\ I_1 + I_2 + I_3 &= 0 \end{aligned}\right\} \qquad (11)$$

397. Special Cases.

A. *Balanced System.*

$$Y_1 = Y_2 = Y_3 = Y$$
$$Z_1 = Z_2 = Z_3 = Z.$$

Substituting this in (6), and transposing:

$$\left.\begin{aligned}
&\left.\begin{aligned} \dot{E}_1 &= \epsilon \dot{E} \\ \dot{E}_2 &= \epsilon^2 \dot{E} \\ \dot{E}_3 &= \dot{E} \end{aligned}\right\} \qquad
\begin{aligned} \dot{E}_1' &= \frac{\epsilon \dot{E}}{1 + 3\,YZ} \\ \dot{E}_2' &= \frac{\epsilon^2 \dot{E}}{1 + 3\,YZ} \\ \dot{E}_3' &= \frac{\dot{E}}{1 + 3\,YZ} \end{aligned} \\
&\left.\begin{aligned} \dot{I}_1' &= \frac{\epsilon \dot{E} Y}{1 + 3\,YZ} \\ \dot{I}_2' &= \frac{\epsilon^2 \dot{E} Y}{1 + 3\,YZ} \\ \dot{I}_3' &= \frac{\dot{E} Y}{1 + 3\,YZ} \end{aligned}\right\} \qquad
\begin{aligned} \dot{I}_1 &= \frac{\epsilon^2 (\epsilon - 1)\, \dot{E} Y}{1 + 3\,YZ} \\ \dot{I}_2 &= \frac{(\epsilon - 1)\, \dot{E} Y}{1 + 3\,YZ} \\ \dot{I}_3 &= \frac{\epsilon (\epsilon - 1)\, \dot{E} Y}{1 + 3\,YZ} \end{aligned}
\end{aligned}\right\} \qquad (12)$$

The equations of the symmetrical balanced three-phase system.

B. *One circuit loaded, two unloaded.*

$$Y_1 = Y_2 = 0, \quad Y_3 = Y,$$
$$Z_1 = Z_2 = Z_3 = Z.$$

Substituted in equations (6):

$$\left.\begin{aligned} \epsilon \dot{E} - \dot{E}_1' + \dot{E}_3' YZ &= 0 \\ \epsilon^2 \dot{E} - \dot{E}_2' + \dot{E}_3' YZ &= 0 \end{aligned}\right\} \text{ unloaded branches.}$$
$$\dot{E} - \dot{E}_3' (1 + 2\,YZ) = 0, \text{ loaded branch.}$$

hence:

$$\left.\begin{aligned}
&\left.\begin{aligned} \dot{E}_1' &= \frac{\dot{E}\,\{\epsilon + (1 + 2\,\epsilon)\,YZ\}}{1 + 2\,YZ} \\ \dot{E}_2' &= \frac{\dot{E}\,\{\epsilon^2 + (1 + 2\,\epsilon^2)\,YZ\}}{1 + 2\,YZ} \end{aligned}\right\} \text{ unloaded;} \\
&\dot{E}_3' = \frac{\dot{E}}{1 + 2\,YZ} \qquad \text{loaded;}
\end{aligned}\right\} \begin{gathered} \text{all three} \\ \text{e.m.fs.} \\ \text{unequal, and} \\ \text{of unequal} \\ \text{phase angles.} \end{gathered} \quad (13)$$

$$\left.\begin{aligned} I_1' &= I_2' = 0 \\ I_3' &= \frac{EY}{1+2\,YZ} \end{aligned}\right\} \qquad (13)$$

$$\left.\begin{aligned} I_1 &= \frac{EY}{1+2\,YZ} \\ I_2 &= -\frac{EY}{1+2\,YZ} \\ I_3 &= 0 \end{aligned}\right\} \qquad (13)$$

C. *Two circuits loaded, one unloaded.*

$$Y_1 = Y_2 = Y, \quad Y_3 = 0,$$
$$Z_1 = Z_2 = Z_3 = Z.$$

Substituting this in equations (6), it is

$$\left.\begin{aligned} \epsilon E - E_1'\,(1 + 2\,YZ) + E_2'YZ &= 0 \\ \epsilon^2 E - E_2'\,(1 + 2\,YZ) + E_1'YZ &= 0 \end{aligned}\right\} \text{loaded branches.}$$

$$E - E_3' + (E_1' + E_2')\,YZ = 0 \quad \text{unloaded branch.}$$

or, since

$$(E_1' + E_2') = -E_3';$$
$$E - E_3' - E_3'YZ = 0,$$
$$E_3' = \frac{E}{1+YZ};$$

thus,

$$\left.\begin{aligned} &\left.\begin{aligned} E_1' &= \frac{E\epsilon\,\{1 + (2+\epsilon)\,YZ\}}{1 + 4\,YZ + 3\,Y^2Z^2} \\ E_2' &= \frac{E\epsilon^2\,\{1 + (2+\epsilon^2)\,YZ\}}{1 + 4\,YZ + 3\,Y^2Z^2} \end{aligned}\right\} \text{loaded branches.} \\ &E_3' = \frac{E}{1+YZ} \qquad\qquad \text{unloaded branch.} \end{aligned}\right\} \qquad (14)$$

As seen, with unsymmetrical distribution of load, all three branches become more or less unequal, and the phase displacement between them unequal also.

CHAPTER XXXIX.

QUARTER-PHASE SYSTEM.

398. In a three-wire quarter-phase system, or quarter-phase system with common return-wire of both phases, let the two outside terminals and wires be denoted by 1 and 2, the middle wire or common return by 0.

It is then,

$\dot{E}_1 = \dot{E}$ = e.m.f. between 0 and 1 in the generator.
$\dot{E}_2 = j\dot{E}$ = e.m.f. between 0 and 2 in the generator.

Let $\dot{I}_1$ and $\dot{I}_2$ = currents in 1 and in 2,
$\dot{I}_0$ = current in 0,
Z_1 and Z_2 = impedances of lines 1 and 2,
Z_0 = impedance of line 0,
Y_1 and Y_2 = admittances of circuits 0 to 1, and 0 to 2,
$\dot{I}_1'$ and $\dot{I}_2'$ = currents in circuits 0 to 1, and 0 to 2,
$\dot{E}_1'$ and $\dot{E}_2'$ = potential differences at circuit 0 to 1, and 0 to 2.

it is then,
$$\left.\begin{aligned} &\dot{I}_1 + \dot{I}_2 + \dot{I}_0 = 0,\\ \text{or,}\quad &\dot{I}_0 = -(\dot{I}_1 + \dot{I}_2);\end{aligned}\right\} \tag{1}$$
that is, $\dot{I}_0$ is common return of $\dot{I}_1$ and $\dot{I}_2$.

Further, we have :
$$\left.\begin{aligned} \dot{E}_1' &= \dot{E} - \dot{I}_1 Z_1 + \dot{I}_0 Z_0 = \dot{E} - \dot{I}_1(Z_1 + Z_0) - \dot{I}_2 Z_0\\ \dot{E}_2' &= j\dot{E} - \dot{I}_2 Z_0 + \dot{I}_0 Z_0 = j\dot{E} - \dot{I}_2(Z_2 + Z_0) - \dot{I}_1 Z_1\end{aligned}\right\} \tag{2}$$
and
$$\left.\begin{aligned} \dot{I}_1 &= Y_1 \dot{E}_1'\\ \dot{I}_2 &= Y_2 \dot{E}_2'\\ \dot{I}_0 &= -(Y_1\dot{E}_1' + Y_2\dot{E}_2')\end{aligned}\right\} \tag{3}$$

Substituting (3) in (2), and expanding,
$$\left.\begin{aligned} \dot{E}_1' &= \dot{E}\,\frac{1 + Y_2Z_2 + Y_2Z_0(1-j)}{(1+Y_1Z_0+Y_1Z_1)(1+Y_2Z_0+Y_2Z_2) - Y_1Y_2Z_0^2}\\ \dot{E}_2' &= j\dot{E}\,\frac{1+Y_1Z_1+Y_1Z_0(1+j)}{(1+Y_1Z_0+Y_1Z_1)(1+Y_2Z_0+Y_2Z_2) - Y_1Y_2Z_0^2}\end{aligned}\right\} \tag{4}$$

Hence, the two e.m.fs. at the end of the line are unequal in magnitude, and not in quadrature any more.

399. SPECIAL CASES:

A. *Balanced System.*

$$Z_1 = Z_2 = Z;$$

$$Z_0 = \frac{Z}{\sqrt{2}};$$

$$Y_1 = Y_2 = Y.$$

Substituting these values in (4), gives:

$$\left.\begin{aligned} \dot{E}_1' &= \dot{E}\frac{1 + \frac{1+\sqrt{2}-j}{\sqrt{2}}YZ}{1+\sqrt{2}\,(1+\sqrt{2})\,YZ+(1+\sqrt{2})\,Y^2Z^2} \\ &= \dot{E}\frac{1 + (1.707 - 0.707\,j)\,YZ}{1 + 3.414\,YZ + 2.414\,Y^2Z^2} \\ \dot{E}_2' &= j\dot{E}\frac{1 + \frac{1+\sqrt{2}+j}{\sqrt{2}}YZ}{1+\sqrt{2}\,(1+\sqrt{2})\,YZ+(1+\sqrt{2})\,Y^2Z^2} \\ &= j\dot{E}\frac{1 + (1.707 + 0.707\,j)\,YZ}{1 + 3.414\,YZ + 2.414\,Y^2Z^2} \end{aligned}\right\} \qquad (5)$$

Hence, the balanced quarter-phase system with common return is unbalanced with regard to voltage and phase relation, or in other words, even if in a quarter-phase system with common return both branches or phases are loaded equally, with a load of the same phase displacement, nevertheless the system becomes unbalanced, and the two e.m.fs. at the end of the line are neither equal in magnitude, nor in quadrature with each other.

B. *One branch loaded, one unloaded.*

$$Z_1 = Z_2 = Z,$$

$$Z_0 = \frac{Z}{\sqrt{2}}.$$

(*a*) $\quad Y_1 = 0, \quad Y_2 = Y,$

(*b*) $\quad Y_1 = Y, \quad Y_2 = 0.$

Substituting these values in (4), gives:

$$(a)\qquad \left.\begin{aligned}
\dot{E}_1' &= \dot{E}\,\frac{1 + YZ\dfrac{1+\sqrt{2}-j}{\sqrt{2}}}{1 + YZ\dfrac{1+\sqrt{2}}{\sqrt{2}}}\\
&= \dot{E}\left\{1 - \frac{j}{1+\sqrt{2}+\dfrac{\sqrt{2}}{YZ}}\right\}\\
&= \dot{E}\left\{1 - \frac{j}{2.414 + \dfrac{1.414}{YZ}}\right\}\\
\dot{E}_2' &= j\dot{E}\,\frac{1}{1 + YZ\dfrac{1+\sqrt{2}}{\sqrt{2}}}\\
&= j\dot{E}\,\frac{1}{1 + 1.707\,YZ}
\end{aligned}\right\} \qquad (6)$$

$$(b)\qquad \left.\begin{aligned}
\dot{E}_1' &= \dot{E}\,\frac{1}{1 + YZ\dfrac{1+\sqrt{2}}{\sqrt{2}}}\\
&= \dot{E}\,\frac{1}{1 + 1.707\,YZ}\\
\dot{E}_2' &= j\dot{E}\,\frac{1 + YZ\dfrac{1+\sqrt{2}+j}{\sqrt{2}}}{1 + YZ\dfrac{1+\sqrt{2}}{\sqrt{2}}}\\
&= j\dot{E}\left\{1 + \frac{j}{1+\sqrt{2}+\dfrac{\sqrt{2}}{YZ}}\right\}\\
&= j\dot{E}\left\{1 + \frac{j}{2.414 + \dfrac{1.414}{YZ}}\right\}
\end{aligned}\right\} \qquad (7)$$

These two e.m.fs. are unequal, and not in quadrature with each other.

But the values in case (*a*) are different from the values in case (*b*).

That means:

The two phases of a three-wire, quarter-phase system are unsymmetrical, and the leading phase, 1, reacts upon the lagging phase, 2, in a different manner than 2 reacts upon 1.

It is thus undesirable to use a three-wire, quarter-phase system, except in cases where the line impedances, Z, are negligible.

In all other cases, the four-wire, quarter-phase system is preferable, which essentially consists of two independent single-phase circuits, and is treated as such.

Obviously, even in such an independent quarter-phase system, at unequal distribution of load, unbalancing effects may take place.

If one of the branches or phases is loaded differently from the other, the drop of voltage and the shift of the phase will be different from that in the other branch; and thus the e.m fs. at the end of the lines will be neither equal in magnitude, nor in quadrature with each other.

With both branches, however, loaded equally, the system remains balanced in voltage and phase, just like the three-phase system under the same conditions.

Thus the four-wire, quarter-phase system and the three-phase system are balanced with regard to voltage and phase at equal distribution of load, but are liable to become unbalanced at unequal distribution of load; the three-wire, quarter-phase system is unbalanced in voltage and phase, even at equal distribution of load.

CHAPTER XL.

BALANCED SYMMETRICAL POLYPHASE SYSTEMS.

400. In most applications of polyphase systems the system is a balanced symmetrical system, or as nearly balanced as possible. That is, it consists of n equal e.m.fs. displaced in phase from each other by $\frac{1}{n}$ period, and producing equal currents of equal phase displacement against their e.m.fs. In such systems, each e.m.f. and its current can be considered separately as constituting a single-phase system, that is, the polyphase system can be resolved into n equal single-phase systems, each of which consists of one conductor of the polyphase system, with zero impedance as return circuit. Hereby the investigation of the polyphase system resolves itself into that of its constituent single-phase system.

So, for instance, the polyphase system shown in Fig. 249, at balanced load, can be considered as consisting of the equal single-phase systems : $0-1$; $0-2$; $0-3$; . . . $0-n$, each of which consists of one conductor, 1, 2, 3, . . . n, and the return conductor, 0. Since the sum of all the currents equals 0, there is no current in conductor 0, that is, no voltage is consumed in this conductor; this is equivalent to assuming this conductor as of zero impedance. This common return conductor, 0, since it carries no current, can be omitted, as is usually the case. With star connection of an apparatus into a polyphase system, as in Fig. 241, the impedance of the equivalent single-phase system is the impedance of one conductor or circuit; if, however, the apparatus is ring connected, as shown diagrammatically in Fig. 242, the impedance of the ring-connected part of the circuit has to be reduced to star connection, in the usual manner of reducing a circuit to another circuit of different voltage, by the ratio

$$c = \frac{\text{ring voltage}}{\text{star voltage}};$$

or, as these voltages are usually called in a three-phase system,

$$c = \frac{\text{delta voltage}}{Y \text{ voltage}}.$$

That is, all ring voltages are divided, all ring currents multiplied with c; all ring impedances are divided, all ring admittances multiplied with the square of the ratio, c^2.

For instance, if in a three-phase induction motor with delta-connected circuits, the impedance of each circuit is

$$\dot{Z} = r - jx,$$

and the voltage impressed upon the circuit terminals $\dot{E}$, and the motor is supplied over a line of impedance, per line wire,

$$\dot{Z}_0 = r_0 - jx_0,$$

the motor impedance, reduced to star connection, or Y impedance, is

$$\dot{Z}' = \frac{r - jx}{c^2} = \frac{1}{3}(r - jx),$$

and the impressed voltage, reduced to Y circuit,

$$\dot{E}' = \frac{\dot{E}}{\sqrt{3}},$$

and the total impedance of the equivalent single-phase circuit is therefore

$$\dot{Z}_0 + \dot{Z}' = (r_0 - jx_0) + \frac{1}{3}(r - jx).$$

Inversely, however, where this appears more convenient, all quantities may be reduced to ring or delta connection, or one of the ring connections considered as equivalent single-phase circuit, of impedance

$$\dot{Z} + c^2\dot{Z}_0 = (r - jx) + 3(r_0 - jx_0).$$

Since the line impedances, line currents and the voltages consumed in the lines of a polyphase system are star, or (in a three-phase system) Y quantities, it usually is more convenient to reduce all quantities to Y connection, and use one of the Y-circuits as the equivalent single-phase circuit.

401. As an example may be considered the calculation of a long-distance transmission line, delivering 10,000 kw., three-phase power at 60 cycles, 80,000 volts and 90 per cent power-factor at 100 miles from the generating station, with approximately 10 per cent loss of power in the transmission line, and with the line conductors arranged in a triangle 6 feet distant from each other.

10,000 kw. total power delivered gives 3,333 kw. per line or single-phase branch (Y power).

3,333 kw. at 90 per cent power-factor gives 3,700 kv-amp.

80.000 volts between the lines gives $80,000 \div \sqrt{3} = 46,100$ volts from line to neutral, or per single-phase circuit.

3,700 kv-amp. per circuit, at 46,100 volts, gives 80 amperes per line.

10 per cent loss gives 333 kw. loss per line, and at 80 amperes, this gives a resistance per line,

$$333,000 \div 80^2 = 52 \text{ ohms},$$

or, 0.52 ohms per mile.

The nearest standard size of wire is No. 0 B. & S., which has a resistance of 0.52 ohms, and a weight of 1680 pounds per mile.

Choosing this size of wire so requires for the 300 miles of line conductor, $300 \times 1680 = 500,000$ pounds of copper.

At 0.52 ohms per mile, the resistance per transmission line or circuit of 100 miles length, is

$$r = 52 \text{ ohms}.$$

The inductance of wire No. 0, with $d = 0.325$ inches diameter, and 6 feet $= 72$ inches distance from the return conductor, is calculated from the formula of line inductance* as, 2.3 mil-henrys per mile; hence, per circuit,

$$L = 0.23 \text{ henry},$$

and herefrom the reactance,

$$\begin{aligned} x &= 2\pi f L \\ &= 88 \text{ ohms}. \end{aligned}$$

* "Theoretical Elements of Electrical Engineering," Second or Third Edition, Section I, §7, instance 1.

The capacity of the transmission line may be calculated directly, or more conveniently it may be derived from the inductance. If C is the capacity of the circuit, of which the inductance is L, then

$$f_1 = \frac{1}{4\sqrt{LC}}$$

is the fundamental frequency of oscillation, or natural period, that is, the frequency which makes the length, l, of the line a quarter-wave length.

Since the velocity of propagation of the electric field is the velocity of light, v, with a wave-length, $4\,l$, the number of waves per second, or frequency of oscillation of the line, is

$$f_1 = \frac{v}{4\,l}$$

and herefrom then follows:

$$\frac{v}{l} = \frac{1}{\sqrt{LC}}.^{*}$$

or,

$$C = \frac{\left(\frac{l}{v}\right)^2}{L};$$

hence, for

$$\begin{aligned} l &= 100 \text{ miles}, \\ v &= 186{,}000 \text{ miles per second}, \\ L &= 0.23 \text{ henry}, \\ C &= 1.26 \text{ mf}. \end{aligned}$$

and the capacity susceptance,

$$b = 2\pi fC = 475 \times 10^{-6}.$$

* Or, if μ = permeability, κ = dielectric constant of the medium surrounding the conductor, it is

$$f_1 = \frac{v}{4\,l\sqrt{\mu\kappa}};$$

hence,

$$\frac{v}{l} = \sqrt{\frac{\mu\kappa}{LC}},$$

Representing, as approximation, the line capacity by a condenser shunted across the middle of the line,

We have, impedance of half the line,

$$Z = \frac{r}{2} - j\,\frac{x}{2} = 26 - 44\,j \text{ ohms.}$$

Choosing the voltage at the receiving end as zero vector,

$$e = 46{,}100 \text{ volts,}$$

at 90 per cent power-factor and therefore 43.6 per cent inductance factor, the current is represented by

$$\underset{.}{I} = 80\,(0.9 - 0.436\,j) = 72 - 35\,j.$$

This gives:

Voltage at receiver circuit, $e = 46{,}100$ volts;

current in receiver circuit, $\underset{.}{I} = 72 + 35\,j$ amp.;

impedance voltage of half the line, $Z\underset{.}{I} = 3410 - 2260\,j$ volts.

Hence, the condenser voltage, $\underset{.}{E}_1 = e + Z\underset{.}{I} = 49{,}510 - 2260\,j$ volts;

and the condenser current, $-jb\underset{.}{E}_1 = -1.1 - 23.8\,j$ amp.;

hence, the total, or generator current, $\underset{.}{I}_0 = \underset{.}{I} - jb\underset{.}{E}_1 = 70.9 + 11.2\,j$ amp.

The impedance voltage of the other half of the line, $Z\underset{.}{I}_0 = 2330 - 2830\,j$ volts;

hence, the generator voltage, $\underset{.}{E}_0 = \underset{.}{E}_1 + Z\underset{.}{I}_0 = 51{,}840 - 5090\,j$ volts;

and the phase angle of the generator current,

$$\tan\theta_1 = \frac{11.2}{70.9} = 0.158; \quad \theta_1 = 9.0°$$

The phase angle of the generator voltage,

$$\tan\theta_2 = -\frac{5090}{51{,}840} = -0.098; \quad \theta_2 = -5.6°;$$

the lag of the generator current, $\theta_0 = \theta_1 - \theta_2 = 14.6°$;

hence the power-factor at the generator, $\cos\theta_0 = 96.7$ per cent.

And the power output, $3\,[\dot{I}, e]^1 = 10{,}000$ kw.;
the power input, $3\,[\dot{I}_0, \dot{E}_0]^1 = 11{,}190$ kw.;
the efficiency $= 89.35$ per cent.;
the volt-ampere output, $3\,ie = 11{,}110$ kv-amp.;
the volt-ampere input, $3\,i_0 e_0 = 11{,}220$ kv-amp.;
ratio: $= 99.02$ per cent.

And the absolute values are.

receiver current, $i = 80$ amp.;
receiver voltage, $e = 46{,}100 \times \sqrt{3} = 80{,}000$ volts;
generator current, $i_0 = 71.8$ amp.;
generator voltage, $e_0 = 52{,}100 \times \sqrt{3} = 90{,}000$ volts;
voltage drop in line, $= 11.1$ per cent.

402. Balanced polyphase systems thus can be calculated as single-phase systems, and this has been done in many preceding chapters, as in those on the induction machines, synchronous machines, etc., that is, apparatus which is usually operated on polyphase circuits.

Only in dealing with those phenomena which are resultants of all the phases of the polyphase system, in the resolution of the polyphase system into its constituent single-phase systems the effective value of the constant has to be used, which corresponds to the resultant effect. This, for instance, is the case in calculating the magnetic field of the induction machine — which is energized by the combination of all phases — or the armature reaction of synchronous machines, etc.

For instance, in the induction machine, from the generated e.m.f., e,— in Chapter XIX — the magnetic flux of the machine is calculated, and from the magnetic flux and the dimensions of the magnetic circuit: length and section of air-gap, and length and section of the iron part, follows the ampere-turns excitation, that is, the ampere turns, F_0, required to produce the magnetic flux.

The resultant m.m.f. of m equal magnetizing coils displaced in position by $\frac{1}{m}$ cycle, energized by m equal currents of an m-phase system, is given by paragraph 361 as

$$F = \frac{nmI}{\sqrt{2}}$$

where

I = current per phase, or per magnetizing coil,
n = number of turns per coil,
m = number of phases.

The exciting current per phase required to produce the resulting m.m.f., F_0, therefore, is

$$I = \frac{F_0 \sqrt{2}}{nm};$$

hence, for a three-phase system,

$$I = \frac{F_0 \sqrt{2}}{3\, n},$$

and for a quarter-phase system, with two coils in quadrature,

$$I = \frac{F_0}{n \sqrt{2}}.$$

In the investigation of the armature reaction of synchronous machines, Chapter XXII, the armature reaction of an m-phase machine is, by paragraph 361,

$$F = \frac{n_0 m I}{\sqrt{2}};$$

where

m = number of phases,
n_0 = number of turns per phase, effective, that is, allowing for the spread of the turns over an arc of the periphery in machines of distributed winding,
I = current per phase,

and when, in Chapter XXII, the armature reaction is given by nI, the number of effective turns, n, is, accordingly, for a polyphase alternator,

$$n = \frac{m}{\sqrt{2}} n_0;$$

hence, in a three-phase machine,

$$n = \frac{3\, n_0}{\sqrt{2}} = 1.5\, n_0 \sqrt{2};$$

in a quadrature-phase machine,

$$n = n_0 \sqrt{2}.$$

APPENDICES.

APPENDIX I.

ALGEBRA OF COMPLEX IMAGINARY QUANTITIES.

INTRODUCTION.

403. The system of numbers, of which the science of algebra treats, finds its ultimate origin in experience. Directly derived from experience, however, are only the absolute integral numbers; fractions, for instance, are not directly derived from experience, but are abstractions expressing relations between different classes of quantities. Thus, for instance, if a quantity is divided in two parts, from one quantity two quantities are derived, and denoting these latter as halves expresses a relation, namely, that two of the new kinds of quantities are derived from, or can be combined to one of the old quantities.

404. Directly derived from experience is the *operation of counting* or of *numeration*,

$$a, \quad a + 1, \quad a + 2, \quad a + 3 \ldots .$$

Counting by a given number of integers,

$$a + \underbrace{1 + 1 + 1 \ldots + 1}_{b \text{ integers}} = c,$$

introduces the operation of *addition*, as multiple counting,

$$a + b = c.$$

It is

$$a + b = b + a;$$

that is, the terms of addition, or addenda, are interchangeable.

Multiple addition of the same terms,

$$\underbrace{a + a + a + \ldots + a}_{b \text{ equal numbers}} = c,$$

introduces the operation of *multiplication*,

$$a \times b = c.$$

It is $$a \times b = b \times a,$$

that is, the terms of multiplication, or factors, are interchangeable.

Multiple multiplication of the same factors,

$$\underbrace{a \times a \times a \times \ldots \times a}_{b \text{ equal numbers}} = c,$$

introduces the operation of *involution*,

$$a^b = c.$$

Since $$a^b \text{ is not equal to } b^a,$$

the terms of involution are not interchangeable.

405. The reverse operation of addition introduces the operation of *subtraction*.

If $$a + b = c,$$

it is $$c - b = a.$$

This operation cannot be carried out in the system of absolute numbers, if

$$b > c.$$

Thus, to make it possible to carry out the operation of subtraction under any circumstances, the system of absolute numbers has to be expanded by the introduction of the *negative number*,

$$-a = (-1) \times a,$$

where (-1) is the *negative unit.*

Thereby the system of numbers is subdivided in the positive and negative numbers, and the operation of subtraction possible for all values of subtrahend and minuend. From the definition of addition as multiple numeration, and subtraction as its inverse operation, it follows,

$$c - (-b) = c + b,$$

thus: $$(-1) \times (-1) = 1;$$

that is, the negative unit is defined by $(-1)^2 = 1$.

406. The reverse operation of multiplication introduces the operation of *division*.

If $$a \times b = c,$$

it is $$\frac{c}{b} = a.$$

In the system of integral numbers this operation can only be carried out if b is a factor of c.

To make it possible to carry out the operation of division under any circumstances, the system of integral numbers has to be expanded by the introduction of the *fraction*,

$$\frac{c}{b} = c \times \left(\frac{1}{b}\right),$$

where $\frac{1}{b}$ is the integer fraction, and is defined by

$$\left(\frac{1}{b}\right) \times b = 1.$$

407. The reverse operation of involution introduces two new operations, since in the involution,

$$a^b = c,$$

the quantities a and b are not reversible.

Thus $$\sqrt[b]{c} = a, \text{ the } \textit{evolution},$$

$$\log_a c = b, \text{ the } \textit{logarithmation}.$$

The operation of evolution of terms, c, which are not complete powers, makes a further expansion of the system of numbers necessary, by the introduction of the *irrational number* (endless decimal fraction), as for instance,

$$\sqrt{2} = 1.414213\ldots$$

408. The operation of evolution of negative quantities, c, with even exponents, b, as for instance,

$$\sqrt[2]{-a},$$

makes a further expansion of the system of numbers necessary, by the introduction of the *imaginary unit*

$$\sqrt{-1}$$

Thus $$\sqrt[2]{-a} = \sqrt[2]{-1} \times \sqrt[2]{a},$$

where: $\sqrt{-1}$ is denoted by j.

Thus, the imaginary unit, j, is defined by

$$j^2 = -1.$$

By addition and subtraction of real and imaginary units, compound numbers are derived of the form,

$$a + jb,$$

which are denoted as *complex imaginary numbers*.

No further system of numbers is introduced by the operation of evolution.

The operation of logarithmation introduces the irrational and imaginary and complex imaginary numbers also, but no further system of numbers.

409. Thus, starting from the absolute integral numbers of experience, by the two conditions:

1st. Possibility of carrying out the algebraic operations and their reverse operations under all conditions,

2d. Permanence of the laws of calculation,

the expansion of the system of numbers has become necessary, into

positive and negative numbers,
integral numbers and fractions,
rational and irrational numbers,
real and imaginary numbers and complex imaginary numbers.

Therewith closes the field of algebra, and all the algebraic operations and their reverse operations can be carried out irrespective of the values of terms entering the operation.

Thus within the range of algebra no further extension of the system of numbers is necessary or possible, and the most general number is

$$a + jb,$$

where a and b can be integers or fractions, positive or negative, rational or irrational.

Any attempt to extend the system of numbers beyond the complex quantity, leads to numbers, in which the factors of a product are not interchangeable, in which one factor of a product may be zero without the product being zero, etc., and which thus can not be treated by the usual methods of algebra, that is, are extra-algebraic numbers. Such for instance are the double frequency vector products of Chapter XV.

ALGEBRAIC OPERATIONS WITH COMPLEX IMAGINARY QUANTITIES.

410. *Definition of imaginary unit:*

$$j^2 = -1.$$

Complex imaginary number:

$$\dot{A} = a + jb.$$

Substituting:

$$a = r \cos \beta,$$

$$b = r \sin \beta,$$

it is $$\dot{A} = r (\cos \beta + j \sin \beta),$$

where $$r = \sqrt{a^2 + b^2},$$

$$\tan \beta = \frac{b}{a},$$

r = vector,

β = amplitude of complex imaginary number, $\dot{A}$.

Substituting:

$$\cos \beta = \frac{\varepsilon^{j\beta} + \varepsilon^{-j\beta}}{2},$$

$$\sin \beta = \frac{\varepsilon^{j\beta} - \varepsilon^{-j\beta}}{2j},$$

it is $$\dot{A} = r\varepsilon^{j\beta},$$

where $\varepsilon = \lim_{n=\infty} \left(1 + \frac{1}{n}\right)^n = \sum_0^\infty \frac{1}{1 \times 2 \times 3 \times \ldots \times k},$

is the basis of the natural logarithms.

Conjugate numbers are called:

$$a + jb = r (\cos \beta + j \sin \beta) = r\varepsilon^{j\beta},$$

and $a - jb = r (\cos [-\beta] + j \sin [-\beta]) = r (\cos \beta - j \sin \beta) = r\varepsilon^{-j\beta},$

it is $$(a + jb)(a - jb) = a^2 + b^2 = r^2.$$

Associate numbers are called:

$$a + jb = r(\cos\beta + j\sin\beta) = r\epsilon^{j\beta},$$

and $$b + ja = r\left(\cos\left[\frac{\pi}{2} - \beta\right] + j\sin\left[\frac{\pi}{2} - \beta\right]\right) = r\epsilon^{j\left(\frac{\pi}{2} - \beta\right)},$$

it is $$(a + jb)(b + ja) = j(a^2 + b^2) = jr^2.$$

If $$a + jb = a' + jb',$$

it is $$a = a',$$
$$b = b'.$$

If $$a + jb = 0;$$

it is $$a = 0,$$
$$b = 0.$$

411. *Addition and Subtraction:*

$$(a + jb) \pm (a' + jb') = (a \pm a') + j(b \pm b').$$

Multiplication:

$$(a + jb)(a' + jb') = (aa' - bb') + j(ab' + ba');$$

or $$r(\cos\beta + j\sin\beta) \times r'(\cos\beta' + j\sin\beta') = rr'(\cos[\beta + \beta'] + j\sin[\beta + \beta']);$$

or $$r\epsilon^{j\beta} \times r'\epsilon^{j\beta'} = rr'\epsilon^{j(\beta + \beta')}.$$

Division:

Expansion of complex imaginary fraction, for rationalization of denominator or numerator, by multiplication with the conjugate quantity:

$$\frac{a + jb}{a' + jb'} = \frac{(a + jb)(a' - jb')}{(a' + jb')(a' - jb')} = \frac{(aa' + bb') + j(ba' - ab')}{a'^2 + b'^2}$$
$$= \frac{(a + jb)(a - jb)}{(a' + jb')(a - jb)} = \frac{a^2 + b^2}{(aa' + bb') + j(ab' - ba')};$$

or, $\dfrac{r\,(\cos\beta + j\sin\beta)}{r'\,(\cos\beta' + j\sin\beta')} = \dfrac{r}{r'}\,(\cos[\beta - \beta'] + j\sin[\beta - \beta']);$

or, $\dfrac{re^{j\beta}}{r'e^{j\beta'}} = \dfrac{r}{r'}e^{j(\beta-\beta')}$

involution:

$$(a + jb)^n = \{r\,(\cos\beta + j\sin\beta)\}^n = \{re^{j\beta}\}^n$$
$$= r^n\,(\cos n\beta + j\sin n\beta) = r^n e^{jn\beta};$$

evolution:

$$\sqrt[n]{a + jb} = \sqrt[n]{r\,(\cos\beta + j\sin\beta)} = \sqrt[n]{re^{j\beta}}$$
$$= \sqrt[n]{r}\left(\cos\frac{\beta}{n} + j\sin\frac{\beta}{n}\right) = \sqrt[n]{r}e^{j\frac{\beta}{n}}.$$

412. *Roots of the Unit:*

$\sqrt[2]{1} = +1, \quad -1;$

$\sqrt[3]{1} = +1, \dfrac{-1 + j\sqrt{3}}{2}, \dfrac{-1 - j\sqrt{3}}{2};$

$\sqrt[4]{1} = +1, \quad -1, \quad +j, \quad -j;$

$\sqrt[6]{1} = +1, \dfrac{+1 + j\sqrt{3}}{2}, \dfrac{-1 + j\sqrt{3}}{2}, -1, \dfrac{-1 - j\sqrt{3}}{2}, \dfrac{+1 - j\sqrt{3}}{2};$

$\sqrt[8]{1} = +1, -1, +j, -j, \dfrac{+1 + j}{\sqrt{2}}, \dfrac{+1 - j}{\sqrt{2}}, \dfrac{-1 + j}{\sqrt{2}}, \dfrac{-1 - j}{\sqrt{2}};$

$\sqrt[n]{1} = \cos\dfrac{2\pi k}{n} + j\sin\dfrac{2\pi k}{n} = e^{\frac{2\pi jk}{n}}, \; k = 0, 1, 2 \ldots . n - 1.$

413. *Rotation:*

In the complex imaginary plane, multiplication with

$$\sqrt[n]{1} = \cos\frac{2\pi}{n} + j\sin\frac{2\pi}{n} = e^{\frac{2\pi j}{n}}$$

means rotation, in positive direction, by $\dfrac{1}{n}$ of a revolution,

multiplication with (-1) means reversal, or rotation by 180°,
multiplication with $(+j)$ means positive rotation by 90°,
multiplication with $(-j)$ means negative rotation by 90°.

414. *Complex imaginary plane:*

While the positive and negative numbers can be represented by the points of a line, the complex imaginary numbers are represented by the points of a plane, with the horizontal axis, $A'OA$, as real axis, the vertical axis, $B'OB$, as imaginary axis. Thus all

the positive real numbers are represented by the points of half-axis $\overline{OA}$ towards the right;

the negative real numbers are represented by the points of half-axis $\overline{OA'}$ towards the left;

the positive imaginary numbers are represented by the points of half-axis $\overline{OB}$ upwards;

the negative imaginary numbers are represented by the points of half-axis $\overline{OB'}$ downwards;

the complex imaginary numbers are represented by the points outside of the coordinate axes.

APPENDIX II.

OSCILLATING CURRENTS.

INTRODUCTION.

415. An electric current varying periodically between constant maximum and minimum values — that is, in equal time intervals repeating the same values — is called an alternating current if the arithmetic mean value equals zero; and is called a pulsating current if the arithmetic mean value differs from zero.

Assuming the wave as a sine curve, or replacing it by the equivalent sine-wave, the alternating current is characterized by the period or the time of one complete cyclic change, and the amplitude or the maximum value of the current. Period and amplitude are constant in the alternating current.

A very important class are the currents of constant period, but geometrically varying amplitude; that is, currents in which the amplitude of each following wave bears to that of the preceding wave a constant ratio. Such currents consist of a series of waves of constant length, decreasing in amplitude, that is, in strength, in constant proportion. They are called oscillating currents in analogy with mechanical oscillations — for instance of the pendulum — in which the amplitude of the vibration decreases in constant proportion.

Since the amplitude of the oscillating current varies, constantly decreasing, the oscillating current differs from the alternating current in so far that it starts at a definite time and gradually dies out, reaching zero value theoretically at infinite time, practically in a very short time, short even in comparison with the time of one alternating half-wave. Characteristic constants of the oscillating current are the period, T, or frequency, $f = \frac{1}{T}$, the first amplitude and the ratio of any two successive

amplitudes, the latter being called the decrement of the wave. The oscillating current will thus be represented by the product of a periodic function, and a function decreasing in geometric proportion with the time. The latter is the exponential function, A^{f-gt}.

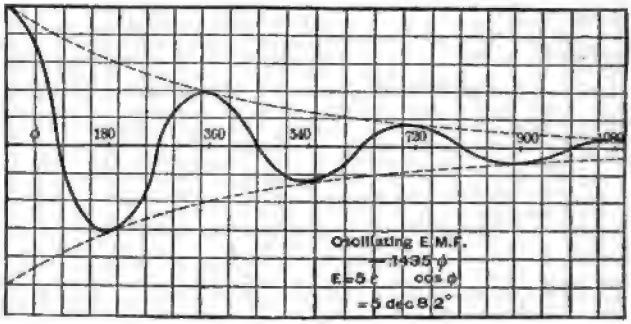


FIG. 259.

416. Thus, the general expression of the oscillating current is

$$I = A^{f-gt} \cos (2 \pi f t - \theta),$$

since

$$A^{f-gt} = A^f A^{-gt} = i \varepsilon^{-bt}.$$

Where ε = basis of natural logarithms, the current may be expressed,

$$I = i \varepsilon^{-bt} \cos (2 \pi f t - \theta) = i \varepsilon^{-a\phi} \cos (\phi - \theta),$$

where $\phi = 2 \pi f t$; that is, the period is represented by a complete revolution.

In the same way an oscillating electromotive force will be represented by

$$E = e^{-a\phi} \cos (\phi - \theta).$$

Such an oscillating electromotive force for the values,

$$e = 5, \quad a = 0.1435 \text{ or } \varepsilon^{-2\pi a} = 0.4, \quad \theta = 0,$$

is represented in rectangular coordinates in Fig. 259, and in polar coordinates in Fig. 260. As seen from Fig. 259, the oscillating wave in rectangular coordinates is tangent to the two exponential curves,

$$y = \pm e \varepsilon^{-a\phi}$$

417. In polar coordinates, the oscillating wave is represented in Fig. 260 by a spiral curve passing the zero point twice per period, and tangent to the exponential spiral,

$$y = \pm e \varepsilon^{-a\phi}.$$

The latter is called the envelope of a system of oscillating waves of which one is shown separately, with the same constants as Figs. 259 and 260, in Fig. 261. Its characteristic feature is:

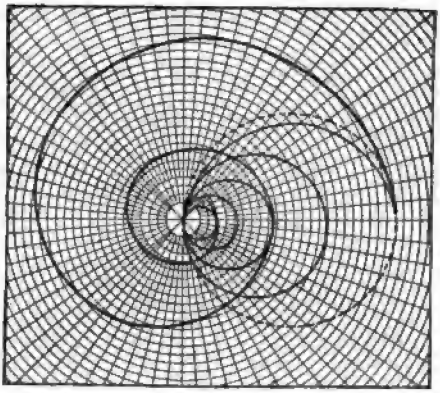


FIG. 260.

The angle which any concentric circle makes with the curve $y = e\varepsilon^{-a\phi}$, is

$$\tan \alpha = \frac{dy}{yd\phi} = -a,$$

which is, therefore, constant; or, in other words: "The envelope of the oscillating current is the exponential spiral, which is characterized by a constant angle of intersection with all concentric circles or all radii vectores." The oscillating current-wave is the product of the sine wave and the exponential or loxodromic spiral.

418. In Fig. 262 let $y = e\varepsilon^{-a\phi}$ represent the exponential spiral;

let $$z = e \cos (\phi - \theta)$$

represent the sine wave;

and let $$E = e\varepsilon^{-a\phi} \cos (\phi - \theta)$$

represent the oscillating wave.

We have then

$$\tan \beta = \frac{dE}{Ed\phi}$$
$$= \frac{-\sin (\phi - \theta) - a \cos (\phi - \theta)}{\cos (\phi - \theta)}$$
$$= - \{\tan (\phi - \theta) + a\};$$

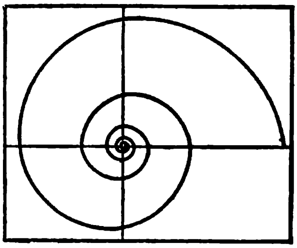


FIG. 261.

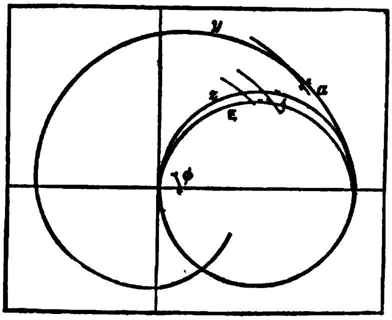


FIG. 262.

that is, while the slope of the sine wave, $z = e \cos (\phi - \theta)$, is represented by

$$\tan \gamma = - \tan (\phi - \theta),$$

the slope of the exponential spiral, $y = e\varepsilon^{-a\phi}$, is

$$\tan \alpha = -a = \text{constant},$$

that of the oscillating wave, $E = e\varepsilon^{-a\phi} \cos (\phi - \theta)$, is

$$\tan \beta = - \{\tan (\phi - \theta) + a\}.$$

Hence, it is increased over that of the alternating sine-wave by the constant, a. The ratio of the amplitudes of two consequent periods is

$$A = \frac{E_{2\pi}}{E_0} = \varepsilon^{-2\pi a}.$$

A is called the numerical decrement of the oscillating wave, a the exponential decrement of the oscillating wave, α the angular decrement of the oscillating wave. The oscillating wave can be represented by the equation,

$$E = e\varepsilon^{-\phi \tan \alpha} \cos (\phi - \theta).$$

In the example represented by Figs. 259 and 260, we have $A = 0.4$, $a = 0.1435$, $\alpha = 8.2°$.

Impedance and Admittance.

419. In complex imaginary quantities, the alternating wave,

$$z = e \cos (\phi - \theta),$$

is represented by the symbol,

$$\dot{E} = e (\cos \theta + j \sin \theta) = e_1 + je_2.$$

By an extension of the meaning of this symbolic expression, the oscillating wave, $E = e\varepsilon^{-a\phi} \cos (\phi - \theta)$, can be expressed by the symbol,

$$\dot{E} = e (\cos \theta + j \sin \theta) \operatorname{dec} \alpha = (e_1 + je_2) \operatorname{dec} \alpha,$$

where $a = \tan \alpha$ is the exponential decrement, α the angular decrement, $\varepsilon^{-2\pi a}$ the numerical decrement.

Inductance.

420. Let $r =$ resistance, $L =$ inductance, and $x = 2 \pi f L =$ reactance.

In a circuit excited by the oscillating current,

$$\dot{I} = i\varepsilon^{-a\phi} \cos (\phi - \theta) = i (\cos \theta + j \sin \theta) \operatorname{dec} \alpha$$
$$= (i_1 + ji_2) \operatorname{dec} \alpha,$$

where $\quad i_1 = i \cos \theta, \quad i_2 = i \sin \theta, \quad a = \tan \alpha.$

We have then,
the electromotive force consumed by the resistance, r, of the circuit,

$$E_r = rI \operatorname{dec} \alpha.$$

The electromotive force consumed due to the inductance, L, of the circuit,

$$E_x = L\frac{dI}{dt} = 2\pi f L\frac{dI}{d\phi} = x\frac{dI}{d\phi}.$$

Hence $E_x = -xi\varepsilon^{-a\phi}\{\sin(\phi-\theta) + a\cos(\phi-\theta)\}$

$$= -\frac{xi\varepsilon^{-a\phi}}{\cos\alpha}\sin(\phi-\theta+\alpha).$$

Thus, in symbolic expression,

$$\dot{E}_x = -\frac{xi}{\cos\alpha}\{-\sin(\theta-\alpha) + j\cos(\theta-\alpha)\}\ \text{dec}\ \alpha$$

$$= -xi(a+j)(\cos\theta + j\sin\theta)\ \text{dec}\ \alpha;$$

that is, $\dot{E}_x = -xI(a+j)\ \text{dec}\ \alpha.$

Hence the apparent reactance of the oscillating-current circuit is, in symbolic expression,

$$X = x(a+j)\ \text{dec}\ \alpha.$$

Hence it contains a power component, ax, and the impedance is

$$Z = (r - X)\ \text{dec}\ \alpha = \{r - x(a+j)\}\ \text{dec}\ \alpha = (r - ax - jx)\ \text{dec}\ \alpha.$$

Capacity.

421. Let r = resistance, C = capacity, and $x_c = \frac{1}{2\pi fC}$ = condensive reactance. In a circuit excited by the oscillating current, I, the electromotive force consumed due to the capacity C is

$$E_{x_c} = \frac{1}{C}\int I dt = \frac{1}{2\pi fC}\int I d\phi = k\int I d\phi;$$

or, by substitution,

$$E_{x_c} = x\int i\varepsilon^{-a\phi}\cos(\phi-\theta)\,d\phi$$

$$= \frac{x}{1+a^2}i\varepsilon^{-a\phi}\{\sin(\phi-\omega) - a\cos(\phi-\theta)\}$$

$$= \frac{xi\varepsilon^{-a\phi}}{(1+a^2)\cos a}\sin(\phi-\theta-\alpha);$$

hence, in symbolic expression,

$$\dot{E}_{x_c} = \frac{x\dot{i}}{(1 + a^2)\cos\alpha}\{-\sin(\theta + \alpha) + j\cos(\theta + \alpha)\}\operatorname{dec}\alpha$$

$$= \frac{x\dot{i}}{1 + a^2}(a + j)(\cos\theta + j\sin\theta)\operatorname{dec}\alpha;$$

hence,

$$\dot{E}_{x_c} = \frac{x}{1 + a^2}(-a + j)\,\dot{I}\operatorname{dec}\alpha;$$

that is, the apparent capacity of the oscillating circuit is, in symbolic expression,

$$C = \frac{x_c}{1 + a^2}(-a + j)\operatorname{dec}\alpha.$$

422. We have then:

in an oscillating-current circuit of resistance, r, inductive reactance x, and condensive reactance x_c, with an exponential decrement a, the apparent impedance, in symbolic expression, is,

$$Z = \left\{r - x(a + j) + \frac{x_c}{1 + a^2}(-a + j)\right\}\operatorname{dec}\alpha$$

$$= \left\{r - a\left(x + \frac{x_c}{1 + a^2}\right) - j\left(x - \frac{x_c}{1 + a^2}\right)\right\}\operatorname{dec}\alpha$$

$$= r_a - jx_a;$$

and, absolute,

$$z_a = \sqrt{r_a^2 + x_a^2}$$

$$= \sqrt{\left[r - a\left(x + \frac{x_c}{1 + a^2}\right)\right]^2 + \left[x - \frac{x_c}{1 + a^2}\right]^2}.$$

Admittance.

423. Let

$$I = i\varepsilon^{-a\phi}\cos(\phi - \theta) = \text{current}.$$

Then from the preceding discussion, the electromotive force consumed by resistance r, inductive reactance x, and condensive reactance x_c, is

$$E = ie^{-a\phi}\left\{\cos(\phi - \theta)\left[r - ax - \frac{a}{1 + a^2}x_c\right] - \sin(\phi - \theta)\left[x - \frac{x_c}{1 + a^2}\right]\right\}$$

$$= iz_a\varepsilon^{-a\phi}\cos(\phi - \theta + \delta),$$

where $$\tan\delta = \frac{x - \dfrac{x_c}{1+a^2}}{r - ax - \dfrac{a}{1+a^2}x_c},$$

$$z_a = \sqrt{\left(x - \frac{x_c}{1+a^2}\right)^2 + \left(r - ax - \frac{a}{1+a^2}x_c\right)^2};$$

substituting $\theta + \delta$ for θ, and $e = iz_a$ we have

$$E = e\varepsilon^{-a\phi}\cos(\phi - \theta),$$

$$I = \frac{e}{z_a}\varepsilon^{-a\phi}\cos(\phi - \theta - \delta)$$

$$= e\varepsilon^{-a\phi}\left\{\frac{\cos\delta}{z_a}\cos(\phi - \theta) + \frac{\sin\delta}{z_a}\sin(\phi - \theta)\right\};$$

hence in complex quantities,

$$\dot{E} = e(\cos\theta + j\sin\theta)\operatorname{dec}\alpha,$$

$$\dot{I} = \dot{E}\left\{\frac{\cos\delta}{z_a} + j\frac{\sin\delta}{z_a}\right\}\operatorname{dec}\alpha;$$

or, substituting,

$$\dot{I} = \dot{E}\left\{\frac{r - ax - \dfrac{a}{1+a^2}x_c}{\left(x - \dfrac{x_c}{1+a^2}\right)^2 + \left(r - ax - \dfrac{a}{1+a^2}x_c\right)^2} + j\frac{x - \dfrac{x_c}{1+a^2}}{\left(x - \dfrac{x_c}{1+a^2}\right)^2 + \left(r - ax - \dfrac{a}{1+a^2}x_c\right)^2}\right\}\operatorname{dec}\alpha.$$

424. Thus in complex quantities, for oscillating currents, we have: conductance,

$$g = \frac{r - ax - \dfrac{a}{1+a^2}x_c}{\left(x - \dfrac{x_c}{1+a^2}\right)^2 + \left(r - ax - \dfrac{a}{1+a^2}x_c\right)^2};$$

susceptance,

$$b = \frac{x - \frac{x_c}{1 + a^2}}{\left(x - \frac{x_c}{1 + a^2}\right)^2 + \left(r - ax - \frac{a}{1 + a^2} x_c\right)^2};$$

admittance, in absolute values,

$$y = \sqrt{g^2 + b^2} = \frac{1}{\sqrt{\left(x - \frac{x_c}{1 + a^2}\right)^2 + \left(r - ax - \frac{a}{1 + a^2} x_c\right)^2}};$$

in symbolic expression,

$$Y = g + jb = \frac{\left(r - ax - \frac{a}{1 + a^2} x_c\right) + j\left(x - \frac{x_c}{1 + a^2}\right)}{\left(x - \frac{x_c}{1 + a^2}\right)^2 + \left(r - ax - \frac{a}{1 + a^2} x_c\right)^2}.$$

Since the impedance is

$$Z = \left(r - ax - \frac{a}{1 + a^2} x_c\right) - j\left(x - \frac{x_c}{1 + a^2}\right) = r_a - jx_a,$$

we have

$$Y = \frac{1}{Z}; \; y = \frac{1}{z_a}; \; g = \frac{r_a}{z_a^2}; \; b = \frac{x_a}{z_a^2};$$

that is, the same relations as in the complex quantities in alternating-current circuits, except that in the present case all the constants, r_a, x_a, z_a, g, z, y, depend upon the decrement a.

Circuits of Zero Impedance.

425. In an oscillating-current circuit of decrement a, of resistance r, inductive reactance x, and condensive reactance x_c, the impedance was represented in symbolic expression by

$$Z = r_a - jx_a = \left(r - ax - \frac{a}{1 + a^2} x_c\right) - j\left(x - \frac{x_c}{1 + a^2}\right),$$

or numerically by

$$z = \sqrt{r_a^2 + x_a^2} = \sqrt{\left(r - ax - \frac{a}{1 + a^2} x_c\right)^2 + \left(x - \frac{x_c}{1 + a^2}\right)^2}.$$

Thus the inductive reactance, x, as well as the condensive reactance, x_c, do not represent wattless electromotive forces as in an alternating-current circuit, but introduce power components of negative sign,

$$-ax - \frac{a}{1+a^2} x_c;$$

that means, in an oscillating-current circuit, the counter electromotive force of self-induction is not in quadrature behind the current, but lags less than 90°, or a quarter period; and the charging current of a condenser is less than 90°, or a quarter period, ahead of the impressed electromotive force.

426. In consequence of the existence of negative power components of reactance in an oscillating-current circuit, a phenomenon can exist which has no analogy in an alternating-current circuit; that is, under certain conditions the total impedance of the oscillating-current circuit can equal zero:

$$Z = 0.$$

In this case we have

$$r - ax - \frac{a}{1+a^2} x_c = 0; \; x - \frac{x_c}{1+a^2} = 0,$$

substituting in this equation,

$$x = 2\pi f L; \; x_c = \frac{1}{2\pi f C};$$

and expanding, we have

$$a = \frac{1}{\sqrt{\frac{4L}{r^2C} - 1}}$$

$$2\pi f = \frac{r}{2L}\sqrt{\frac{4L}{r^2C} - 1} = \frac{r}{2aL}.$$

That is, if in an oscillating-current circuit, the decrement,

$$a = \frac{1}{\sqrt{\frac{4L}{r^2C} - 1}},$$

and the frequency $f = \frac{r}{4\pi a L}$, the total impedance of the circuit is zero; that is, the oscillating current, when started once, will continue without external energy being impressed upon the circuit.

427. The physical meaning of this is: If upon an electric circuit a certain amount of energy is impressed and then the circuit left to itself, the current in the circuit will become oscillating, and the oscillations assume the frequency, $f = \frac{r}{4\pi a L}$, and the decrement,

$$a = \frac{1}{\sqrt{\frac{4L}{r^2C} - 1}}.$$

That is, the oscillating currents are the phenomena by which an electric circuit of disturbed equilibrium returns to equilibrium.

This feature shows the origin of the oscillating currents, and the means of producing such current by disturbing the equilibrium of the electric circuit; for instance, by the discharge of a condenser, by make-and-break of the circuit, by sudden electrostatic charge, as lightning, etc. Obviously, the most important oscillating currents are those in a circuit of zero impedance, representing oscillating discharges of the circuit. Lightning strokes frequently belong to this class.

Oscillating Discharges.

428. The condition of an oscillating discharge is $Z = 0$, that is,

$$a = \frac{1}{\sqrt{\frac{4L}{r^2C} - 1}}, \quad 2\pi f = \frac{r}{2aL} = \frac{r}{2L}\sqrt{\frac{4L}{r^2C} - 1}.$$

If $r = 0$, that is, in a circuit without resistance, we have $a = 0$, $f = \frac{1}{2\pi\sqrt{LC}}$; that is, the currents are alternating with no decrement, and the frequency is that of resonance.

If $\frac{4L}{r^2C - 1} < 0$, that is, $r > 2\sqrt{\frac{L}{C}}$, a and f become imaginary; that is, the discharge ceases to be oscillatory. An electrical discharge assumes an oscillating nature only, if $r < 2\sqrt{\frac{L}{C}}$. In the case $r = 2\sqrt{\frac{L}{C}}$ we have $a = \infty$, $f = 0$; that is, the current dies out without oscillation.

From the foregoing we have seen that oscillating discharges — as for instance the phenomena taking place if a condenser charged to a given potential is discharged through a given circuit, or if lightning strikes the line circuit — are defined by the equation, $Z = 0 \operatorname{dec} \alpha$.

Since

$$\dot{I} = (i_1 + ji_2) \operatorname{dec} \alpha, \qquad \dot{E}_r = \dot{I} r \operatorname{dec} \alpha,$$

$$\dot{E}_x = -x\dot{I}\,(a + j) \operatorname{dec} \alpha, \quad \dot{E}_{x_c} = \frac{x_c}{1 + a^2}\,\dot{I}\,(-a + j) \operatorname{dec} \alpha,$$

we have
$$r - ax - \frac{a}{1 + a^2} x_c = 0,$$
$$-x + \frac{x_c}{1 + a^2} = 0;$$

hence, by substitution,

$$\dot{E}_{x_c} = x\dot{I}\,(-a + j) \operatorname{dec} \alpha.$$

The two constants, i_1 and i_2, of the discharge, are determined by the initial conditions — that is, the electromotive force and the current at the time, $t = 0$.

429. Let a condenser of capacity C be discharged through a circuit of resistance r and inductance L. Let e = electromotive force at the condenser in the moment of closing the circuit — that is, at the time $t = 0$ or $\phi = 0$. At this moment the current is zero, — that is,

$$\dot{I} = ji_2, \quad i_1 = 0.$$

Since $\dot{E}_{x_c} = x\dot{I}\,(-a + j) \operatorname{dec} \alpha = e$ at $\phi = 0$,

we have $x i_2 \sqrt{1 + a^2} = e$ or $i_2 = \frac{e}{x\sqrt{1 + a^2}}$.

Substituting this, we have,

$$\dot{I} = j\frac{e}{x\sqrt{1+a^2}}\operatorname{dec}\alpha, \qquad \dot{E}_r = je\frac{r}{x\sqrt{1+a^2}}\operatorname{dec}\alpha,$$

$$\dot{E}_x = \frac{e}{\sqrt{1+a^2}}(1-ja)\operatorname{dec}\alpha, \; \dot{E}_{x_c} = -\frac{e}{\sqrt{1+a^2}}(1+ja)\operatorname{dec}\alpha,$$

the equations of the oscillating discharge of a condenser of initial voltage, e.

Since $$x = 2\pi fL,$$

$$a = \frac{1}{\sqrt{\frac{4L}{r^2C}-1}},$$

$$2\pi f = \frac{r}{2aL},$$

we have

$$x = \frac{r}{2a} = \frac{r}{2}\sqrt{\frac{4L}{r^2C}-1};$$

hence, by substitution,

$$\dot{I} = je\sqrt{\frac{C}{L}}\operatorname{dec}\alpha, \quad \dot{E}_r = jer\sqrt{\frac{C}{L}}\operatorname{dec}\alpha,$$

$$\dot{E}_x = \frac{er}{2}\sqrt{\frac{C}{L}}\left(\sqrt{\frac{4L}{r^2C}-1}-j\right)\operatorname{dec}\alpha,$$

$$\dot{E}_{x_c} = -\frac{er}{2}\sqrt{\frac{C}{L}}\left(\sqrt{\frac{4L}{r^2C}-1}+j\right)\operatorname{dec}\alpha,$$

$$a = \frac{1}{\sqrt{\frac{4L}{r^2C}-1}}, \qquad f = \frac{r\sqrt{\frac{4L}{r^2C}-1}}{4\pi L},$$

the final equations of the oscillating discharge, in symbolic expression.

Oscillating-Current Transformer.

430. As an example of the application of the symbolic method of analyzing the phenomena caused by oscillating currents, the transformation of such currents may be investigated. If an oscillating current is produced in a circuit including the primary of a transformer, oscillating currents will also exist in the secondary of this transformer. In a transformer let the ratio of secondary to primary turns be p. Let the secondary be closed by a circuit of total resistance, $r_1 = r_1' + r_1''$, where $r_1' =$ external, $r_1'' =$ internal, resistance. The total inductance $L_1 = L_1' + L_1''$, where $L_1' =$ external, $L_1'' =$ internal, inductance; total capacity, C_1. Then the total admittance of the secondary circuit is

$$Y_1 = (g_1 + jb_1) \text{ dec } \alpha = \frac{1}{\left(r_1 - ax_1 - \dfrac{a}{1 + a^2} x_{c1}\right) - j\left(x_1 - \dfrac{x_c}{1 + a^2}\right)},$$

where $x_1 = 2\pi f L_1 =$ inductive reactance: $x_{c1} = \dfrac{1}{2\pi f C} =$ condensive reactance. Let $r_0 =$ effective hysteretic resistance, $L_0 =$ inductance; hence, $x_0 = 2\pi f L_0 =$ reactance; hence,

$$Y_0 = g_0 + jb_0 = \frac{1}{(r_0 - ax_0) - jx_0} = \text{admittance}$$

of the primary exciting circuit of the transformer; that is, the admittance of the primary circuit at open secondary circuit.

As discussed elsewhere, a transformer can be considered as consisting of the secondary circuit supplied by the impressed electromotive force over leads, whose impedance is equal to the sum of primary and secondary transformer impedance, and which are shunted by the exciting circuit, outside of the secondary, but inside of the primary impedance.

Let $r =$ resistance; $L =$ inductance; $C =$ capacity;

hence,

$$x = 2\pi f L = \text{inductive reactance},$$

$$x_c = \frac{1}{2\pi f C} = \text{condensive reactance of the total primary circuit},$$

including the primary coil of the transformer. If $\dot{E}_1' = E_1'$ dec α denotes the electromotive force generated in the secondary of the transformer by the mutual magnetic flux, — that is, by the oscillating magnetism interlinked with the primary and secondary coil, — we have $\dot{I}_1 = \dot{E}_1' Y_1$ dec α = secondary current.

Hence, $\dot{I}_1' = p\dot{I}_1$ dec $\alpha = p\dot{E}_1' Y_1$ dec α = primary load current, or component of primary current corresponding to secondary current. Also, $\dot{I}_0 = \frac{1}{p}\dot{E}_1' Y_0$ dec α = primary exciting current; hence, the total primary current is

$$\dot{I} = \dot{I}_1' + \dot{I}_0 = \frac{\dot{E}_1'}{p} \{Y_0 + p^2 Y_1\} \text{ dec } \alpha.$$

$\dot{E}' = \frac{\dot{E}_1'}{p}$ dec α = generated primary electromotive force.

Hence the total primary electromotive force is

$$\dot{E} = (\dot{E}' + \dot{I}Z) \text{ dec } \alpha = \frac{\dot{E}_1'}{p} \{1 + ZY_0 + p^2 ZY_1\} \text{ dec } \alpha.$$

In an oscillating discharge the total primary electromotive force, $\dot{E} = 0$, — that is,

$$1 + ZY_0 + p^2 ZY_1 = 0;$$

or, after substitution,

$$1 + \frac{\left(r - ax - \frac{a}{1 + a^2} x_c\right) - j\left(x - \frac{x_c}{1 + a^2}\right)}{(r_0 - ax_0) - jx_0}$$
$$+ p^2 \frac{\left(r - ax - \frac{a}{1 + a^2} x_c\right) - j\left(x - \frac{x_c}{1 + a^2}\right)}{\left(r_1 - ax_1 - \frac{a}{1 + a^2} x_{c1}\right) - j\left(x_1 - \frac{x_{c1}}{1 + a^2}\right)} = 0.$$

Substituting in this equation, $x = 2\pi fL$, $x_c = \frac{1}{2\pi fC}$, etc., we get a complex imaginary equation with the two constants, a and f. Separating this equation in the real and the imaginary parts, we derive two equations, from which the two constants, a and f, of the discharge are calculated.

431. If the exciting current of the transformer is negligible, — that is, if $Y_0 = 0$, the equation becomes essentially simplified, —

$$1 + p^2 \frac{\left(r - ax - \frac{a}{1 + a^2} x_c\right) - j\left(x - \frac{x_c}{1 + a_2}\right)}{\left(r_1 - ax_1 - \frac{a}{1 + a^2} x_{c1}\right) - j\left(x_1 - \frac{x_{c1}}{1 + a^2}\right)} = 0;$$

that is,

$$\left(r_1 - ax_1 - \frac{a}{1 + a^2} x_{c1}\right) + p^2\left(r - ax - \frac{a}{1 + a^2} x_c\right) = 0;$$

$$\left(x_1 - \frac{x_{c1}}{1 + a^2}\right) + p^2\left(x - \frac{x_c}{1 + a^2}\right) = 0;$$

or, combined, —

$$(r_1 - 2\,ax_1) + p^2\,(r - 2\,ax) = 0,$$

$$r_1 + p^2 r = 2\,a\,(x_1 + p^2 x),$$

$$x_{c1} + p^2 x_c = (1 + a^2)\,(x_1 + p^2 x).$$

Substituting for x_1, x, x_{c1}, x_c, we have

$$a = \frac{1}{\sqrt{\frac{4\,(L_1 + p^2 L)}{(r_1 + p^2 r)^2\,(C_1 + p^2 C)} - 1}},$$

$$2\,\pi f = \frac{r_1 + p^2 r}{2a\,(L_1 + p^2 L)}$$

$$= \frac{r_1 + p^2 r}{2\,(L_1 + p_2 L)} \sqrt{\frac{4\,(L_1 + p^2 L)}{(r_1 + p^2 r)^2\,(C_1 + p^2 C)} - 1},$$

$$\dot{E} = \frac{\dot{E}_1'}{p}\,\{1 + p^2 Z Y_1\}\ \text{dec}\ \alpha,$$

$$\dot{I} = p\dot{E}_1' Y_1\ \text{dec}\ \alpha,$$

$$\dot{I}_1 = \dot{E}_1' Y_1\ \text{dec}\ \alpha,$$

the equations of the oscillating-current transformer. with $\dot{E}_1'$ as parameter.

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